Quantum Field Theory

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Books

The best book for this course is probably Peskin and Schroeder's very well written "Introduction to Quantum Field Theory"; while described as an "introduction" it is very complete, going beyond where we will go in this course. It uses both canonical quantisation and the newer path-integral quantisation; we shall use the former in this course, but the latter is used in next term's Advanced Quantum Field Theory course.

Ryder's "Quantum Field Theory" is a much more elementary work, but has the advantage of being available in paperback, which may make it a better purchase.

The lecturer retains a fondness for the book she herself learnt from, Bjorken & Drell, but it has mostly been superseded by Peskin and Schroeder. It only covers the canonical quantisation.

Other books worth mentioning are Zee's, and Srednicki's, both using path-integral quantisation, and Weinberg's. The author is a real powerhouse in the field, and as such the book is worth reading, but be aware that it can be very idiosyncratic.

Outline

This course will consist of:

Introduction

Scalar Field Theory - using classical field theory and canonical quantisation

Dirac Field Theory - again using classical field theory and canonical quantisation

Interacting Field Theories - Wick's theorem, Feynman diagrams and Feynman rules

The Electromagnetic Field - covering Quantum Electrodynamics (QED)

Introduction

Quantum field theory is the quantum theory of systems with infinitely many degrees of freedom, especially relativistic fields (which have at least one degree of freedom for every spatial point). QM mostly covers the quantisation of particles, e.g. going from classical mechanics to quantum mechanics to describe things like the Hydrogen atom. The key point of QFT is that states of a field can be interpreted as those of multiparticle relativistic QM, a process sometimes called "second quantisation". It is essential to introduce the concept of a field to avoid giving rise to inconsistencies such as violation of causality. QFT is essential for modern theoretical physics.

Units and Scales

In nature there are three fundamental constants: the speed of light c, Planck's constant $\hbar$, and Newton's constant G. Their dimensions are $[c] = LT^{-1}, [\hbar] = L^2MT^{-1}, [G] = L^3M^{-1}T^{-2}$. Throughout this course we work in natural units , in which $c = \hbar = 1$ and we can express all dimensional quantities in terms of a single scale, mass or equivalently energy ($E = mc^2$). Masses are usually quoted in energy units, e.g. $M_e \simeq \frac{1}{2}$MeV, $M_p \simeq 1$GeV.

Classical Field Theory

We consider dynamical fields - a field is a quantity which is defined at each spacetime point $(t, \mathbf{x})$. We are dealing with a quantity $\phi(t, \mathbf{x})$ having infinitely many degrees of freedom; note that position is now just a label on the field, not a dynamical variable.

For example, the electromagnetic field $\mathbf{E}(t, \mathbf{x}), \mathbf{B}(t, \mathbf{x})$. We can derive these from a single field $A^\mu(t, \mathbf{x}) = (\phi, \mathbf{A})$ (for μ running from 0 to 3) such that this field is a vector in space-time. Then $\mathbf{E} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A}$, $\mathbf{B} = \nabla \times \mathbf{A}$. This ensures that $\nabla \cdot \mathbf{B} = 0, \dot{\mathbf{B}} = -\nabla \times \mathbf{E}$; see the final section of the course for more on the electric field.

Lagrangian Field Theory

In classical mechanics, the action, S, is $\int dtL$, the Lagrangian. In field theory the Lagrangian is $\int d^3\mathbf{x}\mathcal{L}$, where $\mathcal{L}$ is the Lagrangian density, $\mathcal{L}(\phi(t, \mathbf{x}), \partial_\mu\phi)$ (where $\partial_\mu\phi = \frac{\partial}{\partial x^\mu}\phi$). $S = \int dtL = \int dt \int d^3x\mathcal{L}$, which we can call $\int d^4x\mathcal{L}$; d^4x is an integral over all spacetime.

The action is dimensionless, so $[\mathcal{L}] = M^4$ [recall we only have one scale; mass is equivalent to length]. The equations of motion are derived by the principle of least action: we vary the action, keeping end points fixed, and require $\delta S = 0$.

$0 = \delta S = \int d^4x(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{}\partial(\partial_\mu\phi)\delta(\partial_\mu\phi))$; rewriting the second term this is $\int d^4x(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi) + \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi))$. [Can't be write - check] The last term here is a total derivative, which integrates to zero for any variation $\delta\phi(t, \mathbf{x})$ that decays at spatial infinity and vanishes at the initial and final times. So $\delta S = 0$ for variations of this form if $\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$; this is the equation of motion or Euler-Lagrange equation.

Example: The Klein-Gordon Equation

$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2$, where ϕ is a real scalar field and we use the Minkowski metric $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi, \frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$; these are the Euler-Lagrange equations. So the field equations are $\partial_\mu\partial^\mu\phi + m^2\phi = 0$ - the Klein-Gordon equation.

Compare our $\mathcal{L}$ to $L = T - V$ - we identify the kinetic energy $T = \int d^3\mathbf{x}\frac{1}{2}\dot{\phi}^2$ and potential energy $V = \int d^3\mathbf{x}(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2)$.

Lorentz Transformation

A Lorentz Transformation is of the form $x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$. This preserves the relativistic line element $ds^2 = dt^2 - d\mathbf{x}^2$, so $\Lambda^\mu_\sigma \Lambda^\nu_\tau g^{\sigma\tau} = g^{\mu\nu}$, i.e. Lorentz transformations form a Lie group under matrix multiplication (this is covered in the Symmetries of Particles course, or for pure mathematicians, in the Lie Algebras course). For example, rotation about the x^3 axis by angle θ has $\Lambda^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, or a boost along the x^1 axis with velocity $v < 1$ has $\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where $\gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$ (see special relativity).
Lorentz transformations have a representation on the fields, e.g. for $x \to \Lambda x$, under the Lorentz transformation $\phi \to \phi'$, where $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) = \phi(x')$, where $x'^\mu = (\Lambda^{-1})^\mu_\nu x^\nu$ [this is an unusual definition of x', but I believe I have it correctly].

A Lorentz invariant theory is one where the action is unchanged by the Lorentz transformation. E.g. $S = \int d^4x(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi))$, where $U(\phi)$ is a polynomial in ϕ, e.g. $\frac{1}{2}m^2\phi^2$, but might also include terms in ϕ^4 (we shall see later there is a very good reason it could not include terms in ϕ^6 or larger); our integrand is $\mathcal{L}(x)$. $U'(x) = U(\phi'(x)) = U(\phi(x')) = U(x')$. $\partial_\mu\phi' = \frac{\partial}{\partial x^\mu}\phi(x') = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu}\phi(x') = (\Lambda^{-1})^\nu_\mu \partial'_\nu\phi(x')$ where $\partial'_\nu = \frac{\partial}{\partial x'_\nu}$ - the derivative of the scalar field transforms as a vector. So the derivative part of $\mathcal{L}$ $\mathcal{L}'_{\text{deriv}}(x) = (\partial_\mu\phi(x)\partial_\nu\phi(x)g^{\mu\nu})' = g^{\mu\nu}\partial_\mu\phi'\partial_\nu\phi' = g^{\mu\nu}(\Lambda^{-1})^\sigma_\mu(\Lambda^{-1})^\tau_\nu\partial'_\sigma\phi(x)\partial'_\tau\phi(x) = g^{\sigma\tau}\partial'_\sigma\phi(x')\partial'_\tau(\phi(x'))$. $\partial'_\tau\phi(x') = \mathcal{L}_{\text{deriv}}(x')$. $S' = \int d^4x\mathcal{L}(x') = \int d^4x\mathcal{L}(\Lambda^{-1}x)$. Changing variables by $y = \Lambda^{-1}x$, $\det\Lambda = 1$ so our Jacobian is 1; $S' = \int d^4y\mathcal{L}(y) = S$, so the action is invariant.

Therefore, the Klein-Gordon equation is Lorentz invariant; if $\phi(x)$ is a solution, so is $\phi'(x) = \phi(\Lambda^{-1}x)$.

Note: under Lorentz transformation a vector field transforms like $\partial^\mu\phi$, i.e. $A'_\mu(x) = (\Lambda^{-1})^\nu_\mu A_\nu(\Lambda^{-1}x)$.

Noether's Theorem

If a Lagrangian field theory has an infinitesimal symmetry, then there is an associated current, j_μ, which is conserved, i.e. $\partial_\mu j^\mu = 0$, i.e. $\frac{d}{dt}j^0 + \nabla\cdot\mathbf{j} = 0$. A conserved current implies a conserved charge $Q = \int d^3\mathbf{x}j^0$: $\frac{d}{dt}Q = \int d^3\mathbf{x}\frac{d}{dt}j^0 = -\int d^3\mathbf{x}\nabla\cdot\mathbf{j} = 0$ by the divergence theorem, since $j \to 0$ as $|\vec{x}| \to \infty$.

Proof: Consider an infinitesimal variation in ϕ $\phi(x) \to \phi'(x) = \phi(x) + \alpha\delta\phi$ with α infinitesimal. We assume this is a symmetry, i.e. does not affect the field equations, which must be because it leaves the action invariant. The Lagrangian density therefore needs to be invariant up to a total derivative: $\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_\mu X^\mu(x)$ (1) (of course, X^μ may be zero). We have $\mathcal{L}(x) \to \mathcal{L}(x)\alpha\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \alpha\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) = \mathcal{L}(x) + \alpha\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi) + \alpha(\frac{\partial}{\partial\phi}\mathcal{L} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi)$ (2); this last term is 0 by the Euler-Lagrange equations, so $\delta\mathcal{L} = \alpha\partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi)$. Identifying (1) and (2), $j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - X^\mu$ is conserved: $\partial_\mu j^\mu = 0$, assuming the field equations are obeyed.

Summary of previous lecture: $x \to \Lambda x, \phi \to \phi'(x) = \phi(\Lambda^{-1}x), x'^{\mu} = (\Lambda^{-1})^{\mu}_{\nu}x^{\nu}$. $u(\phi) \to u'(\phi(x)) = u(\phi(x'))$. $\partial_\mu \phi' = (\Lambda^{-1})^{\nu}_{\mu}\partial'_{\nu}\phi(x')$ where $\partial'_\nu = \frac{\partial}{\partial x'^{\nu}}$. The action is unchanged; Noether's theorem is that if $\mathcal{L}$ has an infinitesimal symmetry, there is a conserved current j^μ such that $\partial_\mu j^\mu = 0$, and therefore also a conserved charge.

Example: Complex Klein Gordon field $\phi(x)$, treat ϕ and $\phi^\star$ as independent. $\mathcal{L} = \partial_\mu\phi^\star\partial^\mu\phi - m^2\phi^\star\phi$ (note that because this is the complex case there are no factors of $\frac{1}{2}$). Our symmetry is a phase rotation: $\phi \to \phi e^{i\alpha}, \phi^\star \to \phi^\star e^{-i\alpha}$. Recall: we wrote $\phi \to \phi' = \phi + \alpha\delta\phi$. So here $\alpha\delta\phi = i\alpha\phi$ (note that I have been writing δ where I mean Δ throughout; hopefully the difference is unimportant), $\alpha\delta\phi^\star = -i\alpha\phi^\star$; what we previously called X^μ (a misleading label since it need not relate to the x direction at all) must be 0 here. $j^\mu = i((\partial^\mu\phi^\star)\phi - \phi^\star(\partial^\mu\phi))$; we can verify $\partial_\mu j^\mu = 0$ directly from the Klein-Gordon equation. j^μ is interpreted as electric current and $Q = \int d^3\mathbf{x} j^0$ as electric charge.

Example 2: Infinitesimal translation. $x^\mu \to x^\mu + a^\mu, \phi(x) \to \phi(x+a) = \phi(x) + a^\nu\partial_\nu\phi(x)$ (using a Taylor expansion; translation is infinitesimal), and similarly for the derivative. So $\mathcal{L}(x) \to \mathcal{L} + a^\nu\partial_\nu\mathcal{L} = \mathcal{L} + a^\nu\partial_\nu(\delta^\mu_\nu\mathcal{L})$, the δ being a notational convenience. Since the infinitesimal parameter a^ν is a vector, the conserved current is a tensor: $T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi = \delta^\mu_\nu\mathcal{L}$, such that $\partial_\mu T^\mu_\nu = 0$.

T^μ_ν is the energy-momentum tensor. The conserved "charge" will be a vector; the conserved charge associated with time translation is the energy (Hamiltonian) $H = \int T^{00}d^3\mathbf{x} \equiv E = \int(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial_0\phi - \mathcal{L})d^3\mathbf{x}$. Conserved momentum is associated with spatial translations: $p^i = \int T^{0i}d^3\mathbf{x} = -\int\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial_i\phi d^3\mathbf{x}$.

The Hamiltonian

Write $\dot{\phi} = \partial_0\phi$. In Lagrangian field theory, the degrees of freedom are the field values at each spatial point, and we integrate over all space rather than a finite summation. Suppose $L(t) = \int\mathcal{L}(x)d^3\mathbf{x}$ ($\mathcal{L}$ of course being a function of ϕ); the canonical momentum of the field $\phi(x)$ is $\Pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)}$. Then Hamiltonian $H = \int(\Pi(x)\dot{\phi}(x) - \mathcal{L}(x))d^3\mathbf{x}$, with $\dot{\phi}$ eliminated in favour of Π as in classical dynamics. H is the same expression we had for conserved energy earlier.

E.g. $\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\nabla\phi\cdot\nabla\phi - u(\phi)\Pi = \dot{\phi}$. $H = \int d^3\mathbf{x}\mathcal{H}$ where $\mathcal{H}$ is the Hamiltonian density; this is $\int d^3\mathbf{x}(\frac{1}{2}\Pi^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + u(\phi))$. The equations of motion can be rederived via canonical Poisson brackets for Π and ϕ (e.g. Hamilton's equation $\dot{\phi} = \frac{\partial\mathcal{H}}{\partial\Pi}, \dot{\Pi} = \frac{\partial\mathcal{H}}{\partial\phi}$).

Canonical Quantisation of Klein-Gordon Theory

Canonical Quantisation

In QM, canonical quantisation took us from classical to quantum theory. For generalised coordinates q_i and their canonical momentum p_i, the commutation relations are: $[q_i, p_j] = i\delta_{ij}, [q_i, q_j] = [p_i, p_j] = 0$ (recall we are taking $\hbar = 1$ in this course).

In field theory, we do the same for the field $\phi_i(x)$ and its conjugate momentum $\Pi_i(x)$, but since we have a continuous and not discrete system we obtain the Dirac delta function rather than the Kronecker δ. $[\phi(\mathbf{x}), \Pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$

(the (3) denoting a 3D δ-function, a distinction which is important because we shall also use 4D delta functions in the course), $[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\Pi(\mathbf{x}), \Pi(\mathbf{y})] = 0$. At present we are working in the Schrödinger picture where ϕ, Π do not depend on time, but we shall soon move on to the Heisenberg picture and include time.

The Hamiltonian also becomes an operator. Expand the classical Klein-Gordon field: $\phi(t, \mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(t, \mathbf{p})$ (note $\phi^\star(\mathbf{p}) = -\phi(p)$ since $\phi(\mathbf{x})$ is real). The Klein-Gordon equation becomes $(\frac{\partial^2}{\partial t^2} + (|\mathbf{p}|^2 + m^2))\phi(t, \mathbf{p}) = 0$; compare this with the Simple Harmonic Oscillator (SHO) of frequency $E_p = \sqrt{|\mathbf{p}|^2 + m^2}$. Recall (in QM with $\hbar = 1$) the SHO has $H_{\text{SHO}} = \frac{p^2}{2m} + \frac{1}{2}\omega^2\phi^2$; to find the eigenvalues we write $\phi = \sqrt{\frac{1}{2\omega}}(a + a^\dagger), p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$; $[\phi, p] = i, [a, a^\dagger] = 1$. Then $H_{\text{SHO}} = \omega(aa^\dagger + \frac{1}{2})$.

The ground state $|0\rangle$ is such that $a|0\rangle = 0$; it is an eigenstate of H with eigenvalue $\frac{1}{2}\omega$, the zeropoint energy. Since $[H, a^\dagger] = \omega a^\dagger, [H, a] = -\omega a$, we then have states $|n\rangle = (a^\dagger)^n|0\rangle$ (not normalized) with eigenvalues $(n + \frac{1}{2})\omega$. We will do the same thing in QFT.

Apply this to the free scalar field: let $\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}}(a_p e^{i\mathbf{p}\cdot\mathbf{x}} + a_p^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}), \Pi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\sqrt{\frac{E_p}{2}}(-i)(a_p e^{i\mathbf{p}\cdot\mathbf{x}} - a_p^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}})$, by analogy with the SHO. (Recall $E_p = \sqrt{|\mathbf{p}|^2 + m^2}$. It's useful to rearrange this; since we are integrating over all momentum, we can switch the second term around and write it as $\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}}(a_p + a_{-p}^\dagger)e^{i\mathbf{p}\cdot\mathbf{x}}, \Pi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}(-i)\sqrt{\frac{E_p}{2}}(a_p - a_{-p}^\dagger)e^{i\mathbf{p}\cdot\mathbf{x}}$. The canonical commutation relations (CCR) imply $[a_p, a_{p'}] = [a_p^\dagger, a_{p'}^\dagger] = 0, [a_p, a_{p'}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}')$. Conversely, $[\phi(\mathbf{x}), \Pi(\mathbf{x}')] = \int \frac{d^3\mathbf{p} d^3\mathbf{p}'}{(2\pi)^6}(-\frac{i}{2})(\frac{E_{p'}}{E_p})^{\frac{1}{2}}([a_{-p}^\dagger, a_{p'}] - [a_p, a_{-p'}^\dagger])e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{p}'\cdot\mathbf{x}')} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}')}(i) = i\delta^{(3)}(\mathbf{x} - \mathbf{x}')$ [and this is consistent with what it should be].

We can now write the Hamiltonian in terms of creation and annihilation operators $a^\dagger, a$: $H = \int d^3\mathbf{x} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} e^{i(\mathbf{p}+\mathbf{p}')\mathbf{x}}(-\sqrt{\frac{E_p E_p'}{4}}(a_p - a_{-p}^\dagger)(a_{p'} - a_{-p'}^\dagger) + \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{4\sqrt{E_p E_{p'}}}(a_p + a_{-p}^\dagger)(a_{p'} + a_{-p'}^\dagger))$. (This comes from writing $H = \int d^3x \frac{1}{2}\Pi(\mathbf{x})\Pi(\mathbf{x})$ and using $\Pi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\Pi(\mathbf{p})$). Now the $\int d^3x$ becomes $(2\pi)^3\delta^{(3)}(\mathbf{p} + \mathbf{p}')$, the $\int d^3\mathbf{p}'$ gives us that $\mathbf{p}' = -\mathbf{p}$, and recall $E_p^2 = p^2 + m^2$, then $H = \int \frac{d^3\mathbf{p}}{(2\pi)^3} E_p(\frac{1}{2}a_p^\dagger a_p + \frac{1}{2}a_p a_p^\dagger) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} E_p(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger])$. The second term is proportional to $\delta(0)$, an infinite constant - it is the zero point energy of infinitely many oscillator systems. Since experiments only measure energy difference, not total energy, we can ignore this term and set $H = \int \frac{d^3\mathbf{p}}{(2\pi)^3} E_p a_p^\dagger a_p$. This is the normal ordered form; we shall cover this in detail later. The normal orgered form of an operator has all annihilation operators placed to the right.

We can now evaluate $[H, a_p^\dagger] = E_p a_p^\dagger, [H, a_p] = -E a_p$. Postulating that the vacuum state satisfies $a_p|0\rangle = 0 \forall p$ so $H|0\rangle = 0$ and $|0\rangle$ has zero energy (unsurprising since we have just dropped the zero point energy). All other states can be built by the action of creation operators on $|0\rangle$, e.g. $a_p^\dagger|0\rangle$ gives a state with energy E_p, $a_p^\dagger a_{p'}^\dagger|0\rangle$ a state with energy $E_p + E_{p'}$, and so on.

Now interpret the eigenstates. Recall the momentum operator $\mathbf{P} = -\int d^3\mathbf{x}\Pi(\mathbf{x})\nabla\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\mathbf{p}a_p^\dagger a_p$. So $a_p^\dagger$ creates momentum $\mathbf{p}$ and energy $E_p = (|\mathbf{p}|^2 + m^2)^{\frac{1}{2}}$. Similarly $a_p^\dagger a_{p'}^\dagger|0\rangle$ has momentum $\mathbf{p}+\mathbf{p'}$. We interpret $a_p^\dagger|0\rangle$ as a 1-particle space, where the particle has rest mass m, energy E_p and momentum $\mathbf{p}$.

We can now determine the spin statistics of the Klein-Gordon field. Consider the two particle state $a_p^\dagger a_{p'}^\dagger|0\rangle$. Since $a_p^\dagger, a_{p'}^\dagger$ commute this is identical to $a_{p'}^\dagger a_p^\dagger|0\rangle$ in which the two particles are interchanged - so the Klein-Gordon field obeys Bose-Einstein statistics.

The full space is spanned by acting on the vacuum with all possible $a^\dagger$s: $|0\rangle, a_p^\dagger|0\rangle, a_p^\dagger a_{p'}^\dagger|0\rangle, \ldots, a_{p_1}^\dagger \ldots a_{p_n}^\dagger|0\rangle$. This space is called Fock space; it is simply the sum of the n-particle Hilbert spaces. The operator $N = \int \frac{d^3\mathbf{p}}{(2\pi)^3} a_p^\dagger a_p$ counts the number of particles in a given Fock space - it's the number operator, and satisfies $N|p_1 \ldots p_n\rangle = n|p_1 \ldots p_n\rangle$ and $[N, H] = 0$ - so the particle number is conserved (this is true for free field theory).

Normalisation

The vacuum $|0\rangle$ is defined such that $\langle 0|0\rangle = 1$. The 1-particle states $|p\rangle$ are such that $\langle \mathbf{p}|\mathbf{p}\rangle = 2E_p(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})$; the factor of E_p is needed for Lorentz invariance. We can verify that $|\mathbf{p}\rangle = \sqrt{2E_p}a_p^\dagger|0\rangle$. Finally, let us consider $\phi(\mathbf{x})|0\rangle$; recall $\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\sqrt{\frac{1}{2E_p}}(a_p e^{i\mathbf{p}\cdot\mathbf{x}} + a_p^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}})$, so $\phi(\mathbf{x})|0\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{p}\rangle$ - a linear superposition of single-particle states.

It was asked after this lecture: why did we not write everything in covariant form? We were working in the Schrödinger picture, where operators are time-independent. To include the time dependence, we need to use the Heisenberg picture.

The Heisenberg Picture

Consider the Klein-Gordon field in spacetime. In the Heisenberg picture operators become time-dependent and states time-independent. Heisenberg operators obey $i\frac{\partial}{\partial t}O = [O, H]$, or equivalently $O(t) = e^{iHt}O(0)e^{-iHt}$.

In QFT one has operators depending on space and time, so $\phi(x) \equiv \phi(t,\mathbf{x})/e^{iHt}\phi(\mathbf{x})e^{-iHt}$ where $\phi(\mathbf{x})$ is the Schrödinger picture field, and similarly for $\Pi(x)$. The advantage to working with Heisenberg picture fields is that they are closer to being Lorentz invariant - the Klein-Gordon field is Lorentz invariant, and it would be a shame to sacrifice this in quantising it.

Now, $i\frac{\partial}{\partial t}\phi(t,\mathbf{x}) = [\phi(t,\mathbf{x}), H] = [\phi(t,\mathbf{x}), \int d^3\mathbf{x}'(\frac{1}{2}\Pi^2(t,\mathbf{x}') + \frac{1}{2}(\nabla'\phi(t,\mathbf{x}')) + \frac{1}{2}m^2\phi^2(t,\mathbf{x}'))] = \int d^3\mathbf{x}' i\delta^3(\mathbf{x}-\mathbf{x}')\Pi(t,\mathbf{x}') = i\Pi(t,\mathbf{x})$; similarly $i\frac{\partial}{\partial t}\Pi(t,\mathbf{x}) = -i(-\nabla^2+m^2)\phi(t,\mathbf{x})$, so $\frac{\partial^2}{\partial t^2}\phi(t,\mathbf{x}) = (\nabla^2 - m^2)\phi(t,\mathbf{x})$ - we have recovered the Klein-Gordon equation from Heisenberg's equation.

By writing ϕ, Π in terms of $a, a^\dagger$ we get a better understanding of the time dependence. Note $Ha_p = a_p(H - E_p)$ so $H^n a_p = a_p(H-E_p)^n$, and similarly for $a^\dagger$. So $e^{iHt}a_p = a_p e^{iHt}e^{-iE_pt}$ $\therefore$ $e^{iHt}a_p e^{-iHt} = a_p e^{-iE_pt}$, and similarly $e^{iHt}a_p^\dagger e^{-iHt} = a_p^\dagger e^{iE_pt}$. We can then substitute this into our expression for $\phi(t,\mathbf{x})$, obtaining $\phi(t,\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \times (a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x})\mid_{p^0=E_p}$, recalling $p\cdot x = p^0 t - \mathbf{p}\cdot\mathbf{x} = E_p t - \mathbf{p}\cdot\mathbf{x}$; similarly for $\Pi(t,\mathbf{x})$. For the Heisenberg field, $\langle p|\phi(x)|0\rangle = e^{ip\cdot x}$ (this is an exercise, since it may be on the first example sheet).

Just as H evolves an operator in time, so $\mathbf{P}$ translates it in space: $\mathbf{P}a_p = a_p(\mathbf{P}-\mathbf{p})$ etc. The components of $\mathbf{P}$ commute with H and amongst themselves, so $e^{iP\cdot x} = e^{iHt-i\mathbf{p}\cdot\mathbf{x}}$ is well defined, and the Heisenberg field obeys $\phi(x) = e^{iP\cdot x}\phi(0)e^{-iP\cdot x}$.

Commutators and Propagators

Now that we have the Heisenberg fields we need only consider $\phi(x)$, because $\Pi(x) = \dot{\phi}(x)$. We are interested in quantities like $[\phi(x), \phi(y)]$ and $\langle 0|\phi(x)\phi(y)|0\rangle$. Let $\phi(x) = \phi^+(x) + \phi^-(X)$ where $\phi^+(x) = \int \frac{d^3\mathbf{p}}{(2x)^3}\frac{1}{\sqrt{2E_p}}a_p e^{-ip\cdot x}$, called the positive frequency part, and $\phi^-(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_p^\dagger e^{ip\cdot x}$ the negative frequency part. Then clearly ϕ^+s commute with other ϕ^+ and ϕ^- commute with other ϕ^-, so the nonzero part of the commutator will come from $[\phi^+(x)\phi^-(y)] = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2E_p e_{p'}2}}[a_p, a_{p'}^\dagger]e^{i(p\cdot x - p'\cdot y)} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)}$; this quantity is called $D(x-y)$. $D(x-y)$ is a Lorentz invariant function of $(x-y)$; it is nonvanishing for timelike and spacelike $(x-y)$,but does decay like $e^{-m|\mathbf{x}-\mathbf{y}|}$ for large $|\mathbf{x}-\mathbf{y}|$ if $x^0 = y^0$.

Hence $[\phi(x), \phi(y)] = [\phi^+(x), \phi^-(y)] + [\phi^-(x), \phi^+(y)] = D(x-y) - D(y-x)$. If $x-y$ is spacelike, this vanishes, since $x-y$ can be Lorentz transformed to $y-x$. This generalises what we postulated for equal time commutators: the fields $\phi(x), \phi(y)$ are simulatneously measurable if $x-y$ is spacelike.

Since $\phi^+(x)|0\rangle = 0$, $\langle 0|\phi^-(x) = 0$, we have $\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\phi^+(x)\phi^-(y)|0\rangle = \langle 0|[\phi^+(x), \phi^-(y)]|0\rangle = D(x-y)$. $D(x-y)$ is a propagator - it is the amplitude for a particle created at y to propagate to x.

In interacting QFT the most useful quantity is the Feynman Propagator $D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle$ where T denotes time ordered products - the operators are put in time ordering with the latest on the left. We can do this by introducing θ-functions (AKA the Heaviside function - $\theta(\tau) = 1$ if $\tau > 0$, 0 if $\tau < 0$ ($\theta(0) = \frac{1}{2}$)) of the time variable - $D_F(x-y) = \theta(x^0 - y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\phi(y)\phi(x)|0\rangle$, so $D_F(x-y) = \theta(x^0-y^0)D(x-y) + \theta(y^0-x^0)D(y-x)$ ($\star$). This can be written as a four-momentum integral $D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip\cdot(x-y)}$ with p^0 unconstrained, i.e. the p^0 integration deformed slightly off the real axis: $E_p^2 = |\mathbf{p}|^2 + m^2$, so there are poles at $p^0 = \pm E_p$, so we integrate along the contour $\text{Im}p^0 = \epsilon\text{Re}p^0$ rather than the real axis in p^0-space. To verify this, $\frac{1}{p^2-m^2} = \frac{1}{p^{0\,2}-E_p^2} = \frac{1}{(p^0-E_p)(p^0+E_p)}$ - as a function of p^0 this has poles at $p^0 = \pm E_p$ and residues of $\pm\frac{1}{2E_p}$. To perform the p^0 integration, for $x^0 > y^0$ we close the contour in the lower half plane (the integrand is exponentially small on the semicircle). The residue theorem gives $D_f(x-y) = \int \frac{d^3\mathbf{p}}{(2\pi)^4}(-2\pi i)\frac{i}{2E_p}e^{-E_p(x^0-y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)}$. For $y^0 > x^0$, we similarly close the contour in the upper half-plane and find $D_F(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{iE_p(x^0-y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(y-x)}$ (this is correct; note we have gained (or lost) a - sign by taking the integral in the opposite direction along the contour); this agrees with the original definition of a Feynman propagator ($\star$).

The contour we have used lies mostly along the real p^0 axis, with a small half-loop up around the pole at $p^0 = E_p$ and down around the pole $p^0 = -E_p$.

Alternatively we can use Feynman's $i\epsilon$ perscription - we replace the integrand by $\frac{1}{p^2-m^2+i\epsilon}$ (for ϵ real, positive and infinitesimal), perturbing the poles off the real axis - then the pole at E_p moves below the real p^0 axis and that at $-E_p$ moves above the real p^0 axis, so we can just integrate along the real p^0 axis as usual.

The Feynman propagator is a Green function for the Klein-Gordon operator, since $(\partial_x^2+m^2)D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}(-p^2+m^2)e^{ip(x-y)}$ (the $-p^2$ being from the Fourier transformation of the ∂_x^2) which is $-i\int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)} = i\delta^{(4)}(x-y)$.

Lorentz Transformation

We had $x^\mu \to \Lambda^\mu_\nu x^\nu$, and the Klein-Gordon field $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$. This is an active transformation - we consider shifting the field while keeping our coordinates constant (of course this is entirely equivalent to the perhaps more "physically realistic" case of a passive transformation, keeping the field fixed while we shift our coordinates - for which we would have to interchange Λ and Λ^{-1} and redo our equations).

E.g. consider (for simplicity) the non-relativistic case $\phi(\mathbf{x})$. This is a distribution - assume it has an accumulation at $\mathbf{x} = \mathbf{x}_0 = (1,0,0)$. Now perform a rotation $\mathbf{x} \mapsto R\mathbf{x}$ about the z axis, giving a new field $\phi'(\mathbf{x})$. This has an accumulation at $\mathbf{x} = (0,1,0) = R\mathbf{x}_0$. If we want to express $\phi'(\mathbf{x})$ in terms of the old $\phi(\mathbf{x})$, we put ourselves at $\mathbf{x} = (0,1,0)$ to see what the field looked like from $\mathbf{x}_0$: $\mathbf{x}_0 = R^{-1}\mathbf{x}, \phi'(\mathbf{x}) = \phi(R^{-1}\mathbf{x})$.

Our relativistic fields undergo active Lorentz transformations; the Klein-Gordon field is the simplest example. The Klein-Gordon field is a scalar field transforming under a Lorentz Transformation as $x^\mu \to \Lambda^\mu_\nu x^\nu$ as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$. On quantisation the Klein-Gordon field is spin 0. Other, more general fields have more complicated transformations, e.g. the Dirac field which represents a spin-$\frac{1}{2}$ field. To understand the Dirac equation, we need to study the Lorentz group.

For example, the electromagnetic field, a vector field $A_\mu(x)$, has $A^\mu(x) \to \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x)$. In general fields transform as $\phi^a(x) \to D[\Lambda]^a_b\phi^b(\Lambda^{-1}x)$, where the matrices $D[\Lambda]$ form a representation of the Lorentz group, i.e. $D[\Lambda_a]D[\Lambda_b] = D[\Lambda_a\Lambda_b], D[\Lambda^{-1}] = D[\Lambda]^{-1}, D[1] = 1$. Writing $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$ with ω infinitesimal, the condition for a Lorentz transformation $\Lambda^\mu_\sigma\Lambda^\nu_\tau g^{\sigma\tau} = g^{\mu\nu}$ is $\omega^{\mu\nu}+\omega^{\nu\mu} = 0$ i.e. ω is antisymmetric. Now an antisymmetric 4×4 matrix has $4\times\frac{3}{2} = 6$ independent components, i.e. there are 6 [infinitesimal] transformations of the Lorentz group - 3 rotations and 3 boosts. So the basis for these six 4×4 matrices is $(M^{\sigma\tau})^\mu_\nu$ where σ,τ are a pair of antisymmetric indices $\sigma,\tau = 0,1,2,3$, i.e. $M^{01} = M^{10}$ etc. σ,τ label independent matrices, μ,ν are indices identifying matrix rows and columns. These six matrices are given by $(M^{\sigma\tau})^{\mu\nu} = g^{\sigma\mu}g^{\tau\nu} - g^{\tau\mu}g^{\sigma\nu}$. Note that when we lower indices with the Minkowski metric, we pick up some - signs so the matrices with indices lowers are no longer always explicitly antisymmetric, e.g. $(M^{01})^\mu_\nu = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, (M^{12})^\mu_\nu =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

M^{01} generates boosts in the x^1 direction; it's real and symmetric. M^{12} generates rotations in the $(x^1 - x^2)$ plane.

Now $\omega^\mu_\nu = \frac{1}{2}\Omega_{\sigma\tau}(M^{\sigma\tau})^\mu_\nu$ where $\Omega_{\sigma\tau}$ are six [infinitesimal] numbers. $M^{\sigma\tau}$ are the generators of the Lorentz transformations and obey the Lie algebra $[M^{\sigma\tau}, M^{\rho\eta}] = g^{\tau\rho}M^{\sigma\eta} - g^{\sigma\rho}M^{\tau\eta} + g^{\sigma\eta}M^{\tau\rho} - g^{\tau\eta}M^{\sigma\rho}$. A finite (rather than infinitesimal) Lorentz transformation is given by exponentiation: $\Lambda = \exp\frac{1}{2}(\Omega_{\mu\nu}M^{\mu\nu})$.

We saw that the Klein-Gordon field gives rise to particles when quantised, but relatively uninteresting ones - spin 0. We want to consider fields that will give rise to interesting particles of higher spin. So consider fields with their own internal degrees of freedom, which will transform differently - consider the EM field, where under a boost what was once a pure electric field gains a magnetic component.

So we ask: how can fields transform under LTs? To answer this, we need to characterize LTs more thoroughly, and consider the Lorentz group algebra.

Recall that a general LT is $\Lambda = \exp(\frac{1}{2}\Omega_{\mu\nu}M^{\mu\nu})$, where $M^{\mu\nu}$ are matrices $(M^{\mu\nu})^{\sigma\tau} = g^{\mu\sigma}g^{\nu\tau} - g^{\nu\sigma}g^{\mu\tau}$. So $[M^{\sigma\tau}, M^{\rho\eta}] = g^{\tau\rho}M^{\sigma\eta} - g^{\sigma\rho}M^{\tau\eta} + g^{\sigma\eta}M^{\tau\rho} - g^{\tau\eta}M^{\sigma\rho}$. The explanation for why it was ok to consider only infinitesimal group elements and then exponentiate and claim this gives a general group element is in the symmetries and particles course.

We seek other representations of the Lorentz group, which will be given by other matrices satisfying the same algebra, i.e. the same commutation relation.

The Dirac Equation

This is the equation of motion obeyed by a classical spin-$\frac{1}{2}$ field. Upon quantisation it will give rise to spin-$\frac{1}{2}$ fermions. The equation is $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$, where γ^μ for $\mu = 0, 1, 2, 3$ are four 4×4 matrices and ψ is a 4-component complex column vector $\psi_\alpha(x)$, usually called a Dirac spinor (m is implicitly multiplied by a unit matrix). The γ^μ satisfy a Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}1_4$, where $\{\gamma^\mu, \gamma^\nu\}$ is the anticommutator $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$ (and 1_4 is the 4×4 identity). Note this Clifford algebra doesn't actually specify the dimension, but 4 is the smallest number for which (nontrivial) solutions exist. For example, $\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ where σ^i are the Pauli matrices $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (which themselves satisfy $\{\sigma^i, \sigma^j\} = 2\delta^{ij}1_2$). This is called the chiral basis, for reasons which will become clear in a few lectures' time.

We can construct other representations by $U\gamma^\mu U^{-1}$ for any invertible U. However, it is a provable theorem that up to such conjugation, the only irreducible representation for the Clifford algebra is this one.

Claim: Each component ψ_α of the Dirac spinor satisfies the Klein-Gordon equation: $(i\gamma^\nu\partial_\nu + m)(i\gamma^\mu\partial_\mu - m)\psi = 0 = (-\gamma^\nu\gamma^\mu\partial_\nu\mu - m^2)\psi$. By symmetry in the derivatives ($\partial_\mu, \partial_\nu$ commute and the equation is symmetric) this is $(-\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\nu\partial_\mu - m^2)\psi = -(\partial^2 + m^2)\psi_\alpha = 0$. Here we have an implicit unit matrix, so the equation must be true for all components.

Lorentz Invariance of the Dirac Equation

We want to find a Lorentz transformation of the Dirac spinor $\psi'_\alpha(x) = S[\Lambda]_{\alpha\beta}\psi_\beta(\Lambda^{-1}x)$ where $\Lambda = \exp(\frac{1}{2}\Omega_{\mu\nu}M^{\mu\nu})$ and $S[\Lambda] = \exp(\frac{1}{2}\Omega_{\mu\nu}S^{\mu\nu})$ where the $S^{\mu\nu}$ satisfy the Lorentz algebra.

Recall $\psi'_\alpha(x) = S[\Lambda]_{\alpha\beta}\psi_\beta(\Lambda^{-1}x)$, where $\Lambda = \exp(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma})$ where $(M^{\rho\sigma})^{\mu\nu} = g^{\rho\mu}g^{\sigma\nu} - g^{\sigma\mu}g^{\rho\nu}$. $[M^{\rho\sigma}, M^{\tau\nu}] = g^{\sigma\tau}M^{\rho\nu} - g^{\rho\tau}M^{\sigma\nu} + g^{\rho\nu}M^{\sigma\tau} - g^{\sigma\nu}M^{\rho\tau}$, and $S[\Lambda] = \exp(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$. The generators $S^{\rho\sigma}$ are defined by $S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma]$; recall $\{\gamma^\rho, \gamma^\sigma\} = 2g^{\rho\sigma}1_4$.

Claim: $S^{\rho\sigma}$ form a representation of the Lorentz Lie algebra, i.e. $[S^{\mu\nu}, S^{\rho\sigma}] = g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} + g^{\mu\sigma}S^{\nu\rho} - g^{\nu\sigma}S^{\mu\rho}$. The proof of this result is not hard, but contains a lot of tedious algebra - the reader should consult page 84 of David Tong's online notes. A simplifying "trick", which we shall need later, is that one should first show $[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}$. We can write this as $-(M^{\mu\nu})^\rho_\sigma \gamma\sigma$.

Example: how do we know this is a new representation, not the usual 4×4 matrix Λ? Use $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$. For rotations, $S^{ij} = -\frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$; write $\Omega_{ij} = -\epsilon_{ijk}\varphi^k$. Then $S[\Lambda] = \exp(\frac{1}{2}\Omega_{\mu\nu}S^{\mu\nu}) = \begin{pmatrix} e^{i\frac{\varphi\cdot\sigma}{2}} & 0 \\ 0 & e^{i\frac{\varphi\cdot\sigma}{2}} \end{pmatrix}$. So under a 2π rotation about the z axis, $\varphi = (0, 0, 2\pi)$ and $S[\Lambda] = \begin{pmatrix} e^{i\pi\sigma^3} & 0 \\ 0 & e^{i\pi\sigma^3} \end{pmatrix} = -1$.

For boosts, $S^{0i} = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$, write $\Omega_{0i} = \chi_i$. Then $S[\Lambda] = \begin{pmatrix} e^{i\frac{\chi\cdot\sigma}{2}} & 0 \\ 0 & e^{i\frac{\chi\cdot\sigma}{2}} \end{pmatrix}$.

Finally, we can show that the Dirac equation is invariant under a Lorentz transformation. $(i\gamma^\mu\partial_\mu - m)\psi'(x) = (i\gamma^| mu\partial_\mu - m)S[\Lambda]\psi(\Lambda^{-1}x) = S[\Lambda](iS[\Lambda]^{-1}\gamma^\mu S[\Lambda]\partial_\mu - m)\psi(\Lambda^{-1}x)$. What is $S[\Lambda]^{-1}\gamma^\mu S[\Lambda]$? We shall use the "usual Lie algebra trick": we work infinitesimally, then exponentiate. Infinitesimally, this is $(1 - \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})\gamma^\mu(1 + \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}) = \gamma^\mu - \frac{1}{2}\Omega_{\rho\sigma}[S^{\rho\sigma}, \gamma^\mu] + O(\Omega^2) = \gamma^\mu + \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu_\nu\gamma^\nu$, which exponentiating becomes $\Lambda^\mu_\nu\gamma^\nu$. So $(i\partial_\mu\gamma^\mu - m)\psi'(x) = S[\Lambda](i\Lambda^\mu_\nu\gamma^\nu\partial_\mu - m)\psi(\Lambda^{-1}x) = S[\Lambda](i\Lambda^\mu_\nu\gamma^\nu(\Lambda^{-1})^\lambda_\mu\tilde{\partial}_\lambda - m)\psi(\tilde{x})$, where $\tilde{\partial} = \frac{\partial}{\partial\tilde{x}}$ and $\tilde{x} = \Lambda^{-1}x$. This is $S[\Lambda](i\gamma^\nu\tilde{\partial}_\nu - m)\psi(\tilde{x}) = 0$.

Constructing an Action for the Dirac Equation

We could try $\mathcal{L} = \psi^\dagger(i\partial_\mu\gamma^\mu - m)\psi$ (Notation: we write $\not{\partial}$ for $\gamma^\mu\partial_\mu$, and later we shall see $\not{A} = \gamma^\mu A_\mu$). This has a problem - it's not Lorentz invariant, because $\psi \to S[\Lambda]\psi \Rightarrow \psi^\dagger \to \psi^\dagger S[\Lambda]$, but $S[\Lambda]^\dagger S[\Lambda] \neq 1$. In fact, there are no (finite dimesional) unitary representations of the Lorentz group: to see this, note that $(\gamma^0)^2 = 1$, so γ^0 must have real eigenvalues, but $(\gamma^i)^2 = -1$ so γ^i must have imaginary eigenvalues. So while it is possible to pick γ^0 Hermitian, we would then have γ^i anti-hermitian and so $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ can never be Hermitian for all possible μ, ν (nor anti-Hermitian).

To get around this, note $\gamma^0\gamma^\mu\gamma^0 = \gamma^0 = (\gamma^0)^\dagger$ for $\mu = 0$ and $-\gamma^i = (\gamma^i)^\dagger$ for $\mu = i$, i.e. it is $(\gamma^\mu)^\dagger$. With this in mind, we define the <u>Dirac conjugate</u> $\overline{\psi} = \psi^\dagger\gamma^0$. We construct a Lorentz invariant action by $S = \int d^4x\overline{\psi}(i\not{\partial} - m)\psi$. This is Lorentz invariant if $S[\Lambda]^\dagger\gamma^0 S[\Lambda] = \gamma^0$ (e.g. $m\overline{\psi}\psi = m\psi^\dagger\gamma^0\psi \to m\psi^\dagger S^\dagger\gamma^0 S\psi$), but this property follows (by some algebra) from $(S^{\mu\nu})^\dagger = \frac{1}{4}[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}] = -\gamma^0 S^{\mu\nu}\gamma^0$.

Chiral Spinors

Recall that a Dirac spinor transforms as $\psi'_\alpha(x) = S[\Lambda]_{\alpha\beta}\psi_\beta(\Lambda^{-1}x)$. In the chiral basis $\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ we found that $S[\Lambda] = \begin{pmatrix} e^{\frac{i}{2}\varphi\cdot\sigma} & 0 \\ 0 & e^{\frac{i}{2}\varphi\cdot\sigma} \end{pmatrix}$ for rotations and $S[\Lambda] = \begin{pmatrix} e^{\frac{1}{2}\chi\cdot\sigma} & 0 \\ 0 & e^{-\frac{1}{2}\chi\cdot\sigma} \end{pmatrix}$ for boosts. This is block-diagonal in both cases, so the representation $S[\Lambda]$ must be a reducible representation of the Lorentz group. It decomposes into two irreducible representations acting on the Weyl spinors $u_\pm$, with $\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$. Under rotations, $u_\pm \to e^{\frac{i}{2}\varphi\cdot\sigma}u_\pm$ while under boosts, $u_\pm \to e^{\pm\frac{1}{2}\chi\cdot\sigma}u_\pm$. The Dirac Lagrangian is $\overline{\psi}(i\partial\!\!\!/ - m)\psi$, where $\overline{\psi} = \psi^\dagger\gamma^0$; this becomes $\begin{pmatrix} u_+^\dagger & u_-^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \begin{pmatrix} -M & i\sigma^\mu\partial_\mu \\ i\overline{\sigma_\mu}\partial_\mu & -M \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$, where $\sigma^\mu = (1, \sigma^i), \overline{\sigma^\mu} = (1, -\sigma^i)$ ($\overline{\sigma^\mu}$ is just notation, unrelated to the Dirac conjugate of σ^μ). This is $iu_-^\dagger\sigma^\mu\partial_\mu u_- + i\sigma_+^\dagger\overline{\sigma^\mu}\partial_\mu u_+ - M(u_+^\dagger u_- + u_-^\dagger u_+)$.

If we have $M = 0$, then the two Weyl spinors decouple; they satisfy the Weyl equations $i\overline{\sigma^\mu}\partial_\mu u_+ = 0, i\sigma^\mu\partial_\mu u_- = 0$.

We shall see when we quantise that M really does represent a mass - the field gives rise to particles of mass $|M|$ (observe that M may be positive or negative). For massless particles, one can "get away" with only using the Weyl spinors. In fact it is necessary to use the Weyl spinors for all particles, because particles are "really" massless; it is only the Higgs mechanism which gives them mass. And the weak force does act differently on the two parts of the Dirac spinor.

γ^5

In the chiral representation of the γ matrices, the $S[\Lambda]$ come out to be block diagonal. In another basis for γ^μ the representation of the Lorentz group must still divide into two parts. To help see how this works we introduce a "fifth" γ matrix, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ (there is no real standard as to whether it should be this or $-i\gamma^0\gamma^1\gamma^2\gamma^3$, but most courses in Part III use the definition given here, as do Peskin and Schroeder; note that Tong's notes use the opposite convention). This satisfies $\{\gamma^\mu, \gamma^5\} = 0$ and $(\gamma^5)^2 = 1_4$ (this is actually a standard trick that can be done for any even dimensional Clifford algebra representation). It also satisfies $[S^{\mu\nu}, \gamma^5] = 0$. Since $(\gamma^5)^2 = 1$, we can form a projection operator $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$ such that $P_\pm^2 = P_\pm$ and $P_+P_- = 0$. Then we can define the chiral or Weyl spinors as $\psi_\pm = P_\pm\psi$; these are often called the left-handed and right-handed spinors. (e.g. if we use $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma^\mu} & 0 \end{pmatrix}$, we have $\gamma^5 = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix}$.

A note on interactions

There are five Lorentz invariant Fermi bilinears we can construct:

$\overline{\psi}\psi$ is a scalar; we have seen this before as e.g. the $\overline{\psi}M\psi$ term.

$\overline{\psi}\gamma^\mu\psi$ is a vector; we have also seen this before as e.g. the $\overline{\psi}\gamma^\mu\partial_\mu\psi$ term (the derivative doesn't affect the contraction).

$\overline{\psi}S^{\mu\nu}\psi$ is a 2-tensor.

$\overline{\psi}\gamma^\mu\gamma^5\psi$ is an axial vector or pseudovector.

$\overline{\psi}\gamma^5\psi$ is a pseudoscalar.

We can see that this is all there is because ψ has 16 degrees of freedom (it's a 4×4 matrix), and here we have decomposed it into Lorentz invariant representations with a total of 1+4+6+4+1=16 components.

Theories are called vector-like if they treat the $\pm$ spinors the same way (e.g. QCD), and otherwise they are called chiral (e.g. weak interactions).

Symmetries and Currents

The Lagrangian $\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under several symmetries:

Spacetime translations; these give that the energy-momentum tensor $T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi$ is invariant (up to a total derivative). (There are a few subtleties: this is not symmetric in μ, ν, but we can "massage" it into a form that is. To express it in this simple form it was necessary to use the equations of motion, but that's ok - the whole point is that it is invariant under motion according to those equations).

Lorentz transformations give $(j^\mu)^{\rho\sigma} = x^\sigma T^{\mu\rho} - x^\rho T^{\mu\sigma} - i\overline{\psi}\gamma^\mu S^{\rho\sigma}\psi$ is conserved. This is an expression for angular momentum - the extra final term will give rise to spin when we quantise (the angular momentum of a stationary particle is not zero)

Vector symmetry $\psi \to e^{i\alpha}\psi$. The current is $j^\mu_v = \overline{\psi}\gamma^\mu\psi$ where the v is for vector; the reader may check $\partial_\mu j^\mu_v = 0$. The associated charge $Q = \int j^0_v d^3x = \int d^3x\overline{\psi}\gamma^0\psi = \int d^3x\psi^\dagger\psi$; this will be the fermion number, number of electrons etc.

Axial symmetry: when $m = 0$, $\psi \to e^{i\gamma_5\alpha}\psi$ is a symmetry, $j^\mu_A = \overline{\psi}\gamma^\mu\gamma^5\psi$ (the reader may check $\partial_\mu j^\mu_A = 2im\overline{\psi\gamma^5\psi}$, which is 0 when $m = 0$). This is not seen in nature, because it is an anomaly - while preserved by the classical field, it is not preserved by the quantum version.

Summary

The Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \equiv (i\partial\!\!\!/ - m)\psi(x)$. γ^μ are 4×4 matrices which anti-commute, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, the Dirac algebra. We have chosen the basis $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$. The Dirac conjugate $\overline{\psi} = \psi^\dagger\gamma^0 \Rightarrow \mathcal{L} = \overline{\psi}(x) = (i\gamma^\mu\partial_\mu - m)\psi(x)$. We saw the following quantities: a scalar $\overline{\psi}\psi \equiv S(x)$, a vector $\overline{\psi}\gamma^\mu\psi \equiv j^\mu(x)$, a tensor $\overline{\psi}S^{\mu\nu}\psi = T^{\mu\nu}$ ($S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$). j^μ is a conserved (Noether) current. Let $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Then $P_\pm = (\frac{1\pm\gamma_5}{2}), \psi_{\begin{pmatrix} R \\ L \end{pmatrix}} = P_\pm\psi$.

Plane Wave Solutions of the Dirac Equation

In an error on the lecturer's part, this section used the basis $\gamma^0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

We look for solutions of the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$; try $\psi(x) = u(\mathbf{p})e^{-ip\cdot x} = u(\mathbf{p})e^{-iEt+i\mathbf{p}\cdot\mathbf{x}}$. Substituting this into the equation, $(\gamma \cdot p - m)u(\mathbf{p}) = 0$ where $\gamma \cdot p = \gamma^\mu p_\mu \equiv p\!\!\!/$. For nontrivial solutions, we require $p^2 - m^2 = 0$ (to see this, multiply through by $\gamma \cdot p + m$ and use

$(\gamma \cdot p)^2 = p^2)$. Positive energy solutions have $E = E_p = \sqrt{|\mathbf{p}|^2 + m^2}$. There are also negative energy solutions, which can be written as $\psi(x) = v(\mathbf{p})e^{ip\cdot x}$, where $(\gamma \cdot p + m)v(\mathbf{p}) = 0$.

Returning to the positive enrgy case, $\gamma \cdot p - m = \begin{pmatrix} E-m & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -E-m \end{pmatrix}$. Write $u(\mathbf{p}) = \begin{pmatrix} \chi \\ \phi \end{pmatrix}$, then $(E-m)\chi - (\sigma\cdot\mathbf{p})\phi = 0$, $\sigma\cdot\mathbf{p}\chi - (E+m)\phi = 0$ where $E = E_p$. (Note $(\sigma\cdot\mathbf{p})(\sigma\cdot\mathbf{p}) = |\mathbf{p}|^{\mathbf{2}}$). The solution is $u(\mathbf{p}) = \sqrt{E_p+m}\begin{pmatrix} \chi \\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi \end{pmatrix}$, with χ a constant 2-spinor (the square root is just a normalization factor for convenience).

We have seen such 2-spinors already; they are just those for spin-$\frac{1}{2}$ particles. So define χ_S for $S = \pm\frac{1}{2}$ (i.e. spin up or down) by $\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then a complete set of solutions to the Dirac equation for positive energy is $u_s(\mathbf{p})e^{-ip\cdot x} = \sqrt{E_p+m}\begin{pmatrix} \chi_S \\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi_S \end{pmatrix} e^{-ip\cdot x}$. The negative energy solutions are $\psi = v(p)e^{ip\cdot x}$ for $v_S(p) = \sqrt{E_p+m}\begin{pmatrix} frac\sigma\cdot\mathbf{p}E_p+m\chi_S \\ \chi_S \end{pmatrix} e^{-ip\cdot x}$ for χ_S as above.

Spin Sums

In practical calculations we may have to sum spin over two spin states of a fermion if both such states occur, or if the spin state is not measured.

The following orthogonality relation (holds): $\overline{u_S}(\mathbf{p})u_{S'} = 2m\delta_{SS'} = -\overline{v_S}(p)v_{S'}(p)$ where $S, S' \in \pm\frac{1}{2}$. $\overline{u_S}(p)v_{S'}(\mathbf{p}) = 0$ for a given $p = (E_p, \mathbf{p})$ and $\overline{u} = u^\dagger\gamma^0$ as before.

Example: $\overline{u_S(\mathbf{p})}u_{S'}(\mathbf{p}) = \sqrt{E+m}(\chi_S^\dagger, -\chi_S^\dagger\frac{\sigma\cdot\mathbf{p}}{E+M})\begin{pmatrix} \chi_{S'} \\ \frac{\sigma\cdot\mathbf{p}}{E+m}\chi_{S'} \end{pmatrix}\sqrt{E+m} = (E+m)(1 - \frac{|\mathbf{p}|^2}{(E+m)^2}\chi_S^\dagger\chi_{S'}^\dagger = \frac{1}{E+m}(E^2+2Em+m^2-|\mathbf{p}|^2)\delta_{SS'} = 2m\delta_{SS'}$; similarly $u_S^\dagger(\mathbf{p})u_{S'}(\mathbf{p}) = 2E\delta_{SS'}$, since $(\gamma\cdot p - m)u_S(\mathbf{p}) = 0 \Rightarrow (\gamma\cdot p + m)u_S(\mathbf{p}) = 2mu_S(\mathbf{p})$ and $(\gamma\cdot p + m)v_S(\mathbf{p}) = 0 \Rightarrow (\gamma\cdot p - m)v_S(\mathbf{p}) = -2mv_S(\mathbf{p})$. So $\sum_S u_S(\mathbf{p})\overline{u_S}(\mathbf{p}) = \gamma\cdot p + m, \sum_S v_S(\mathbf{p})\overline{v_S}(\mathbf{p}) = \gamma\cdot p - m$; to verify this, act with both sides of each equation on the complete basis $\{u_{S'}(p), v_{S'}(p)\}$.

Note $\frac{1}{2m}(\gamma\cdot p + m)$ and $-\frac{1}{2m}(\gamma\cdot p - m)$ are projection operators onto positive and negative energy solutions with 3-momentum $\mathbf{p}$.

The identities $\overline{u_S}(p)u_{S'}(p) = \delta_{SS'}2m = -\overline{v_S}v_{S'}$, $u_S^\dagger(P)u_{S'}(p) = \delta_{SS'}2E, \sum_S u_S(p)\overline{U_S}(p) = \gamma\cdot p + m, \sum_S v_S(p)\overline{v_S}(p) = \gamma\cdot p - m$ hold in <u>any basis</u>. In our chiral basis $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\psi = u(p)e^{-ip\cdot x}, (\gamma\cdot p - m)u(p) = 0$. $(\gamma\cdot p - m) = \begin{pmatrix} -m & \sigma^\mu p_\mu \\ \overline{\sigma^\mu}p_\mu & -m \end{pmatrix}$ where $\sigma^\mu = (1,\sigma), \overline{\sigma^\mu} = (1,-\sigma)$. Let $u = \begin{pmatrix} \chi \\ \phi \end{pmatrix}$ as before, then $-m\lambda + \sigma^\mu p_\mu\phi = 0, \overline{\sigma^\mu}p_\lambda\lambda - m\phi = 0$. Let $\chi = (\sigma\cdot p)\xi$, then the second equation gives $(\overline{\sigma}\cdot p)(\sigma\cdot p)\xi = m\phi \Rightarrow ({p^0}^2 - \mathbf{p}^2)\xi = m\phi \therefore m\xi = \phi, u(\mathbf{p}) = c\begin{pmatrix} (\sigma\cdot p)\xi \\ m\xi \end{pmatrix}$. Letting $x1 = \sqrt{\sigma\cdot p}\chi$ we have $u(\mathbf{p}) = \begin{pmatrix} \sqrt{\sigma\cdot p}\chi \\ \sqrt{\overline{\sigma}\cdot p}\chi \end{pmatrix}$ (absorbing the normalization factor, and with χ as before).

For eigenstates of σ^3, if $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (spin up) $\equiv \chi_+$, $u(p) = \begin{pmatrix} \sqrt{E - p^3}\chi_+ \\ \sqrt{E + p^3}\chi_+ \end{pmatrix}$ and $\chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_-$ has $u(p) = \begin{pmatrix} \sqrt{E + p^3}\chi_- \\ \sqrt{E - p^3}\chi_- \end{pmatrix}$. Similarly, $v(p) = \begin{pmatrix} \sqrt{\sigma \cdot p}\chi \\ -\sqrt{\overline{\sigma} \cdot p}\chi \end{pmatrix}$, where $v(p)$ is a negative energy solution $\psi = v(p)e^{ip\cdot x}$.

Quantisation of the Dirac Field

The Dirac Lagrangian density $\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi$. The field conjugate to $\psi_\alpha(\mathbf{x})$ is $\Pi_\alpha(\mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\alpha(\mathbf{x})} = i(\overline{\psi}\gamma^0)_\alpha = i\psi^\dagger_\alpha(\mathbf{x})$. We expect the field operators ψ_α, Π_α to create and annihilate spin-$\frac{1}{2}$ particles obeying the Pauli principle. The state of two identical particles of spin $\frac{1}{2}$ changes sign when the particle locations are interchanged, so no more than one particle can exist in a given spin state at a given time.

Impose the canonical anti-commutation relations on the Dirac field at equal time (Peskin and Schroeder demonstrate that imposing the CCR instead leads to completely unphysical results). $\{\psi_\alpha(\mathbf{x}), \psi^\dagger_\beta(\mathbf{x})\} = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x} - \mathbf{x}')$ for $x^0 = x^{0\prime}$. Equivalently $\{\psi_\alpha(\mathbf{x}), \overline{\psi_\beta}(\mathbf{x}')\} = (\gamma^0)_{\alpha\beta}\delta^{(3)}(\mathbf{x} - \mathbf{x}')$ and $\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{x}')\} = \{\overline{\psi}_\alpha(\mathbf{x}), \overline{\psi}_\beta(\mathbf{x}')\} = 0$.

The Hamiltonian for the Dirac field is $H = \int d^3\mathbf{x}(\Pi_\alpha(\mathbf{x})\partial_0\psi_\alpha(\mathbf{x}) - \mathcal{L})$. $\overline{\psi}$ has no conjugate as the Dirac equation contains only one time derivative. Substituting Π into the Hamiltonian, $H = \int d^3\mathbf{x}(i\overline{\psi}(\mathbf{x})\gamma^0\partial_0\psi(\mathbf{x}) - \mathcal{L})$. $\mathcal{L}$ is first order in time derivatives, and so the time derivatives drop out and $H = \int d^3\mathbf{x}(-i\overline{\psi}\gamma \cdot \nabla\psi(\mathbf{x}) + m\overline{\psi}(\mathbf{x})\psi(\mathbf{x}))$ at fixed x^0. Expand the Dirac field in terms of plane wave modes, $\psi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a^S_p u_S(\mathbf{p})e^{ip\cdot x} + b^{\dagger S}_p v_S(\mathbf{p})e^{-ip\cdot x})$. (The S stands for spin; choosing to call the second coefficient $b^\dagger$ rather than b is convention).

We require $a^\dagger, b^\dagger$ to create particles of positive energy. Since ψ has four components, so do $u_S(\mathbf{p}), v_S(\mathbf{p})$.

The Dirac conjugate field is $\overline{\psi}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a^{\dagger S}_{\mathbf{p}}\overline{u_S}(\mathbf{p})e^{-ip\cdot x} + b^S_{\mathbf{p}}\overline{v_S}(\mathbf{p})e^{ip\cdot x})$. The anti-commutation relations above are equivalent to: $\{a^S_{\mathbf{p}}, a^{S'\dagger}_{\mathbf{p}'}\} = \{b^S_{\mathbf{p}}, b^{S'\dagger}_{\mathbf{p}'}\} = (2\pi)^3\delta(\mathbf{p} - \mathbf{p}')\delta^{SS'}$, $\{a, a\} = \{b, b\} = \{a^\dagger, a^\dagger\} = \{b^\dagger, b^\dagger\} = 0$.

Now substitute into the Hamiltonian and use the Dirac equation; the Hamiltonian becomes $H = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\sum_S E_{\mathbf{p}}(a^{S\dagger}_{\mathbf{p}}a^S_{\mathbf{p}} - b^S_{\mathbf{p}}b^{S\dagger}_{\mathbf{p}})$ (compare with solving the SHO). Then (using the anticommutation relations) $Ha^{S\dagger}_{\mathbf{p}} = a^{S\dagger}_{\mathbf{p}}(H + E_{\mathbf{p}})$, $Hb^{S\dagger}_{\mathbf{p}} = b^{S\dagger}_{\mathbf{p}}(H + E_{\mathbf{p}})$ and similarly for $a^S_{\mathbf{p}}, b^S_{\mathbf{p}}$ (but with $-E_{\mathbf{p}}$ rather than $+E_{\mathbf{p}}$). Hence $a^{\dagger S}_{\mathbf{p}}, b^{\dagger S}_{\mathbf{p}}$ are creation operators for particles with positive energy $E_{\mathbf{p}}$. We define the vacuum to satisfy $a^S_{\mathbf{p}}|0\rangle = 0 = b^S_{\mathbf{p}}|0\rangle$.

The Hamiltonian can be written in normal ordered form, using the anticommutation relations for $b^\dagger$s, as $H = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\sum_S E_{\mathbf{p}}(a^{S\dagger}_{\mathbf{p}}a^S_{\mathbf{p}} + b^{\dagger S}_{\mathbf{p}}b^S_{\mathbf{p}})$; we have discarded an infinite negative constant $-(2\pi)^3\delta^{(3)}(\mathbf{0})$ (again, we cannot measure the absolute vacuum energy).

in normal ordered form the vacuum energy is zero - we have discarded a zero point energy. Normal ordering shifts the creation operators to the left of the annihilation operators. For Dirac fields the anti-commutation relations introduce a negative sign each time two operators are exchanged.

A normalized 1-particle state is $|p,s\rangle = \sqrt{2E_{\mathbf{p}}}a_{\mathbf{p}}^{S\dagger}|0\rangle$ or $|p,s\rangle = \sqrt{2E_p}b_p^{S\dagger}|0\rangle$; we shall later look at the difference between these. A 2-particle state is $|p_1,s_1;p_2,s_2\rangle = \sqrt{4E_{p_1}E_{p_2}}a_{p_1}^{s_1\dagger}a_{p_2}^{s_2\dagger}|0\rangle = -\sqrt{4E_{p_1}E_{p_2}}a_{p_2}^{s_2\dagger}a_{p_1}^{s_1\dagger}|0\rangle = -|p_2,s_2;p_1,s_1\rangle$ illustrating Fermi-Dirac statistics; two identical particles cannot have the same momentum and spin.

Anti-Commutator and Dirac Propagator

The Heisenberg field $\psi(x)$ is introduced as in K-G theory and obeys $i\frac{\partial\psi}{\partial t} = [\psi, H]$, which is equivalent to the Dirac equation (H is essentially AB, then $[\psi, AB] = [\psi, A]B + A[\psi, B] = (\psi A - A\psi)B + A(\psi B - B\psi) = \{\psi, A\}B - A\{\psi, B\}$, so the reader should not be worried that we are using a commutator here even though our theory is characterized by anticommutation relations). The solution is $\psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_S a_p^S u_S(p)e^{-ip\cdot x} + b_p^{S\dagger}v_S(p)e^{ip\cdot x}, \overline{\psi}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_S(a_p^{S\dagger}\overline{u_S}(p)e^{ip\cdot x} + b_p^S\overline{v_S}(p)e^{-ip\cdot x})$, and recall $p^0 = E_p$.

We can now compute the anticommutation relations of the Heisenberg field to find the propagators, $iS_{\alpha\beta}(x-y) = \{\psi_\alpha(X), \overline{\psi_\beta}(y)\}$ with x^0, y^0 not necessarily equal (x, y are now 4-vectors). We can drop the spinor indices and express this as $iS(x-y) = \{\psi(x), \overline{\psi}(y)\}$. Substitute in the expansions for ψ and $\overline{\psi}$ and evaluate; $iS(x-y) = \iint \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{1}{\sqrt{2E_p 2E_{p'}}}\sum_{S,S'}(\{a_p^S, a_{p'}^{S'\dagger}\}u_S(p)\overline{u_{S'}}(p')e^{-ip\cdot x+ip'\cdot y} + \{b_{\mathbf{p}}^{\dagger S}b_{p'}^{S'}\}v_S(p)\overline{v_{S'}}(p')e^{ip\cdot x-ip'\cdot y})$. Using the anticommutation relations and integrating with respect to p' using the resulting $\delta(p-p')$ this is $\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}\sum_S(u_S(p)\overline{u_S}(p)e^{-ip\cdot(x-y)} + v_S(p)\overline{V_S}(p)e^{ip\cdot(x-y)}) = (i\gamma\cdot\partial_x + m)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)} - (i\gamma\cdot\partial_x + m)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{ip\cdot(x-y)}$, since we know what the spin sums are; $\sum_S u_S(p)\overline{u_S}(p) = \gamma\cdot p + m, \sum_S v_S(p)\overline{v_S}(p) = (\gamma\cdot p - m)$, and we can write p as $\pm i\partial_x$ in these particular expressions. Then this $= (i\gamma\cdot\partial_x + m)(D(x-y) - D(y-x))$ where $D(x-y)$ is the K-G propagator we calculated earlier.

Now $(i\gamma\cdot\partial_x - m)S(x-y) = 0$ since $(\partial_x^2 + m^2)e^{\pm ip\cdot(x-y)} = 0$ for any p on (the) mass-shell $p^2 = m^2$, and $S(x-y)$ vanishes for spacelike $x-y$ since the K-G (propagator) does.

The Feynman Propagator

We can calculate the vacuum expectation value in a similar fashion, $\langle 0|\psi_\alpha(x)\overline{\psi_\beta}(y)|0\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}(\gamma\cdot p + m)_{\alpha\beta}e^{-ip\cdot(x-y)}$ - only $a, a^\dagger$ contribute, as $b|0\rangle = 0 = \langle 0|b^\dagger$ (and ignoring for now the other coefficients, $\psi = a + b^\dagger, \overline{\psi} = a^\dagger + b$ so $\langle 0|\psi\overline{\psi}|0\rangle = \langle 0|(a + b^\dagger)(a^\dagger + b)|0\rangle = \langle 0|aa^\dagger|0\rangle$). Schematically, therefore, $\langle 0|\psi_x\overline{\psi_y}|0\rangle$ is $\langle 0|aa^\dagger|0\rangle = \langle 0|\{a, a^\dagger\}|0\rangle$ (as $a^\dagger a|0\rangle = 0$), and then this is $(2\pi)^3\delta_{SS'}\delta^{(3)}(\mathbf{p} - \mathbf{p}')$.

The Feynman propagator is calculated for the time ordered product, as in the K-G case. $S_F(x-y) = \langle 0|T\psi(x)\overline{\psi}(y)|0\rangle, = \langle 0|\psi(x)\overline{\psi}(y)|0\rangle$ for $x^0 > y^0$, $\langle 0| - \overline{\psi}(y)\psi(x)|0\rangle$ for $y^0 > x^0$, the - sign being chosen to give continuity as y^0 increases through x^0, if $\mathbf{x} \neq \mathbf{y}$, using the anticommutation relations. This also means $\{\psi(x), \overline{\psi}(y)\} = 0$ outside the light-cone.

S_F has integral representation $S_F = i\int \frac{d^4}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{\gamma\cdot p + m}{p^2 - m^2 + i\epsilon}$; as in the

K-G case, we can reduce to a 3-momentum integral by contour integration, and compare the result with $\langle 0|\psi(x)\overline{\psi}(y)|0\rangle$ etc. above.

Now, $(i\gamma \cdot \partial_x - m)S_F(x-y)i\delta^{(4)}(x-y)$, so S_F is a Green's function for the Dirac operator (use $(\gamma \cdot p - m)(\gamma \cdot p + m) = p^2 - m^2$).

Particles and anti-particles (interpretation)

Classical Dirac theory has a conserved charge $N = \int \psi^\dagger \psi d^3\mathbf{x}$, so $eN = e\int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_S (a_{\mathbf{p}}^{S\dagger} a_{\mathbf{p}}^S + b_{\mathbf{p}}^S b_{\mathbf{p}}^{S\dagger})$; use the anticommutation relations and ignore an infinite constant, then this $= e\int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_S (a_{\mathbf{p}}^{S\dagger} a_{\mathbf{p}}^S - b_{\mathbf{p}}^{S\dagger} b_{\mathbf{p}}^S)$. Acting on the vacuume we get zero: $b|0\rangle = a|0\rangle = 0$, and we have $eNa^\dagger = a^\dagger(eN+e), eNb^\dagger = b^\dagger(eN - E)$ (and similarly with the signs reversed for a, b). So $a^\dagger$ creates a particle of charge e, $b^\dagger$ creates a particle of charge $-e$; in fact e is the electron charge (and so negative). So Dirac theory is a theory of free electrons of charge e and positrons of charge $-e$; N is the number of electrons - the number of positrons. Electrons and positrons have the same mass and spin states, but opposite charge and magnetic moment; the positron, the antiparticle of the electron, was predicted by the Dirac equation, and then discovered (experimentally) a few years later, by Anderson in 1932.

Last time: we introduced the conserved charge N. $a^\dagger$ creates particles of charge e, $b^\dagger$ creates particles of charge $-e$ - electrons and positrons respectively.

Dirac's Hole Theory

Quantised Dirac theory has two types of positive energy particles. Dirac originally interpreted it as having one type of particle with both positive and negative energy states, i.e. taking $\psi = e^{-ip\cdot x}p(p), i\frac{\partial \psi}{\partial t} = E_p \psi$ and $\psi = e^{ip\cdot x}v(p), i(\frac{\partial \psi}{\partial t} = -E_p\psi$ to be positive and negative energy states respectively. Schematically we then have a spectrum of energy states, with energy levels spaced m apart both above and below zero. For a given p, s there are two states, one $+E_{\mathbf{p}}$ and one $-E_{\mathbf{p}}$.

The anticommutation relations mean that each level can either be occupied once or empty. If an electron state had just one positive energy level filled in, then in collisions the electron could lose energy and become of negative energy, leading to an unstable theory. So Dirac postulated that the vacuum state had all negative energy levels filled and the positive energy levels empty. Then e.g. the 1-electron state is one with all negative and one positive energy level filled, while that which we see as a 1-positron state is a "hole state" with all but one negative energy level filled. By the addition of energy to the vacuum we can raise from one of the negative energy levels to a positive one - creating an electro and positron.

For us, $b^\dagger$ creates a particle of positive energy and positive charge (the positron); for Dirac $b^\dagger$ destroys a particle of negative energy and negative charge; these are of course mathematically equivalent, but make for somewhat different physical interpretations.

The vacuum is annihilated by all the b operators (for Dirac, b creates a particle of negative energy, so this tells us that all the negative energy states must already be filled in the vacuum state.

Dirac hole theory is not very convenient in the study of QFT, but is useful in condensed matter physics for describing states below a Fermi surface.

Interacting Quantum Field Theory

We have discussed in detail the quantisation of Klein-Gordon and Dirac theory. The free particle states are eigenstates of the Hamiltonian, we have had no interactions and no scattering. In the real world we get these; also, we know that spin-1 particles exist, so there must be more Lagrangians "out there". Hence we need a framework to include them.

So far we have seen $\mathcal{L}_{\text{K-G}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \mathcal{L}_{\text{Dirac}} = \overline{\psi}(i\partial\!\!\!/ - m)\psi$. We know $\mathcal{L}_{\text{e.m.}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic Lagrangian; quantisation of this will give us photons, which we will need to describe electron scattering.

We have covered: linear field equations, exact quantisation, particles of any momentum, multiparticle states with no scattering, momenta of all particles conserved.

Note: Action S is dimensionless: $[\mathcal{L}] = [M]^4$. So the canonical dimension of all fields is as follows: the scalar field ϕ of $[M]$, spinor field ψ of $[M]^{\frac{3}{2}}$, vector field A_μ of $[M]$. Keep this in mind when considering interactions.

Particle interactions are included by adding cubic or higher terms to $\mathcal{L}$. The interaction term involves only fields at the same point (local interactions), to preserve causality.

i) Scalar ϕ^4 field theory: $\mathcal{L} = \mathcal{L}_{\text{K-G}} + \frac{\mu}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4$ (Note that μ has dimensions $[M]$. Note also that higher order interaction terms are not possible; if we tried to divide two terms or some such to get a term of the correct dimension, while that might be ok at this level we would find that ultimately this would not give a renormalizable theory). λ must be ¿0 for a stable theory.

ii) Yukawa theory $\mathcal{L} = \mathcal{L}_{\text{K-G}} + \mathcal{L}_{\text{Dirac}} - g\overline{\psi}\psi\phi$. This ws originally a theory of the interaction of nucleons (ψ terms) and pions (ϕ terms); now these terms appear in the standard model, where ψ are spin-$\frac{1}{2}$ particles (quarks and leptons) and ϕ is the Higgs. iii) QED $\mathcal{L} = \mathcal{L}_{\text{e.m.}} + \mathcal{L}_{\text{Dirac}} - e\overline{\psi}\gamma^\mu A_\mu\psi$. This is the electrodynamics of particles of charge $\pm e$ and photons.

The Interaction Picture and S-matrix

For simplicity we start with the simplest interacting theory, scalar ϕ^4 theory $\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4(x)$. The Hamiltonian is $H = H_0 + D_{\text{int}}$ with $H_{\text{int}} = +\frac{\lambda}{4!}\int d^3\mathbf{x}\phi^4(\mathbf{x})$ $(\equiv -\int \mathcal{L}_{\text{int}})$, where $H_0 = \int d^3\mathbf{x}(\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2)$ the K-G Hamiltonian.

The equation of motion in $(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$. For $\lambda << 1$ we can use perturbation theory - to include interactions we go from the Heisenberg picture to the interaction picture by evolving operators using H_0 (rather than H); this is similar to perturbation theory in quantum mechanics.

Why did we analyse without a $\frac{\mu}{3!}\phi^3$ term? a) Auch a term makes it harder to renormalize in more advanced treatments b) For a complex scalar field, no such term is possible, as there should be a symmetry $\phi \to \phi e^{i\alpha}, \phi^\star \to \phi^\star e^{-i\alpha}$ c) We want to consider the simplest possible case; it is easy to generalise.

The interaction Hamiltonian enters in two places: the Heisenberg field $\phi(x) = e^{iHt}\phi(\mathbf{x})e^{-iHt}$, and the definition of the vacuum $|\Omega\rangle$. We need to express $\phi(x)$ and $|\Omega\rangle$ in terms of quantities we know how to deal with - i.e. free field theory operators and the free vacuum $|0\rangle$. When λ is small we can use the free H_0 and include interactions (H_{int}) perturbatively.

Interaction Picture

If A is a Schrodinger picture operator, the interaction picture operator is $A_I(t) = e^{iH_0t}Ae^{-iH_0t}$, e.g. $\phi_I(t,\mathbf{x}) = e^{iH_0t}\phi_S(\mathbf{x})e^{-iH_0t}$. So $\phi_I(t,\mathbf{x})$ obeys $(\partial^2+m^2)\phi_I(x) = 0$ with solutions $\phi_I(t,\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a^I_{\mathbf{p}}e^{-ip\cdot x} + a^{I\dagger}_{\mathbf{p}}e^{ip\cdot x})$ as in the free field theory with $\phi_I(t=0,\mathbf{x}) = \phi(\mathbf{x})$.

As before $[a^I_{\mathbf{p}}, a^{i\dagger}_{\mathbf{p}'}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$, with the other commutators vanishing. The state $|0\rangle$ satisfies $a^I_p|0\rangle = 0$ and is the vacuum of the free theory, not the interacting one. Interaction picture states $|\Phi\rangle_I$ are related to Schrodinger picture states $|\Phi\rangle$ by $|\Phi\rangle = e^{-iH_0t}|\Phi\rangle_I$. The Schrodinger equation $i\frac{d}{dt}(e^{-iH_0t}|\Phi\rangle_I) = (H_0+H_{\text{int}})(e^{-iH_0t}|\Phi\rangle_I)$ implies $H_0e^{-iH_0t}|\Phi\rangle_I+e^{-iH_0t}i\frac{d}{dt}|\Phi\rangle_I = H_0e^{-iH_0t}|\Phi\rangle_I+H_{\text{int}}e^{-iH_0t}|\Phi\rangle_I$; cancelling the first terms $i\frac{d}{dt}|\Phi\rangle_I = e^{iH_0t}H_{\text{int}}e^{-iH_0t}|\Phi\rangle_I$, i.e. $i\frac{d}{dt}|\Phi\rangle_I = H_i|\Phi\rangle_I$ ($\star$). H_I is the interaction Hamiltonian in the interaction picture (time-dependent).

Clearly, $H_I = \frac{\lambda}{4!}\int d^3\mathbf{x}\phi^4_I(x)$, for ϕ the Heisenberg field of K-G theory. From now on we work in the interaction picture and drop the subscript I from Φ.

We can derive the solution to ($\star$). Write $\Phi(t)\rangle = U(t,t_0)|\Phi(t_0)\rangle$ where $U(t,t_0)$ is the unitary time evolution operator; it satisfies $u(t_1,t_2)U(t_2,t_3) = U(t_1,t_3), U(t,t) = 0, i\frac{d}{dt}U = H_IU$. Since H_I is an operator, we can't just write don the solution as $U \sim \exp(-iH_it)$; we have to worry about operator ordering. However, we can write down the solution as a power series in λ, valid for small λ.

Formally, $U(t,t_0) = 1+(-i)\int_{t_0}^t dt'H_i(t')U(t')$; substitute in the first term to get an infinite series. $U(t,t_0) = 1+(-i)\int_{t_0}^t dt'H_I(t')+(-i)^2\int_{t_0}^t dt'\int_{t_0}^{t'} dt''H_I(t')H_I(t'')+ \ldots$ (†). In this, all the factors of H_I stand in time order, with the latero n the left. Hence we have $\int_{t_0}^t dt'\int_{t_0}^{t'} dt''H_I(t')H_I(t'') = \frac{1}{2}\int_{t_0}^t dt'\int_{t_0}^t dt''T(H_I(t')H_I(t''))$, the factor of $\frac{1}{2}$ coming from the change in the integration region in the $t't''$ plane.

Hence we can write $U(t,t_0)$ in compact form $U(t,t_0) = T(\exp(-i\int_{t_0}^t dt'H_I(t')))$, where the time ordering is defined as the Taylor series with each term time ordered. This is Dyson's formula.

The S-matrix relevant to scattering is $S = \lim_{t\to\infty} U(t)$¿ Formally S and U are unitary since H_I is Hermitian. So $S = T\exp(-i\int_{-\infty}^{\infty} dtH_I(t)) = T\exp(-\frac{i\lambda}{4}\int_{-\infty}^{\infty} d^4x\phi^4(x)) \equiv 1 + \frac{-i}{1!}\int_{-\infty}^{\infty} dt'H_I(t') + \frac{(-i)^2}{2!}\int_{-\infty}^{\infty} dt'\int_{-\infty}^{\infty} dt''T(H_I(t')H_I(t'')) + \ldots$. Since λ is small, in perturbation theory we keep only a finite number of terms in the formally infinite series; in future courses, when we have covered remormalization, we can work exactly.

Wick's theorem - scalar field

From the Dyson formula we can see that we need to compute time-ordered products of fields in order to compute correlation functions. Wick's theorem expresses time-ordered products in terms of normal-ordered products; it's then easier to calculate correlation functions.

We wrote $\phi(x) = \phi^+(x) + \phi^-(x)$ where $\phi^\pm$ were the positive and negative energy parts: $\phi^+(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_\mathbf{p}e^{-ip\cdot x}$ (for positive frequency), $\phi^-(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_\mathbf{p}^\dagger e^{ip\cdot x}$. Normal ordering puts ϕ^- to the left of ϕ^+ and is denoted by ::, i.e. $|\phi^+\phi^- := \phi^-\phi^+ = |\phi^-\phi^+ :$.

Consider two time-ordered fields $T(\phi(x)\phi(y)) = \phi(x)\phi(y)\ (x^0 > y^0) = (\phi^+(x)+\phi^-(x))(\phi^+(y)+\phi^-(y)) = \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^-(y) + [\phi^+(y), \phi^-(x)]$ but this commutator is known; this is $: \phi(x)\phi(y) : +[\phi^+(x), \phi^-(y)] =: \phi(x)\phi(y) : +D(x-y)$. Similarly for $y^0 > x^0$ $T(\phi(x)\phi(y)) =: \phi(x)\phi(y) : +[\phi^+(y), \phi^-(x)] =: \phi(x)\phi(y) : +D(y-x)$; note the symmetry of normal-ordered products. So $T(\phi(x)\phi(y)) = |\phi(x)\phi(y)| + D_F(x-y)$ where $D_F(x-y)$ is the Feynman propagator $\delta(x^0-y^0)D(x-y) + \delta(y^0-x^0)D(y-x)$. This is consistent with the vacuum expectation value $\langle 0|T\phi(x)\phi(y)|0\rangle = D_F(x-y)$. This is denoted with the <u>contraction</u> $\overbrace{\phi(x)\phi(y)} = D_F(x-y)$. So the relation between time and normal ordering for two fields is $T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + \overbrace{\phi(x)\phi(y)}$. We can generalise this to arbitrary fields: $T\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\} =: \phi(x_1)\dots\phi(x_n) :$ +all possible contractions. This is known as <u>Wick's theorem</u>.

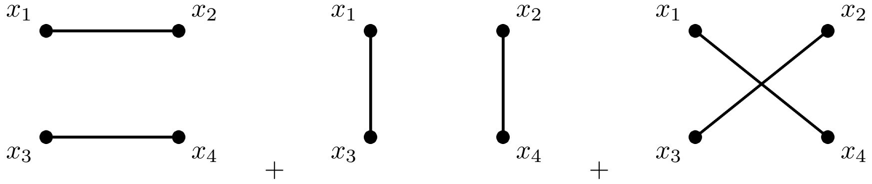
For four fields, writing $\phi(x_1)$ as ϕ_1 and similarly, $T\{\phi_1\phi_2\phi_3\phi_4\} =: \phi_1\phi_2\phi_3\phi_4 : + \overbrace{\phi_1\phi_2} : \phi_3\phi_4 : +$[unreadable]$+ \overbrace{\phi_2\phi_3} : \phi_1\phi_4 : + \overbrace{\phi_2\phi_4} : \phi_1\phi_3 : + \overbrace{\phi_3\phi_4} : \phi_1\phi_2 : + \overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\phi_3} : \phi_2\phi_4 : + \overbrace{\phi_1\phi_4} : \phi_2\phi_3 :$, where of course $\overbrace{\phi_{n_1}\phi_{n_2}} = D_F(x_{n_1} - x_{n_2})$. We can derive this result via $T(\phi_1\phi_2\phi_3\phi_4) = (\phi_1^+\phi_1^-)\dots$, but it's better to prove by induction.

In vacuum expectation values only fully contracted terms survive, the others give zero, i.e. $\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle = D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3)$. Assuming this is true for $\phi_2\dots\phi_m$, and we have proven it for $\phi_1\phi_2$, we add ϕ_1 with $x_1^0 > x_k^0 \forall 2 \le k \le m$; $T(\phi_1\dots\phi_m) = \phi_1(: \phi_2 \cdots \pm i_m :$ +all contractions not involving $\phi_1) = (\phi_1^+ + \phi_1^-)(: \phi_2\dots\phi_m :$ +contractions). We need to include $\phi_1^+ + \phi_1^-$ in the contractions; for ϕ_1^- this is easy since it is already on the left and hence normal ordered. We need to put the ϕ_1^+ terms in normal ordered form; this gives us $\phi_1^+ N(\phi_2\dots\phi_m) =: \phi_1^+\phi_2\dots\phi_m : + \overbrace{\phi_1\phi_2} : \phi_3\dots\phi_m : + \overbrace{\phi_1\phi_3} : \phi_2\phi_4\dots\phi_m : + \dots$ involving contractions $\overbrace{\phi_1\phi_k}$ for $2 \le k \le m$ for this time ordering.

Important consequences of Wick's theorem are: $\langle 0|T\phi_1\dots\phi_m|0\rangle = 0$ if m is odd, $\langle 0|T\phi_1\dots\phi_m|0\rangle = \sum_{\text{pairing of } m \text{fields}} D_F(x_{i_1}-x_{i_2})D_F(x_{i_3}-x_{i_4})\dots D_F(x_{i_{m-1}} - x_{i_m})$.

Feynman Diagrams

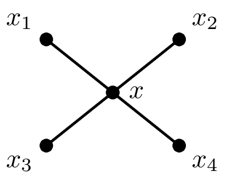
Now we know what $\langle 0|T\phi_1\dots\phi_n|0\rangle$ is we can express this diagramatically; draw dots for x_i and join pairs of points by lines in all possible ways; each line corresponds to D_F. E.g. $\langle 0 : T\phi_1\phi_2\phi_3\phi_4|0\rangle =$



These diagrams suggest an interpretation: particles are created at two spacetime points, each propagates to one of the other points and is then annihilated. The total amplitude is the sum of all diagrams - 3 in this case.

It gets more complicated when we have fields at the same spacetime point e.g. $\langle 0|T\phi_1\phi_2\ldots\phi_m S|0\rangle$, where S is the interaction term involving fields at the same point. The nth term in the perturbation expansion of S gives $\frac{1}{n!}(-\frac{i\lambda}{4!})^n\int\langle 0|T\phi_1\phi_2\ldots\phi_m\phi^4(x)\phi^4(y)\ldots|0\rangle d^4xd$ where there are n terms $x, y, \ldots$. Wick's theorem tells us to contract pairs of fields in all possible ways; rather than doing the general case we will consider some special examples. i) (reordering the terms due to limitations of LaTeX; on the blackboard this was written $\phi_1\phi_2\phi_3\phi_4\phi(x)\phi(x)\phi(x)\phi(x)$ with the contractions all above crossing each other) $-\frac{i\lambda}{4!}\int\langle 0|T\phi_1\phi_2\phi_3\phi_4\phi^4(x)|0\rangle =$ $-\frac{i\lambda}{4!}\int \overbrace{\phi_1\phi(x)}\,\overbrace{\phi_2\phi(x)}\,\overbrace{\phi_3\phi(x)}\,\overbrace{\phi_4\phi(x)}\,d^4x+$permutations$-\frac{i\lambda}{4!}\int \overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi(x)}\,\overbrace{\phi_4\phi(x)}\,\overbrace{\phi(x)\phi(x)}\,d^4x+$ permutations$-\frac{i\lambda}{4!}\int \overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4}\,\overbrace{\phi(x)\phi(x)}\,\overbrace{\phi(x)\phi(x)}\,d^4x+$permutations $= -i\lambda\int D_F(x_1-x)D_F(x_2-x)D_F(x_3-x)D_F(x_4-x)d^4x - \frac{i\lambda}{2}D_F(x_1-x_2)\int D_F(x_3-x)D_F(x_4-x)D_F(x-x)d^4x+$similar$-\frac{i\lambda}{8}D_F(x_1-x_2)D_F(x_3-x_4)\int D_F(x-x)D_F(x-x)d^4x$, the factor of $-\frac{i\lambda}{2}$ being because there are 12 ways of pairing ϕ_3, ϕ_4 with $\phi(x)$ and the $-\frac{i\lambda}{8}$ being because there are 3 ways of pairing $\phi(x)$ and $\phi(x)$.

Now $D_F(x-x) = D_F(0)$ is divergent. We can represent our integral with Feynman diagrams:



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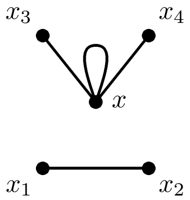

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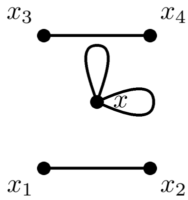
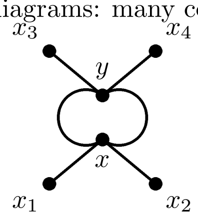
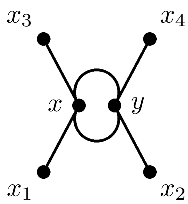


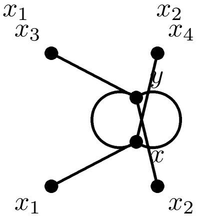
+ 2 similar.

There is a symmetry factor of 2 for each closed loop (by exchanging the ends); there is a factor of 8 for the last diagram since as well as exchanging the ends of either of the two closed loops we can also exchange the two loops.

ii) A second example is the second order correlation function $\frac{1}{2!}(-\frac{i\lambda}{4!})^2\int\langle 0|T\phi_1\phi_2\phi_3\phi_4\phi^4(x)\phi^4(y)|0\rangle d^4x$. There are many types of terms here, but the only (new) ones are the completely contracted ones of the form $\overbrace{\phi_1\phi(x)}\,\overbrace{\phi_2\phi(x)}\,\overbrace{\phi_3\phi(y)}\,\overbrace{\phi_4\phi(y)}\,\overbrace{\phi(x)\phi(y)}\,\overbrace{\phi(x)\phi(y)}$. There are diagrams: many contractions give a diagram:



. The other distinct diagrams are:


and


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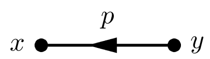
For the first diagram the Feynman rles give $(-\frac{i\lambda}{2})^2\int D_F(x_1-x)D_F(x_2-x)D_F(x_3-y)D_F(x_4-y)d^4xd^4y$ (the combinatorial factor is $\frac{1}{2!}(\frac{1}{4!})^2\times 12\times 12\times 2\times 2 = \frac{1}{2}$, the last four factors comming respectively from the numbers of arrangements of (x_1x_2x), (x_3x_4y), (xy) and [exchanging the two links between them]). We are now in a position to write down the Feynman rules:

Summary of Feynman Rules

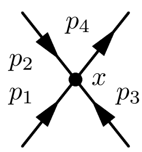
For correlation functions, $\langle 0|T\phi(x_1)\ldots\phi(x_n)\exp(-i\int d^4x\frac{\lambda}{4!}\phi^4(x))|0\rangle$ = the sum of all diagrams with n external points and any number of internal vertices corrected by propagator lines. Each diagram is an integral combining: 1. for each propagator (a line from y to z) $D_F(x-z)$ 2. For each vertex (where propagators meet) x, $i-\lambda\int d^4x$ 3. For each external point x_i (on the end of a propagator), 1. 4. Divide everything by the symmetry factor.

These are the position space Feynman rules. Since $D_F(x-y)=\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}$ it is simple to express Feynman rules in terms of momenta.

Assign a momentum p to each propagator (the direction chosen arbitrarily, at least for now).

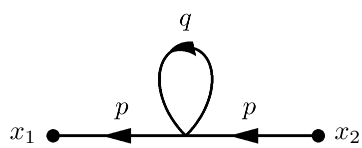


Assign e^{ipy} to the y vertex and e^{-ipx} to the x vertex, $\frac{i}{p^2-m^2+i\epsilon}$ to the line, and integrate $\int\frac{d^4p}{(2\pi)^4}$. So the whole thing is $=D_F(x-y)=\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{ip\cdot(x-y)}$.

The integral at an internal vertex 

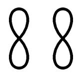
is $\int d^4xe^{-ip_1\cdot x}e^{-ip_2\cdot x}e^{-ip_3\cdot x}e^{ip_4\cdot x}=(2\pi)^4\delta^{(4)}(p_1+p_2+p_3-p_4)$; this imposes momentum conservation at the vertex. These delta functions make some integrals trivial, and for each where they do, the $(2\pi)^4$ cancels out. We are left with the momentum space rules for each diagram:

1. For each propagator (assigned momentum p), $\frac{i}{p^2-m^2+i\epsilon}$ 2. For each (internal) vertex, $-i\lambda$ 3. For each external point x_i, $e^{-ip\cdot x_i}$ 4. Impose momentum conservation at each vertex 5. Integrate over each undetermined momentum $\int\frac{d^4p}{(2\pi)^4}$ 6. Divide by symmetry factor.

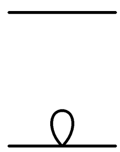


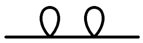
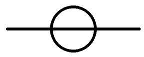
E.g. the connected part of $-\frac{\lambda}{4}\int\langle 0|T\phi_1\phi_2\phi^4(x)|0\rangle d^4x$ is $=\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\frac{i}{p^2-m^2+i\epsilon}\frac{i}{q^2+m^2+i\epsilon}(-i\lambda)e^{-ip\cdot(x_1-x_2)}$ (the symmetry factor of 2 coming from the loop). Here the q integral is just a constant; the p integral is a correction to the propagator between x_1 and x_2 (in the AQFT course we shall cover this in more detail - renormalization).

Vacuum Bubbles and Connected Diagrams

This section is not covered well in Tong's notes on the course. In ϕ^4 theory the expansion of $\langle 0|S|0\rangle$ is $1+$  $+$  $+$  $+$ 

Here the first diagram is at first order and the next three diagrams give the second order terms; diagrams with more loops in would be higher order terms in the expansion. These diagrams are called vacuum bubbles; the combinatorial factors are such that $\langle 0|S|0\rangle$ is the exponential of the sum of distinct vacuum types exp( $+$  $+\ldots$).

Now $\langle 0|T\phi_1 \ldots \phi_n S|0\rangle$ is the sum over diagrams with n external points. A typical diagram has some vacuum bubbles - e.g. one of the second order terms in a two point function is . The vacuum bubbles add up to this same exponential. So $\langle 0|T\phi_1 \ldots \phi_n S|0\rangle$ is the sum over connected diagrams (by which we mean diagrams where every point of the diagram is connected to at least one external point; note this does not necessarily mean connected in a graph-theoretical sense e.g.  is connected), plus $\langle 0|S|0\rangle$. So e.g. the sum of connected diagrams for a two point correlation function, up to second order, is $+$

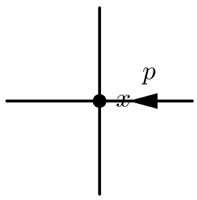
 +  +  +  .

So far we have considered the bare vacuum $|0\rangle$. The true vacuum $|\Omega\rangle$ of the interacting theory is normalized as $\langle\Omega|\Omega\rangle$. Then $\langle\Omega|T\phi_1^H\dots\phi_n^H|\Omega\rangle = \frac{\langle 0|T\phi_1\dots\phi_n|0\rangle}{\langle 0|S|0\rangle}$ = the sum of connected diagrams with n external points. The ϕ_i^H here are Heisenberg fields, ϕ_i are in the interaction picture. So removing vacuum bubbles sets the vacuum correctly. S evolves interaction picture fields to Heisenberg fields.

Scattering Amplitudes

Let $|i\rangle$ and $|f\rangle$ be the initial and final ($t = \pm\infty$) multi-particle states in the interaction picture. The amplitude to find the final state is proportional to $\langle f|S|i\rangle$. The perturbation expansion of S implies $\langle f|S|i\rangle = \delta_{fi} + (2\pi)^4\delta^{(4)}(p_f - p_i)iT_{fi}$ where p_f and p_i are the final and initial (total) 4-momentum. So the delta-function [gives us that] momentum is conserved (due to translational invariance); $T_{fi} = \langle f|T|i\rangle$ measures the "genuine" scattering amplitudes and should be finite. $|T|^2$ is used to compute the experimental cross-section.

Consider a $2 \to n-2$ amplitude is ϕ^4 theory, $\langle p_3\dots p_n|S|p_1p_2\rangle$ where all external momenta satisfy $p^2 = m^2$. The kth term in S is $\frac{1}{k!}(-\frac{i\lambda}{4})^k \int T\phi^4(x)\phi^4(y)\dots\phi^4(z)d^4xd^4y\dots d^4z$, where there are k terms $x, y, \dots, z$.

So two annihilation operators ϕ^+ and $n-2$ creation operators ϕ^- are needed to destroy the "in-particles" and create the "out-particles". Define new contractions $\overbrace{\phi(x)\dots|p\rangle} = e^{-ip\cdot x}$, equivalent to an incoming particle 

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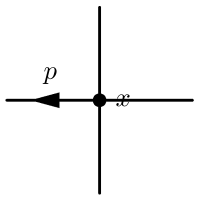
and $\overbrace{\langle p|\dots\phi(x)} = e^{ip\cdot x}$, equivalent to an outgoing particle 

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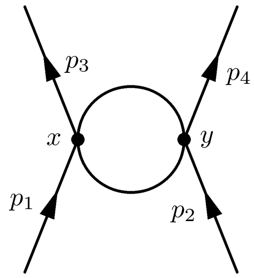
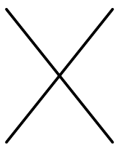
with p the momentum attached to the external line that ends in the x vertex. This is motivated by the result that $\phi^+(x)|p\rangle = e^{-ip\cdot x}|0\rangle$ and its Hermitian conjugate, which we computed earlier.

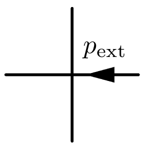
The position space Feynman rules for scattering amplitudes are like those for correlation functions, but with no external points, instead momentum on incoming and [outgoing] lines. E.g. 

has elements $e^{-ip_1\cdot x}, e^{-ip_2\cdot x}, e^{ip_3\cdot x}, e^{ip_4\cdot x}$ and propagator factor $D_F(x-y)$, vertex $-i\lambda\int d^4x$ etc.

Now use the momentum representation for D_F and integrate out vertex positions and all momentum conservation δ-functions. There remains an overall factor $(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)$. Hence, the momentum-space rules for [the] calculation are: sum over all diagrams with the given external legs:

1. For each internal propagator , $\frac{i}{p^2-m^2+i\epsilon}$.

2. For each vertex , $-i\lambda$.

3. For each evternal line 

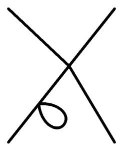
(with momentum going in our out as appropriate), 1.

4. Impose momentum conservation at each vertex.

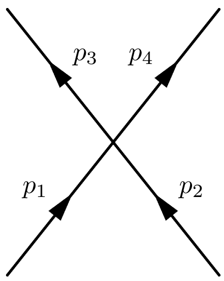
5. Integrate over each undetermined loop momentum k, $\int\frac{d^4k}{(2\pi)^4}$.

6. Divide by symmetry factor.

This process doesn't quite give the scattering amplitude for the physical theory, because $|p_1p_2\rangle$ is a state of the free theory; to ensure the scattering amplitude is correct for the interacting theory we exclude all diagrams with vacuum bubbles.

So scattering takes place in the true vacuum $|\Omega\rangle$ and not $|0\rangle$, and we exclude diagrams with vertices and lops attached to the external legs, e.g. .

E.g. for 2 particle → 2 particle scattering, our iT matrix ($= iT_{fi}$) is to second order given by the diagrams



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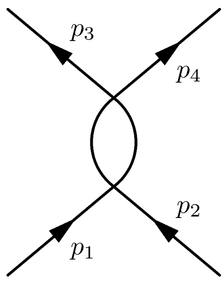

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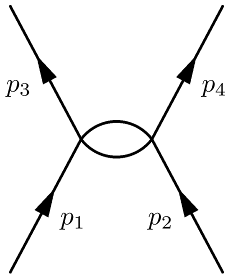

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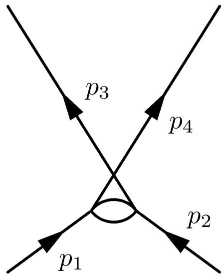
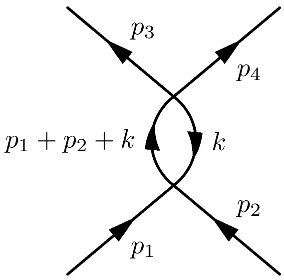


, where $p_1 + p_2 = p_3 + p_4$.

Define Lorentz invariant quantities $s = (p_1 + p_2)^2, t = (p_1 - p_3)^2, u = (p_1 - p_4)^2$ the Handelstam variables; these last three diagrams are respectively called the s-channel, t-channel and u-channel. We have $s + t + u = 4m^2$ (or in general $\sum_i m_i^2$); this is most easily proven in the CM frame.



$= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k+p_1+p_2)^2 - m^2 + i\epsilon}$; call this $\mathcal{M}$. This is a Lorentz invariant function of $p_1 + p_2$, i.e. a function of $s = (p_1 + p_2)^2$, so we can call it $(-i\lambda)^2 v(s)$. Then the third and fourth diagrams above involve the same function applied to t and u; to second order in λ, $iT = i\lambda + (-i\lambda)^2(v(s) + v(t) + v(u))$.

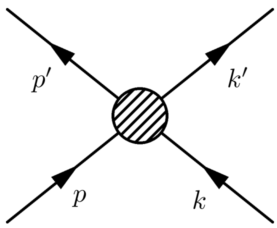
Feynman Rules for Fermions

The treatment of correlation functions generalises to fermions. Lorentz invariance means that H_I is a product of an even number of spinor fields. To apply Wick's theorem, we generalize the definitions of time and normal ordering to fermions; previously we had $T(\psi(x)\overline{\psi}(y)) = \psi(x)\overline{\psi}(y)$ for $x^0 > y^0$, $\overline{\psi}(y)\psi(x)$ for $y^0 > x^0$ and $S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} = \langle 0|T\psi(x)\overline{\psi}(y)|0\rangle$.

For more than two spinors, $T(\psi_1\psi_2\psi_3\psi_4) =$ e.g. $(-1)^3\psi_3\psi_1\psi_4\psi_2$ for $x_3^0 > x_1^0 > x_4^0 > x_2^0$, where we've picked up a - sign for each interchange of fermion fields (using the anticommutation relations). Similarly, $T(\psi(x)\psi(x)) =: \psi(x)\overline{\psi}(y) : + \overbracket{\psi(x)\overline{\psi}(y)}$ where $\overbracket{\psi(x)\overline{\psi}(y)} = S_F(x-y)$ and $\overbracket{\psi(x)\psi(y)} = \overbracket{\overline{\psi}(x)\overline{\psi}(y)} = 0$ (note $: \overbracket{\psi_1\psi_2\overline{\psi_3}}\,\overline{\psi_4} = -\overbracket{\psi_1\overline{\psi_3}} : \psi_2\overline{\psi_4} = -S_F(x_1-x_3) : \psi_2\overline{\psi_4} :$ and similarly.

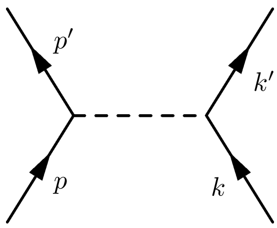
Let's consider a concrete example: Yukawa Theory. This is the interaction of a scalar of mass μ and a fermion of mass m; it's relevant to Higgs theory. $\mathcal{L} = \mathcal{L}_{\text{KG}} + \mathcal{L}_{\text{Dirac}} - g\overline{\psi}\psi\phi$.

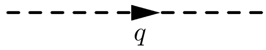
Consider two-fermion scattering (where we suppress the spin label $S = \pm\frac{1}{2}$):



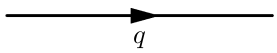
[Note: at this point there was now a substantial delay between the lectures and writing up my notes, so I fear some errors may have crept in]

From the S-matrix the leading contribution to order g^2 is $\frac{1}{2!}(-ig)^2\langle p', k'|T\int\int \overline{\psi}(x)\psi(x)\phi(x)\overline{\psi}(y)\psi(y)\phi(y)d$ (the ψs destroy and $\overline{\psi}$s create; recall $\psi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_S a_p^S u^S(p)e^{-ip\cdot x} + b_p^{\dagger S}v^S(p)e^{ip\cdot x}, \overline{\psi} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_S a_p^{\dagger S}\overline{u}^S(p)e^{ip\cdot x} + b_p^S\overline{v}^S(p)e^{-ip\cdot x})$. $\overbracket{\psi(x)|p}, s\rangle = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\sum_S a_{p'}^{S'}u^{S'}(p')e^{-ip'\cdot x}\sqrt{2E_p}a_p^{S\dagger}|0\rangle = e^{-ip\cdot x}u^S(p)|0\rangle$ for contraction of ψ with the initial state fermions; similarly for $\overline{\psi}$ with the final state fermions (we are considering fermions rather than antifermions). A typical matrix element is e.g. $\langle p', k'|\frac{1}{2!}(-ig)^2\int d^4x\overline{\psi}\psi\phi\int d^4y\overline{\psi}\psi\phi|p, k\rangle$ with contractions over it: the p' with the first $\overline{\psi}$, the k' with the second $\overline{\psi}$, the first ψ with the k, the second ψ with the p, and the two ϕ together. This gives: $(-ig)^2\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-\mu^2+i\epsilon}(2\pi)\delta^{(4)}(p'-p+q(2\pi)^4\delta^{(4)}(k'-k-q)\overline{u}(p')u(p)\overline{u}(k')u(k)$, the factors of $\frac{1}{2!}$ being cancelled by a factor of 2 from interchanging x and y. This is $i(2\pi)^4\delta^{(4)}(\sum p)\frac{-g^2}{q^2-\mu^2+i\epsilon}\overline{u}(p')u(p)\overline{u}(k')u(k)$. The $\delta^{(4)}(\sum p)$ imposes $p - p' = q = k - k'$. This gives us the Feynman rules: in diagrams, solid lines denote fermion, dashed lines denote scalar:

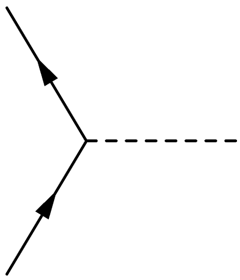




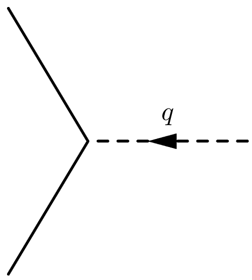
1) Propagators $\overbracket{\phi(x)\phi(y)}$ are $= \frac{i}{q^2-\mu^2+i\epsilon}$;



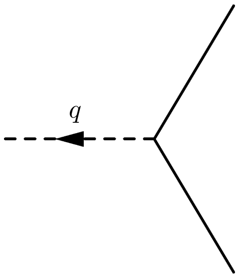
$\overbracket{\psi(x)\overline{\psi}(y)}$ (not present in this diagram) is $= \frac{i(\not p+M)}{p^2+M^2+i\epsilon}$.

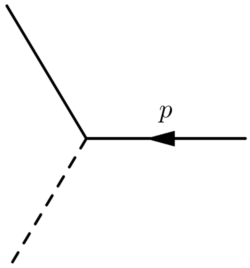


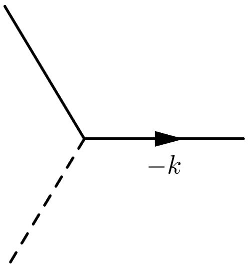
2) Vertices give $-ig$; this is the only possible vertex for this interaction.

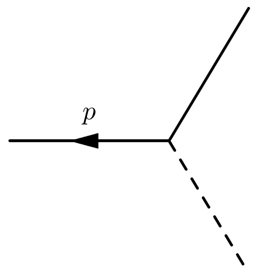


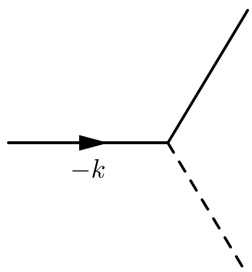
3) External leg contractions: $\overbracket{\phi|q}\rangle = 1$, $\langle \overbracket{q|\phi}$

= 

$= 1$, $\overbrace{\psi|p}, s\rangle =$ 

$= u^S(p)$, fermion <u>in</u>, $\overbrace{\overline{\psi}|k}, s\rangle =$ 

$= \overline{v}^S(k)$, an anti-fermion going <u>in</u> - the line is drawn the way it is because Feynman's convention was that antifermions are fermions going backwards in time (and this necessitates the minus sign on the momentum). (Note that particles going in are on the right and particles going out on the left in these diagrams, in contrast to the usual English convention). Finally $\langle p, \overbrace{s|\overline{\psi}} =$ 

$= \overline{u}^S(p)$, a fermion going out, and $\langle \overbrace{k, s|\psi} =$ 

$= v^S(k)$, an anti-fermion going out.

4) Impose momentum conservation.

5) Integrate over each undetermined loop momentum.

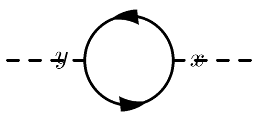
6) Include an overall sign factor for the diagram.

Comments: a) the $\frac{1}{n!}$ factor from the Taylor expansion is always cancelled by the $n!$ ways of interchanging vertices to get the same contraction, so no symmetry factor is required. In Yukawa theory the three fields $\overline{\psi}, \psi, \phi$ are distinct.

b) Arrows on fermion lines flow consistently through the diagram; this ensures fermion number conservation. Momentum is always in the direction of the arrow.

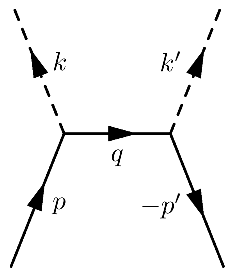
c) Dirac indices are contracted at each vertex $\mathcal{L}_{\text{int}} \sim \overline{\psi}\ \alpha\psi_\alpha\phi$. Hence, the $\not{P} + m$ terms in the propagator get matrix multiplied together or contracted with external spinors.

d) Around a closed fermion loop there is a trace and additional minus sign:



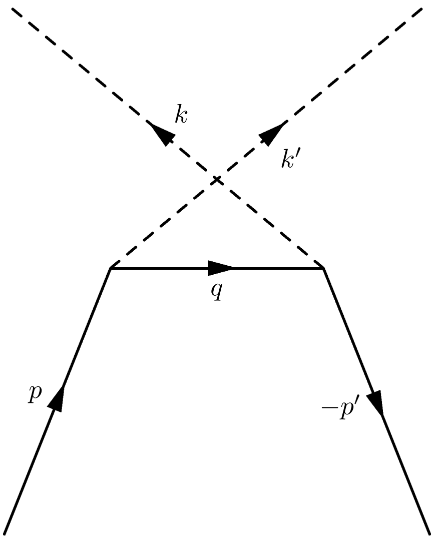
involves $\overline{\psi}_\alpha(x)\,\psi_\alpha(x)\overline{\psi}_\beta(y)\,\psi_\beta(y) = -\,\psi_\beta(y)\overline{\psi}_\alpha(x)\,\psi_\alpha(x)\overline{\psi}_\beta(y) = -\mathrm{tr}(S_F S_F)$.

Example: Writing $\langle f|S-1|i\rangle = iT_{fi} = i\mathcal{M}(2\pi)^4\delta^{(4)}(\sum p_i - \sum p_f)$ where $\sum p_i$ is the sum over incoming momentum, $\sum p_f$ the sum over final momentum, we can write down the scattering amplitude for several examples:



Fermion-antifermion annihilation into two mesons:
$= \langle k', k| \int d^4x \phi\overline{\psi}\psi \int d^4y \phi\overline{\psi}\psi |p, p'\rangle$ where we contract k' with the first ϕ, k with the second ϕ, the first $\overline{\psi}$ with p', the first ψ with the second $\overline{\psi}$ and the second ψ with p [written in words because I can't do crossing contraction brackets in LaTeX]. This $\sim \int d^4x \int d^4y e^{ik'\cdot x}\overline{v}(p')e^{-ip'\cdot x} \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q}+m)}{q^2-m^2+i\epsilon} e^{-iq\cdot(x-y)}u(p)e^{-ip\cdot y}e^{ik\cdot y}$. The integrals over x and y give delta functions, forcing q to flow from y to x as shown. So $i\mathcal{M} \sim (-ig)^2 \frac{\overline{v}(p')i(\gamma^\mu(p-k)_\mu+M)u(p)}{(p-k)^2-M^2}$, as we would get by the Feynman

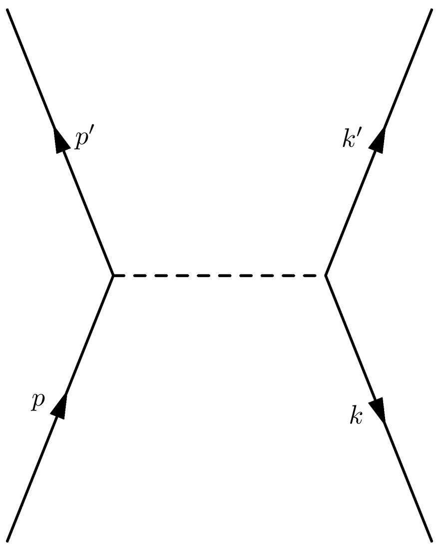
rules. There is another diagram,



,

$i\mathcal{M} \sim (-ig)^2 \frac{\overline{v}(p') i(\gamma^\mu (p-k')_\mu + m) u(p)}{(p-k')^2 - m^2}$.

Fermion-fermion scattering: $\psi\psi \to \psi\psi$. To lowest order we have the two diagrams:



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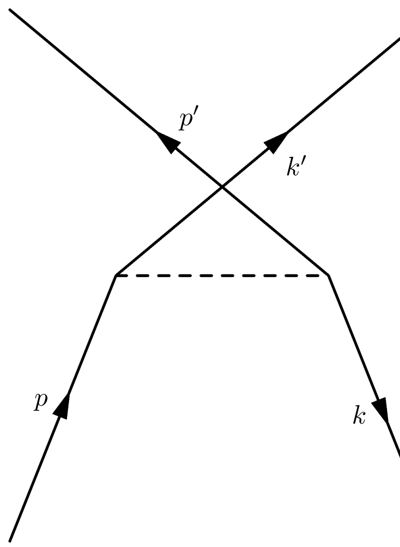
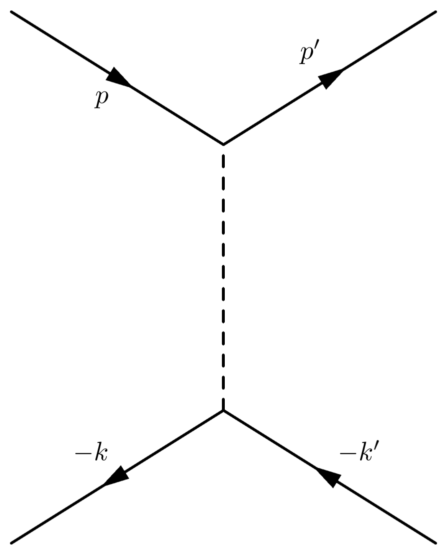


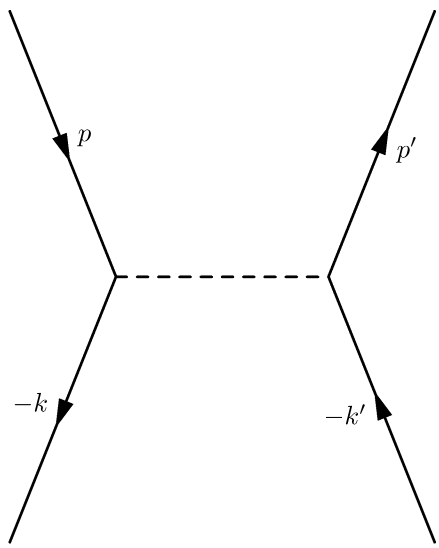
$i\mathcal{M} \sim (-ig)^2((\overline{u}(p')u(p)\frac{i}{(p-p')^2-\mu^2}\overline{u}(k')u(k)) - (\overline{u}(p')u(k)\frac{i}{(p-k')^2-\mu^2}\overline{u}(k')u(p)))$

(the $-$ sign reflecting Fermi statistics, and one i coming from the propagator).

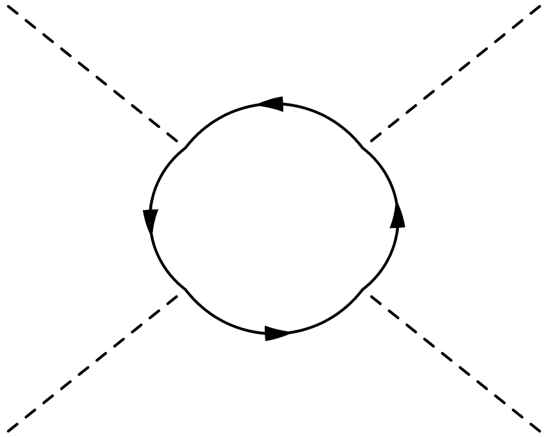


Fermion-antifermion scattering: $\psi\overline{\psi} \to \psi\overline{\psi}$: two diagrams,

+



. $i\mathcal{M} = (-ig)^2(\frac{-i\overline{u}(p')u(p)\overline{v}(k)v(k')}{(p-p')^2-\mu^2} + \frac{i\overline{v}(k)u(p)\overline{u}(p')v(k')}{(p+k)^2-\mu^2})$, with the minus signs coming from spinor contractions.



Finally, meson scattering:
$= \bar{\psi}\,\psi\bar{\psi}\,\psi\bar{\psi}\,\psi\bar{\psi}\,\psi = (-1)\mathrm{tr}(\psi\bar{\psi}\,\psi\bar{\psi}\,\psi\bar{\psi}\,\psi\bar{\psi}) = (-1)\mathrm{tr}(S_F S_F S_F S_F)$; note we have -1 for a closed fermion loop.

Quantum Electrodynamics

This is QED; we set $c = 1$, so $\mu_0 = \epsilon_0 = 1$. Maxwell's equation: we have the gauge potential $A^\mu = (A^0, \boldsymbol{A}) = (\phi, \boldsymbol{A})$. $A_\mu = (A_0, -\boldsymbol{A})$. The electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; this is antisymmetric in (μ, ν). The electric field $E_i = F_{0i} = \partial_0 A_i - \partial_i A_0$, i.e. $\boldsymbol{E} = -\dot{\boldsymbol{A}} - \boldsymbol{\nabla} A_0$. The magnetic field $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$, i.e. $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. The Lagrangian density is $\mathcal{L}_{\text{e.m.}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{E} - \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{B}$. The field equation (equation of motion) $\partial_\mu(\frac{\partial \mathcal{L}_{\text{e.m.}}}{\partial(\partial_\mu A_\nu)} =)0 \Rightarrow \partial_\mu F^{\mu\nu} = 0$ (1). There are also the Bianchi identities: $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ (2) (expand out and use symmetry of partial derivatives). From (1) we get $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$, $\dot{\boldsymbol{E}} = \boldsymbol{\nabla}\times\boldsymbol{B}$ and (2) $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, $\frac{\partial}{\partial t}\boldsymbol{B} = -\boldsymbol{\nabla}\times\boldsymbol{E}$.

Note: i) $\mathcal{L}_{\text{e.m.}}$ doesn't depend on $\dot{A}_0$, so A_0 is not really dynamical and its variational equation can be solved. $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0 \Rightarrow \boldsymbol{\nabla}^2 A_0 + \boldsymbol{\nabla}\cdot\dot{\boldsymbol{A}} = 0$ has solution $A_0(\boldsymbol{x}) = \int d^3\boldsymbol{x}' \frac{\boldsymbol{\nabla}\cdot\dot{\boldsymbol{A}}(\boldsymbol{x}')}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}$. ii) $F_{\mu\nu}$ (and hence $\mathcal{L}_{\text{e.m.}}$ and the Maxwell equations) are invariant under the gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu\chi(x), F_{\mu\nu} \to \partial_\mu(A_\nu + \partial_\nu\chi) - \partial_\nu(A_\mu + \partial_\mu\chi) = F_{\mu\nu}$. A gauge transformation has no physical effect; it just gives a different description of the same field configuration. But this means we have to fix the gauge when making calculations; since physical quantities are gauge invariant, we choose the gauge so as to simplify calculation. Two such choices are a) Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ and b) Lorentz gauge $\partial_\mu A^\mu = 0$; we shall use the former.

Coulomb gauge: $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0 \therefore \boldsymbol{\nabla}\cdot\dot{\boldsymbol{A}} = 0 \therefore A_0 = 0$ so $\boldsymbol{E} = -\dot{\boldsymbol{A}}$. Coulomb gauge breaks Lorentz invariance, but does exhibit the physical degrees of freedom. Since the three components of $\boldsymbol{A}$ satisfy $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$, this leaves two independent degrees of freedom; these are the polarization states of the photon.

Quantisation of the Electromagnetic Field

The conjugate momentum $\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0}$ - a consequence of A_0 not being dynamical. $\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} \equiv E^i$, so the Hamiltonian $H = \int d^3\boldsymbol{x} \pi^i \dot{A}_i - \mathcal{L} = \int d^3\boldsymbol{x}\frac{1}{2}(\boldsymbol{\pi} \cdot \boldsymbol{\pi} + \boldsymbol{B} \cdot \boldsymbol{B}) = \int d^3\boldsymbol{x}\frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{E} + \boldsymbol{B} \cdot \boldsymbol{B})$ ($\boldsymbol{\nabla} \cdot \boldsymbol{\pi} = 0$ from $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$). We could add $A_0(\boldsymbol{\nabla} \cdot \boldsymbol{E})$ so A_0 acts as a Lagrange multiplier imposing $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$.

Equation of motion: for $\boldsymbol{A}$, $\partial_\mu \partial^\mu \boldsymbol{A} = 0$, with wave-like solution $\boldsymbol{A} = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \boldsymbol{\epsilon}(\boldsymbol{p}) e^{ip\cdot x}$ and $p^2 = {p^0}^2 - \boldsymbol{p}^2 = 0$. Then $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0 \Rightarrow \boldsymbol{\epsilon} \cdot \boldsymbol{p} = 0$. This gives the two physical polarization states of the photon, and $\boldsymbol{\epsilon}(\boldsymbol{p})$ is a linear combination of two orthonormal vectors $\boldsymbol{\epsilon}_r$ $(r = 1, 2)$ with $\boldsymbol{\epsilon}_r \cdot \boldsymbol{p} = 0, \boldsymbol{\epsilon}_r(p) \cdot \boldsymbol{\epsilon}_s(p) = \delta_{rs}$.

We impose the canonical commutation relations on H: $[A_i(\boldsymbol{x}), A_j(\boldsymbol{x'})] = 0 = [\pi_i(\boldsymbol{x}), \pi_j(\boldsymbol{x'})], [A_i(\boldsymbol{x}), \pi_j(\boldsymbol{x'})] = i(\delta_{ij} - \nabla^{-2}\partial_i\partial_j)\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x'})$ [Davis insists that the ∇^{-2} is correct - i.e. this is $\frac{\partial_i \partial_j}{\nabla^2}$], and the RHS ensures $\boldsymbol{\nabla} \cdot \boldsymbol{A}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{\pi}$ have nontrivial commutators with everything else.

Proceed with mode expansion: $\boldsymbol{A}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2|p|}} \sum_{r=1,2} \boldsymbol{\epsilon}_r(\boldsymbol{p})(a^r_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^{r\dagger}_{\boldsymbol{p}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}), \Pi(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}(-i)\sqrt{\frac{|\boldsymbol{p}|}{2}} \sum_{r=1,2} \boldsymbol{\epsilon}_r(\boldsymbol{p})(a^r_p e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^{r\dagger}_p e^{-i\boldsymbol{p}\cdot\boldsymbol{x}})$, where $[a^r_{\boldsymbol{p}}, a^s_{\boldsymbol{k}}] = [a^{r\dagger}_{\boldsymbol{p}}, a^{s\dagger}_{\boldsymbol{k}}] = 0, [a^r_{\boldsymbol{p}}, a^{s\dagger}_{\boldsymbol{k}}] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{k})\delta^{rs}$ and there is a completeness relation for polarization states $\sum_{r=1,2} \epsilon^i_r(\boldsymbol{p})\epsilon^j_r(\boldsymbol{p}) = \delta^{ij} - \frac{p^i p^j}{|\boldsymbol{p}|^2}$. After normal ordering as before, $H = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}|\boldsymbol{p}| \sum_{r=1,2} a^{r\dagger}_{\boldsymbol{p}} a^r_{\boldsymbol{p}}$, we can now obtain the photon propagator: $\langle 0|TA_i(x)A_j(y)|0\rangle = D^{\text{tr}}_{ij}(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - i\epsilon}(\delta^{ij} - \frac{p^i p^j}{|\boldsymbol{p}|^2})e^{-ip\cdot(x-y)}$, where tr denotes that this is the transverse part of the photon. However, this is not the final form, as we will see shortly when we consider interactions.

Inclusion of Matter

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi$ where $D_\mu = \partial_\mu + ieA_\mu$ - the gauge invariant derivative, where e is the EM coupling. The field equations are $\partial_\mu F^{\mu\nu} = e\overline{\psi}\gamma^\nu\psi$ i.e. $ej^\mu = e\overline{\psi}\gamma^\mu\psi$ is the EM current, and $i\gamma^\mu D_\mu\psi - m\psi = 0$.

Gauge invariance: under the transformation $\psi(x) \to e^{ie\chi(x)}\psi(x), A_\mu(x) \to A_\mu(x) - \partial_\mu\chi$, the covariant derivative transforms like ψ: $D_\mu\psi \to (\partial_\mu + ieA_\mu - ie\partial_\mu\chi)e^{ie\chi}\psi = e^{ie\chi}D_\mu\psi - ie\partial_\mu\chi e^{ie\chi}\psi + ie\partial_\mu\chi e^{ie\chi}\psi = e^{ie\chi}D_\mu\psi$. But $\overline{\psi} \to \overline{\psi}e^{-ie\chi(x)}$, so $\overline{\psi}D_\mu\psi$ is invariant. So $\mathcal{L}$ and the field equations are gauge invariant.

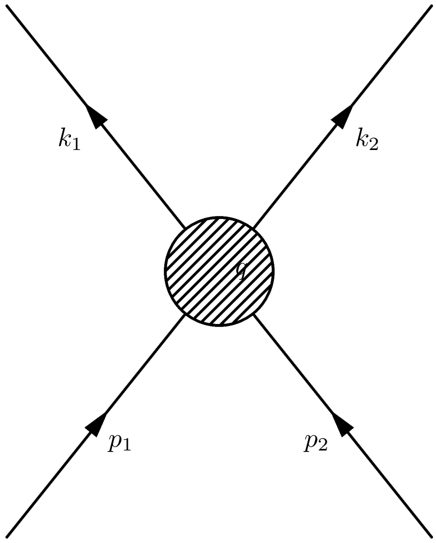
Again we use Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. The A_0 equation is $\partial_i F^{0i} = ej^0$ (i.e. no longer zero), so $-\nabla^2 A_0 = e\overline{\psi}\gamma^0\psi = e\psi^\dagger\psi$. The RHS is the electric charge density e_ρ. Hence $A_0(x) = e\int d^3\boldsymbol{x'}\frac{\rho(\boldsymbol{x'})}{4\pi(\boldsymbol{x}-\boldsymbol{x'})}$ and $\int \boldsymbol{E}^2 = \int(\dot{\boldsymbol{A}} + \boldsymbol{\nabla}A_0)^2 = \int \dot{\boldsymbol{A}}^2 + (\boldsymbol{\nabla}A_0)^2$ - the cross terms vanish on integration by parts, using $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$.

So $\int \boldsymbol{E}^2 = \int \dot{\boldsymbol{A}}^2 - \int A_0 \boldsymbol{\nabla}^2 A_0 = \int \dot{\boldsymbol{A}}^2 + e^2 \iint \frac{\rho(x)\rho(x')}{4\pi|x - x'|} d^3\boldsymbol{x} d^3\boldsymbol{x'}$. This is <u>non-local</u>, but the nonlocality is an artifact of Coulomb gauge.

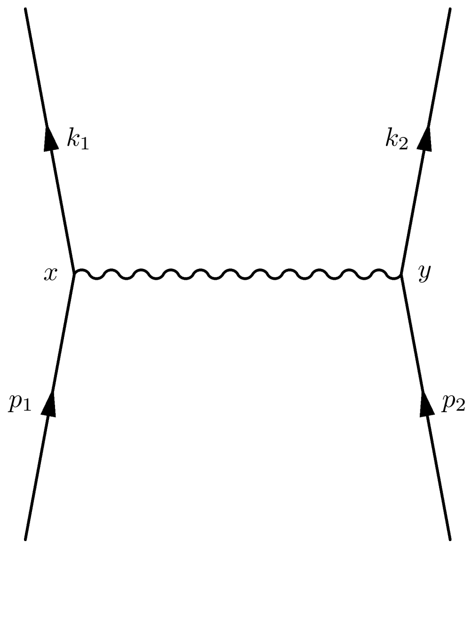
The conjugate of $\boldsymbol{A}$ and ψ: $\pi = \frac{\partial \mathcal{L}}{\partial \dot{A}} = -\boldsymbol{E}, \pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\overline{\psi}\gamma^0$ $\therefore$ $H = \int(\pi \cdot \dot{A} + \pi_\psi\dot{\psi} - \mathcal{L})d^3\boldsymbol{x}$ $\therefore$ $H_{\text{QED}} = \int(\frac{1}{2}\boldsymbol{E} \cdot \boldsymbol{E} + \frac{1}{2}\boldsymbol{B} \cdot \boldsymbol{B} - i\overline{\psi}\boldsymbol{\gamma} \cdot \boldsymbol{\nabla}\psi + m\overline{\psi}\psi)d^3x + \frac{e^2}{2}\iint \frac{\rho(\boldsymbol{x})\rho(\boldsymbol{x'})}{4\pi|\boldsymbol{x}-\boldsymbol{x'}|}d^3\boldsymbol{x}d^3\boldsymbol{x'} - e\int \boldsymbol{j} \cdot \boldsymbol{A} d^3x$, where $\rho = \overline{\psi}\gamma^0\psi, \boldsymbol{j} = \overline{\psi}\boldsymbol{\gamma}\psi, \boldsymbol{\pi} = -\boldsymbol{E}$. Note: there are two types of integration terms now - this gives the Lorentz invariant propagator.

Lorentz Invariant Propagator

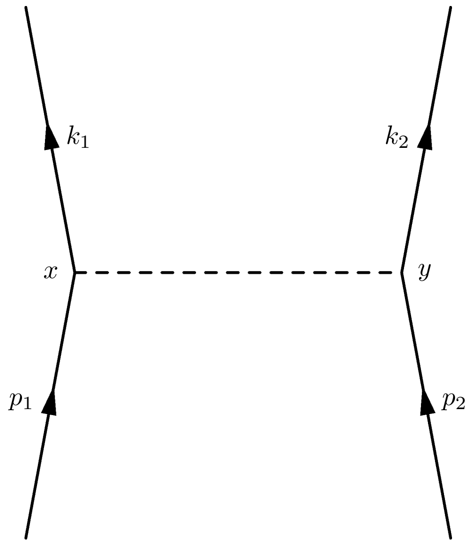
Consider the process of two electron scattering:



$q = p_2 - k_2 = -p_1 + k_1$. We expect this diagram to involve a) spinors on external legs b) vertices $e\gamma^\mu$ c) photon propagators $D_{\mu\nu}(x-y)$. From our Hamil-



tonian we have: i) ver-



tices $-e\gamma^\mu$, transverse propagator $D_{ij}(x-y)$, and ii) vertices $\pm e\gamma^0$, instantaneous (coulomb) interaction $\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\delta(x^0-y^0)$. We can combine these two into the coulomb gauge propagator $D^{\text{coulomb}}_{\mu\nu}(x-y)$ and vertices $e\gamma^\mu$, where $D^{\text{coulomb}}_{00}(x-y) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\delta(x^0-y^0)$. In momentum space (recall) this is $\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{|\boldsymbol{p}|^2}$ $\therefore$ the coulomb propagator is $D^{\text{coulomb}}_{\mu\nu}(q) = \frac{i}{q^2+i\epsilon}(\delta_{ij}-\frac{q_iq_j}{|\boldsymbol{q}|^2})$ for $\mu = i \neq 0, \nu = j \neq 0$, and $\frac{i}{|\boldsymbol{q}|^2}$ for $\mu=\nu=0$, 0 otherwise.

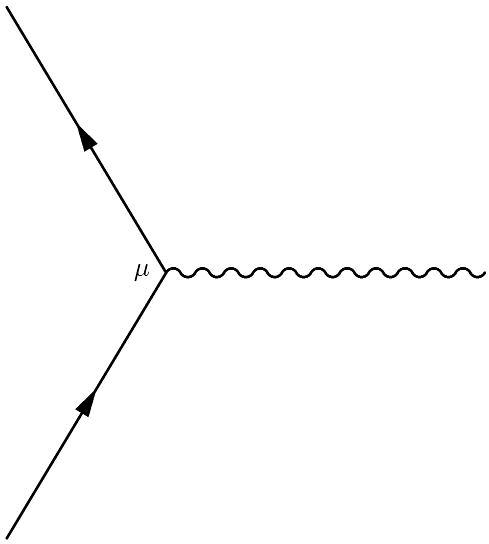
Now consider the cortribution to the scattering amplitude from the left vertex. It is of the form $e\overline{\mu}(k_1)\gamma^\mu u(p_1)$. Current conservation implies $(k_1 - p_1)_\mu\overline{u}(k_1)\gamma^\mu u(p_1) = 0$, as the reader may verify directly from the Dirac equation.

So the propagator occurs in the combination $c^\mu D^{\text{coulomb}}_{\mu\nu}d^\nu$ where $c^\mu = \overline{u}(k_1)\gamma^\mu u(p_1), d^\nu = \overline{u}(k_2)\gamma^\nu u(p_2)$, and $q_\mu c^\mu = c_0c_0 - \boldsymbol{q}\cdot\boldsymbol{s} = 0, q_\mu d^\mu = 0$. So $c^\mu D^{\text{coulomb}}_{\mu\nu}d^\nu = i(\frac{\boldsymbol{c}\cdot\boldsymbol{d}}{q^2} - \frac{{q^0}^2c_0d_0}{q^2|\boldsymbol{q}|^2} + \frac{c_0d_0}{|\boldsymbol{q}|^2}) = \frac{i}{q}(\boldsymbol{c}\cdot\boldsymbol{d} - c_0d_0)$ using $q_0^2 - \boldsymbol{q}^2 = q^2$, which is $-\frac{i}{q^2}c^\mu d_\mu$.

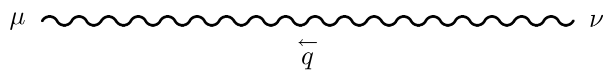
So in this diagram, and in general, we replace $D^{\text{coulomb}}_{\mu\nu}$ by the manifestly invariant propagator $D_{\mu\nu}(q) = -\frac{i}{q^2}g_{\mu\nu}$. This can be generalized to $D_{\mu\nu}(q^2) = -\frac{i}{q^2}(g_{\mu\nu} + \lambda\frac{q_\mu q_\mu}{q^2})$ (we can view $D_{\mu\nu}$ as a function either of q directly or of the invariant q^2) for calculational convenience - by current conservation, the additional term doesn't contribute. $\lambda = 0$ gives the Feynman gauge proppagator, $\lambda = -1$ is the Landau gauge propagator. We can now write down the Feynman rules for QED:

Feynman Rules for QED

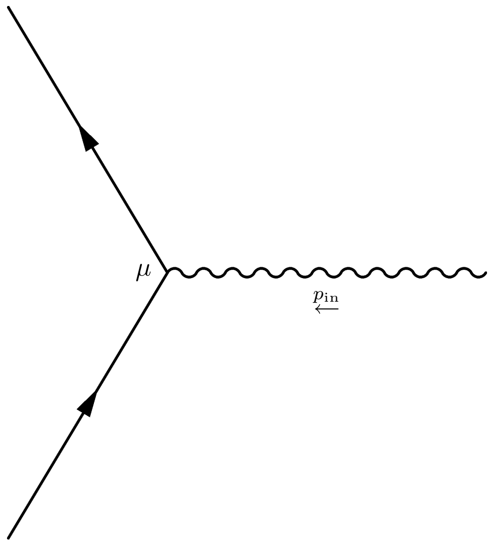
We use solid lines for electrons or positrons and "wiggly" lines for photons. The rules for Fermions are as before. The additional rules are:



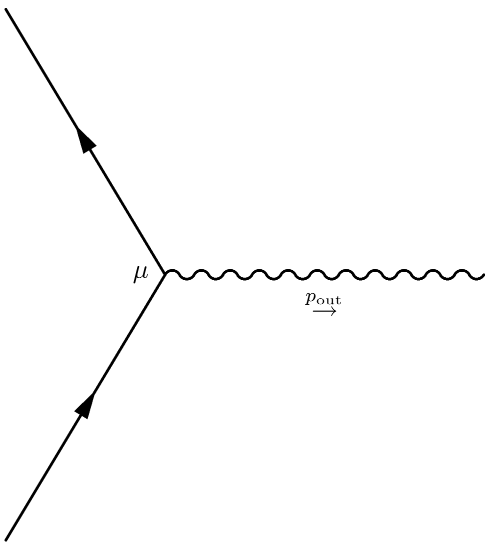
i) Vertices $= ie\gamma^\mu$.



ii) The photon propagator
$= \frac{-ig_{\mu\nu}}{q^2+i\epsilon}$.

iii) External photon lines 

$=\epsilon_\mu^{\text{in}}$,



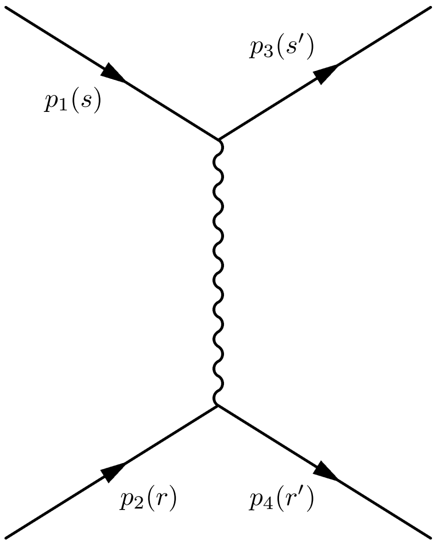
$=\epsilon_\mu^{\text{out}}$. ϵ is the polarization vector of the incoming or outgoing photon.

In coulomb gauge $\epsilon_0 = 0, \boldsymbol{\epsilon} \cdot \boldsymbol{p}_{\text{ext}} = 0$. One contracts over all Lorentz indices in a diagram.

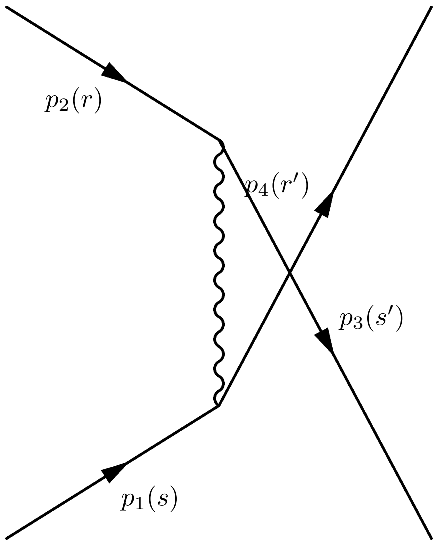
Scattering Processes in QED

There are many important processes we can now calculate at tree level using the Feynman rules. Loop diagrams are higher order and require regularization; see the next lecture.

1) Electron scattering, $e^-e^- \to e^-e^-$. Two diagrams contribute: the t-



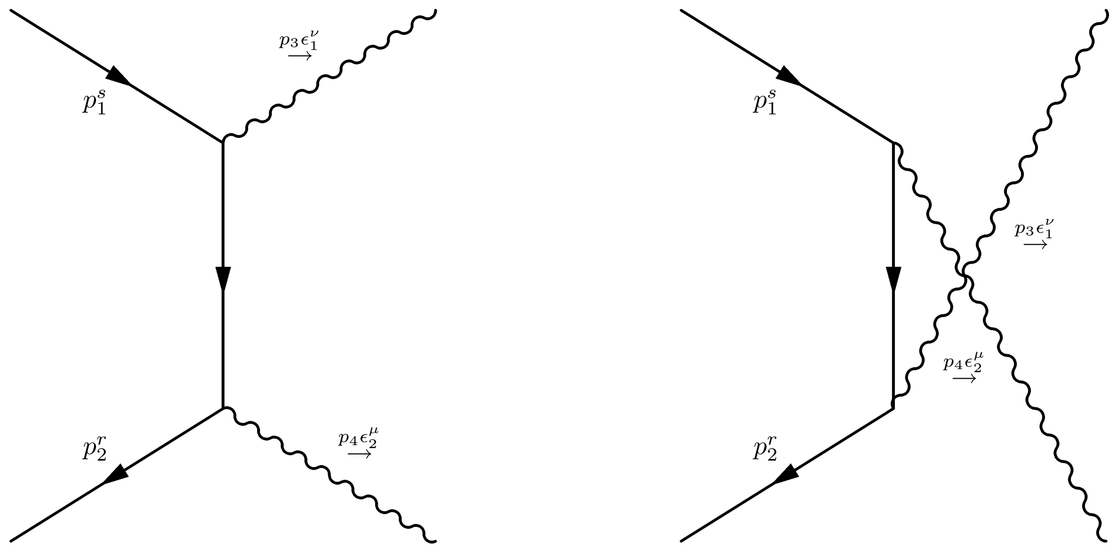
channel with propaga-



tor $\propto \frac{1}{(p_1-p_3)^2} = \frac{1}{t}$, and the u-channel with propagator $\propto \frac{1}{(p_1-p_4)^2} = \frac{1}{u}$. $i\mathcal{M} = -i(-ie)^2((\overline{u}^{s'}(\boldsymbol{p}_3)\gamma^\mu u^s(\boldsymbol{p}_1))\frac{g_{\mu\nu}}{(p_3-p_1)^2}(\overline{u}^{r'}(\boldsymbol{p}_4)\gamma^\nu u^r(\boldsymbol{p}_2)) - (\overline{u}^{r'}(\boldsymbol{p}_4)\gamma^\mu u^s(\boldsymbol{p}_1))\frac{g_{\mu\nu}}{(p_1-p_4)^2}(\overline{u}^{s'}(\boldsymbol{p}_3)\gamma^\nu u^r(\boldsymbol{p}_2)))$.

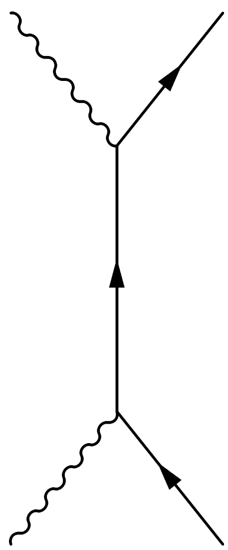
2) Electron-Positron Annihilation

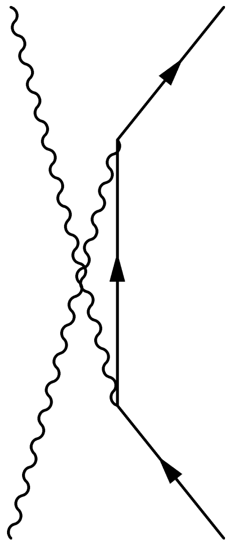
e^+e^- form a mound (positronium) state, which decays into two photons:



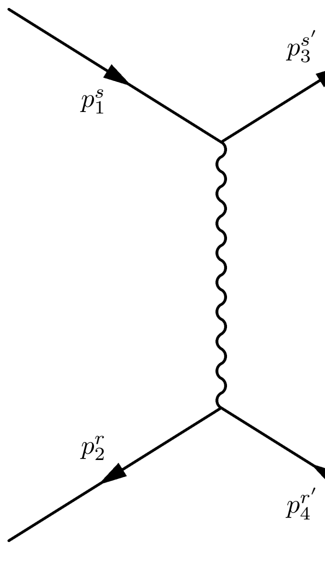
+

$= i(-ie)^2\overline{v}^r(\boldsymbol{p}_2)(\gamma_\mu \frac{\not{p}_1-\not{p}_3+m}{(p_1-p_3)^2-m^2}\gamma_\nu + \gamma_\nu \frac{\not{p}_1-\not{p}_4+m}{(p_1-p_4)^2-m^2}\gamma_\mu)u^s(\boldsymbol{p}_1)\epsilon_1^\nu(\boldsymbol{p}_3)\epsilon_2^\mu(\boldsymbol{p}_4)$; see the standard model course for further calculations, where we simplify this and calculate the base state.

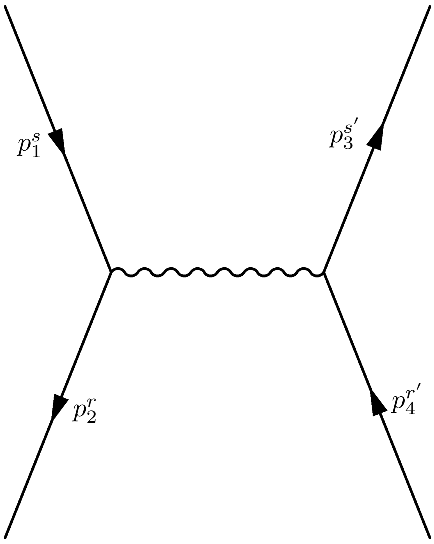
3) Compton scattering:  ,



4) Electron-Positron scattering (Bhabba Scattering): $e^-e^+ \to e^-e^+$.



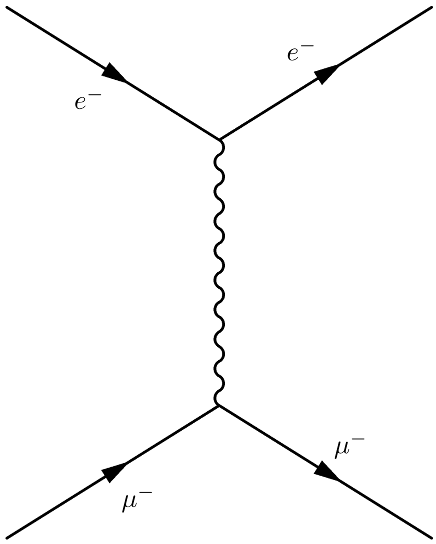
the t-channel, and



, the s-channel. $i\mathcal{M} = i(-ie)^2(-(\overline{u}^{s'}(p_3)\gamma^\mu u^s(p_1))\frac{1}{(p_1-p_3)^2}(\overline{v}^{r'}(p_4)\gamma_\mu v^r(p_2))+(\overline{v}^r(p_2)\gamma^\mu u^s(p_1))\frac{1}{(p_1+p_2)^2}(\overline{u}^{s'}(p_3)\gamma_\mu v$

Muons are “heavy electrons”.

5) Electron-muon scattering: $e^-\mu^- \to e^-\mu^-$,



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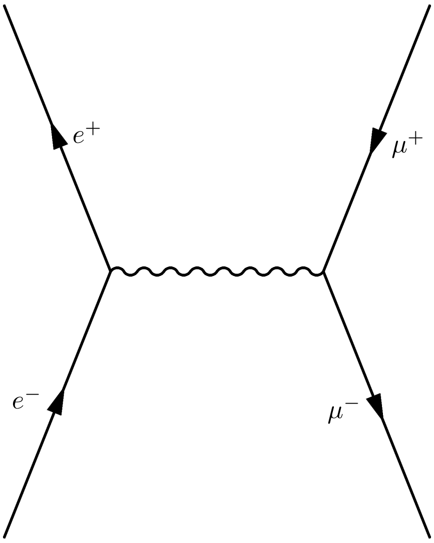
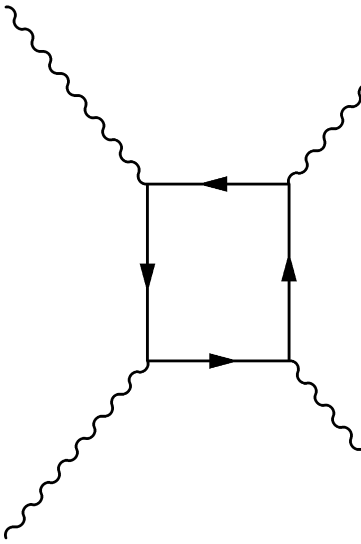
Compare with electron scattering; here we have only one diagram, the t-channel.



$e^+e^- \to \mu^+\mu^-$: ; again we have only one diagram, here the s-channel.



We can consider other processes, e.g. photon scattering $\gamma\gamma \to \gamma\gamma$: This diagram is $O(e^4)$; it is finite by gauge invariance.

This appears to be the end of the course.