Percolation and Combinatorics

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This is intended as a fun course. The subject is quite new (around 50 years old), but there is a large amount of literature on it. Unlike in previous years, notes will not be given out, since the course will follow the lecturer's recent book *Percolation* quite closely.

Basic Concepts

1. Percolation was originally developed to study the passage of liquids through porous media, but in our context it is the study of random sampling of lattice-like infinite graphs; we are mostly interested in the component structure. Specifically, we take Λ, the ground graph, to be some infinite graph, and select vertices or edges at random to obtain Λ_p (where p is the probability of selecting an individual vertex or edge). The theory was founded by Broadbeut and Hammersly in 1957.

2. Terminology; in this field vertices are known as sites and edges as bonds; a site is open if we hae selected it, and a subgraph is open if we have selected all its sites and bonds. We shall clook in particular at open paths, known in this field as "open self-avoiding walks". An open cluster is a component of Λ_p.

Some cases: Λ_p^b is bond percolation, where we select edges as random and take the graph formed by all vertices of Λ and these edges. Λ_p^s is site percolation, where we select vertices at random and take the subgraph induced by these vertices. $\overrightarrow{\Lambda}$ is where our graphs are taken to be oriented; we may allow Λ to be a multigraph.

3. To formalize the above a little: $\Lambda = (V, E)$; we'll define bond percolation. A configuration is a map $\omega : E \to \{0, 1\}$ $e \mapsto \omega_e$. Ω is the stet of all configurations, the state space. We define Σ to be the σ-field on Ω generated by cylindrical sets $C(F, \sigma) = \{\omega \in \Omega : \omega_f = \sigma_f \forall f \in F\}$, where $F \subset E$ is finite and $\sigma : F \to \{0, 1\}$. ω_e is called the state of e, either open or closed.

Probabilities: $\boldsymbol{p} = (p_e)_{e\in E}, 0 \leq p_e \leq 1$. $\mathbb{P}_p(C(F, \sigma)) = \prod_{\sigma_e=1,e\in F} p_e \prod_{\sigma_e=0,e\in F}(1-p_e)$ $(\star)$. Let $\mathbb{P}^b_{\boldsymbol{p}}$ be the measure on Σ generated by $(\star)$.

For $F_1, F_2 \subset E$ finite, $F_1 \cap F_2 = \emptyset$, $\sigma : F_1 \cup F_2 \to \{0, 1\}$, then $\mathbb{P}_{\boldsymbol{p}}(\omega_f = \sigma_f \forall f \in F_1 \mid \omega_f = \sigma_f \forall f \in F_2) = \mathbb{P}_{\boldsymbol{p}}(\omega_f = \sigma_f \forall f \in F_1)$ $(\star\star)$!

We often have $\boldsymbol{p} = (p)$, in which case we write $\mathbb{P}_p$. Obviously, we define site percolation measure the same way; for the moment, everything we say applies equally to both cases.

4. A natural coupling. Take $(X_e)_{e\in E}$ independent RVs, each uniformly distributed on $[0, 1]$. $(X_e) \to \Lambda_{\boldsymbol{p}}$ with edge-set $\{e \in E : p_e \geq X_e\}$. If $\boldsymbol{p} \leq \tilde{\boldsymbol{p}}$ (i.e. $\boldsymbol{p}_e \leq \tilde{\boldsymbol{p}}_e \forall e$), then $\Lambda_{\boldsymbol{p}} \subset \Lambda_{\tilde{\boldsymbol{p}}}$.

5. Critical probability. Fix a site $x \in \Lambda$; let C_x be the open cluster of x, $|C_x|$ is the number of sites in C_x. Define $\vartheta_x(p) = \mathbb{P}_p(C_x \text{ is infinite})$; $\vartheta_x(p)$ is monotone increasing. Taking, as we almost always do in this course, Λ to be a connected, locally finite, infinite graph (usually $\mathbb{Z}^d$), $\vartheta_x(p) = 0 \Leftrightarrow \vartheta_y(p) = 0$ for any $y \in \Lambda$: take a path L from x to y, say of length l, then $\vartheta_y(p) \geq p^l \vartheta_x(p)$. So, define the critical probability $p_H(\Lambda) = \sup\{p : \vartheta_x(p) = 0\}$; this is independent of x.

6. Let E be the event that there is an infinite open cluster.

0 Theorem

(this should really be theorem -5 or so, and is here only because the lecturer would be embarassed to deliver a lecture with no theorems): $\mathbb{P}_p(E) = 0$ if $p < p_H$, 1 if $p > p_H$: i) suppose $p < p_H$; $\mathbb{P}_p(E) \leq \sum_x \vartheta_x(p) = 0$. ii) Suppose $p > p_H$, then $\mathbb{P}_p(E) \geq \theta_x(p) > 0$. Hence by Kolmogrov's 0-1 laws, $\mathbb{P}_p(E) = 1$. Kolmogrov's 0-1 laws: For (X_i) independent RVs, Σ the σ-field generated by (X_i), let $A \in \Sigma$ be independent of $X_1, \ldots, X_n \forall n$. Then $\mathbb{P}(A)$ is 0 or 1. We apply this with $(X_e) = (\omega_e)$; E is independent of any finite set of ω_e.

7. Another critical probability: Define $\chi_x(p) = \mathbb{E}_p(|C_x|)$. This is monotone increasing (by the coupling). $\chi_x(p)$ and $\chi_y(p)$ are about the same (i.e. they are the same up to a constant factor, here p^l). $p_T(\Lambda) = \sup\{p : \chi_x(p) < \infty\}$, independent of x; this is the Temperly critical probability.

Clearly we have $p_H(\Lambda) \geq p_T(\Lambda)$: $\chi_{v_0} = \mathbb{E}(|C_{v_0}|) = \sum_{n=1}^{\infty} \mathbb{P}(|C_{v_0}| \geq n)$ and every summand is $\geq \vartheta_{v_0}$. If $\Lambda = T$ a tree, clearly $p_T = p_H$.

Often we consider Λ homogenous: $\forall x, y \in \Lambda \exists \phi \in \mathrm{Aut}(\Lambda) : \phi(x) = y$.

8. Path counting. Given Λ, write $\mu_l(\Lambda; x) =$ the number of self-avoiding walks (i.e. paths) of length l, starting at x. If Λ is homogenous, can define $\mu_l(\Lambda) = \mu_l(\Lambda; x) \forall x$. We have $\mu_{k+l} \leq \mu_k + \mu_l$ by concatenation (all $k+l$-paths are the concatenation of a k-path and an l-path but the converse is not true). Thus $\lambda = \lim_{l \to \infty} \mu_l^{\frac{1}{l}}$ exists; this is the connective constant of Λ.

Theorem: For Λ as always, suppose $\mu_l(\Lambda; x) \leq (\lambda + o(1))^l$ as $l \to \infty$. Then $p_T^b(\Lambda) \geq \frac{1}{\lambda}$: $\chi_x(p) = \mathbb{E}_p(|C_x|) = \sum_y \mathbb{P}_p(y \in C_x) = \sum_y \mathbb{P}_p(\exists \text{ open } x - y \text{ path}) = \mathbb{E}_p(\#\text{ open paths starting at} x) = \sum_l \mu_l(\Lambda; x) p^l$. Suppose $p < \frac{1}{\lambda}$, then $\chi_x(p) \leq C + \sum_{l \geq l_0} c^l < \infty$, by taking c with $p\lambda < c < 1$ and l_0 large enough that for $l \geq l_0$, $\mu_l(\Lambda; x) p \leq c^l$. So $p \leq p_T(\Lambda)$ and so $p_T(\lambda) \geq \frac{1}{\lambda}$.

9. The Bethe lattice is the k-regular infinite tree; it is used a bit in physics. We shall consider T_k, the rooted k-branching tree (which has a root; each node on level n branches k times so there are k times as many nodes on level $n+1$), which closely resembles the Bethe lattice of degree $k+1$. This is in some sense the simplesd example base graph to consider; it is obviously not worth considering graphs of maximum degree < 3.

Theorem: i) if the maximum degree $\Delta(\Lambda) = \Delta \geq 3$ then $p_T \geq \frac{1}{\Delta - 1}$ ii) For $k \geq 2$, $p_T^b(T_k) = p_H^b(T_k) = \frac{1}{k}$: i) $\mu_l(\Lambda; x) \leq \Delta(\Delta - 1)^{l+1} = (\Delta - l + o(1))^l$ so we are done by the previous theorem. ii) Consider $T_{k,n}$, the subgraph "up to" level n (e.g. $T_{2,3}$ contains the root, the two first-level vertices, the four second-level vertices and the eight third-level vertices). Consider bond percolation with probability p. Let $\pi_k = \mathbb{P}_p(\exists \text{open path from } v_0 \text{ to the set of leaves}$. We have $1 - \pi_{m+1} = (1 - p\pi_m)^k$: for there to not be any paths to level $m+1$, for each of the nodes at level 1, the subtree they are the root of must not contain any paths to level m, or there must be no edge from the root to that node. So $\pi_{m+1} = 1-(1-p\pi_m)^k$; call this $f(\pi_m)$, i.e. $f(x) = 1-(1-px)^k$ for $0 \le x \le 1$. What does this f look like? $f'(x) = kp(1-px)^{k-1}$; $f''(x) = -k(k-2)p^2(1-px)^{k-2}$. So f is monotone increasing and concave. If $f'(0) = kp > 1$, then by considering the graph and the fact that $f'(1) < 1$, there is a unique x_0, $0 < x_0 < 1$, with $f(x_0) = x_0$. $\pi_0 = 1 > x_0$, and if $\pi_n > x_0$ then $\pi_{n+1} = f(\pi_n) > x_0$. Hence, if $p > \frac{1}{k}$ then $\vartheta_{v_0}(p) \ge x_0 > 0$. Consequently $p > p_H$, so we have $p_H \le \frac{1}{k}$, and hence $p_T = p_H = \frac{1}{k}$.

10. Note: more is true. As an exercise for the reader, it is acutally relatively easy to calculate $\vartheta_{v_0}(\frac{1}{k})$. Indeed, $\chi_{v_0}(\frac{1}{k}) = \infty, \vartheta_{v_0}(\frac{1}{k}) = 0$ for T_k: $\chi_{v_0}(\frac{1}{k}) = \mathbb{E}_{\frac{1}{k}}(|C_{v_0}|) = \mathbb{E}_{\frac{1}{k}}(\#\text{ open paths starting at } v_0) = \sum_{l=0}^{\infty} k^l (\frac{1}{k})^l = \sum 1 = \infty$. For ϑ, $f_{k,\frac{1}{k}}(x) = 1-(1-\frac{x}{k})^k, \pi_0 = 1, \pi_1 = f(\pi_0), \ldots$ (recall $\pi_n = \mathbb{P}(\text{component goes down to level } n)$). Then $x - f(x) > 0$ on $(0,1]$ and is increasing since $1-f'(x) = 1-(1-\frac{x}{k})^{k-1} > 0$. Hence, if $\pi_n \ge \epsilon > 0$ then $\pi_n - f(\pi_n) \ge \epsilon - f(\epsilon) > 0$. Hence there can be at most $\frac{1}{\epsilon - f(\epsilon)} + 1$ π_ns which are $\ge \epsilon$; thus $\pi_n \to 0$, so $\vartheta^b_{\frac{1}{k}}(T_k) = 0$.

[11.] Plane graphs: Consider $\Lambda \subset \mathbb{R}^2$, and Λ 3-connected; this implies, though it is a difficult theorem, that the drawing of Λ on the sphere is unique. For example, a subgraph of $\mathbb{Z}^2$. i) Every cycle C separates $\mathbb{R}^2$ into its interior and exterior, as can be shown easily using winding numbers. ii) Eulers formula implies $K_{3,3}$ and K_5 are nonplanar. We know but will never use that a graph is nonplanar iff it contains a topological copy of one of these. iii) Let a, b, c, d be vertices around a cycle C in that order. Then Λ does not contain vertex-disjoint paths from a to c and b to d both in the interior or both in the exterior of C, by e.g. if both are in the interior, draw an exterior point connected to a, b, c, d, then we have a planar K_5. Or we can also prove using $K_{3,3}$: add points e between a and b, f between c and d, joined by an exterior path.

12. Dual graphs: For Λ a 3-connected plane graph, we construct a graph $\Lambda^\star$, the dual of Λ, by assigning a vertex to every face of (the map of) Λ, and for every bond f of Λ, joining the vertices of $\Lambda^\star$ corresponding to the faces bordering f by an edge $f^\star$ (deleting f, two faces "unite"; join by $f^\star$ the vertices of these faces). E.g. a triangular lattice becomes a hexagonal lattice; the correspondence is always between vertices and faces, edges and edges, and faces and vertices. E.g. $\Lambda = \mathbb{Z}^2$ the square lattice has $\Lambda^\star = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ - this lattice is "self-dual".

13. A basic property of $\mathbb{Z}^2$. For $C \subset \mathbb{Z}^2$ finite, $\mathbb{Z}^2 \setminus C$ has a unique infinite component C_∞. The outer boundary $\partial_\infty C$ of C is formed by the bonds dual to the bonds between C and C_∞.

Lemma 3: For a finite $C \subset \mathbb{Z}^2$ which is the vertex set of a connected graph, the outer boundary $\partial_\infty C$ is a cycle containing C in its interior: let $\overrightarrow{F}$ be the set of bonds from C to C_∞ (considered as directed inthis way), and $\overrightarrow{F}^\star$ the set of dual bonds, with $\overrightarrow{f}^\star$ obtained from $\overrightarrow{f}$ by rotating it through $\frac{\pi}{2}$ (in the positive sense). We claim that for any oriented bond $\overrightarrow{f}^\star = \overrightarrow{uv} \in \overrightarrow{F}^\star$, there is a unique $\overrightarrow{vw} \in \overrightarrow{F}^\star$ leaving v. Proving this is fiddly rather than difficult and the details are left as an exercise; use $R = \mathbb{Z}^2 \setminus (C \cup C_\infty)$ and consider which of these three sets some vertices may be in; in particular, say v lies in the square $abcd$ with f being the bond ab; then e.g. if $c \in C_\infty$ we are done, if $d \in C$ done and so on. The difficult case is when $c \in C, d \in C_\infty$, but this cannot arise, as we must then have a path on the outside from a to c and another from b to d, contradicting planarity. Thus $\partial_\infty C \supset$ some cycle S; C is in the interior of S, the infinite component is outside. If $\overrightarrow{f}^\star \in \partial_\infty C$, say $\overrightarrow{f} = yz$, then $y \in C$ and $z \in$ the infinite component of the rest. But then $\overrightarrow{f}$ crosses S, so $\overrightarrow{f}^\star \in S$. Thus $\partial_\infty C \subset S$ and we have the result.

14. Simple bounds on $p_H^b(\mathbb{Z}_2)$: Theorem 4: $\frac{1}{3} \le p_H^b \le \frac{2}{3}$: i) $\Delta(\mathbb{Z}^2) = 4$ ii) If the component of the origin is finite, we must have a cycle around it in the dual graph, and none of the bonds dual to this open. We will show the probability of a large cycle is small. Take $0 < p < 1$ and consider bond percolation on $\mathbb{Z}^2$; let L_k be the path from $(0,0)$ to $(k,0)$ along the x axis. Let A_k be the event that L_k is open; $\mathbb{P}_p(A_k) = p^k$. As we always do, we define $f^\star \in \Lambda^\star$ open iff the corresponding $f \in \Lambda = \mathbb{Z}^2$ is closed. Let B_k be the event that there is no open cycle in $\Lambda^\star$ surrounding L_k. By Lemma 3, if C_0 is finite then its outer boundary is an open cycle in $\Lambda^\star$ surrounding C_0; hence $A_k \cap B_k$ is a subset of the event that C_0 is infinite.

Take $p > \frac{2}{3}$. Then $\mathbb{P}(\{C_0 \text{ is } \infty\}) \ge \mathbb{P}(A_k \cap B_k)$. A_k, B_k depend on disjoint sets of bonds (A_k on L_k, B_k on the complement of L_k), so this is $\mathbb{P}(A_k)\mathbb{P}(B_k) = \mathbb{P}(B_k)p^k$. We claim $\mathbb{P}(B_k) > 0$ if k is large enough. The probability a cycle of length $2l$ in $\Lambda^\star$ is open is $(1-p)^{2l}$; the number of cycles in $\Lambda^\star$ of length $2l$ surrounding 0 is $\le l3^{2l-1}$ - the cycle must cross the positive x axis in one of l places, then if we proceed around the cycle there are at most 3 possible ways to go at each step. Therefore, $\mathbb{P}(\overline{B_k}) \le \sum_{l \ge k+2} \mathbb{P}(\exists \text{ a cycle of length } 2l \text{ surrounding } 0) \le \sum_{l\ge k+2} \mathbb{E}(\# \text{ cycles of length } 2l \text{ surrounding } 0) \le \sum_{l \ge k+2} l3^{2l-1}(1-p)^{2l} < \sum_{l \ge k} l(3(1-p))^{2l}$. $p > \frac{2}{3}$ so this is a convergent geometric series, so $\mathbb{P}(\overline{B_k}) < 1$ if k is large enough, i.e. for large k $\mathbb{P}(B_k) > 0$. Thus $p \ge p_H^b(\mathbb{Z}^2)$ and we have the result.

15. Remarks: i) This is Peierl's argument, given in 1936. ii) Write $\lambda_m(\mathbb{Z}^2)$ for the number of paths in $\mathbb{Z}^2$ starting at 0 of length n. Then $\lambda_n^{\frac{1}{n}} \to \lambda = \lambda(\mathbb{Z}^2)$ the connective constant of $\mathbb{Z}^2$. Our proof actually shows $\frac{1}{\lambda} \le p_T^b(\mathbb{Z}^2) \le p_H^b(\mathbb{Z}^2) \le 1 - \frac{1}{\lambda}$. Current results give $2.62 \le \frac{1}{\lambda} \le 2.68$; for a while in the early stages of the subject it was hoped that connective constants would "give us everything", and physicists are still studying them. iii) $\frac{1}{2\lambda - 1} \le p_T^b(\mathbb{Z}^d) \le p_H^b(\mathbb{Z}^d) \le \frac{2}{3}$, this last by considering a $\mathbb{Z}^2$ subset of $\mathbb{Z}^d$.

16. Oriented percolation. We take $\overrightarrow{\Lambda}$ an oriented multigraph, locally finite; $\overrightarrow{C}_x$ is the open out-cluster of x, we define $p_T^b(\overrightarrow{\Lambda}; x), p_H^b(\overrightarrow{\Lambda}; x)$ in the obvious way in relation to this. If $\overrightarrow{\Lambda}$ is homogenous then these are independent of x; if $\overrightarrow{\Lambda}$ is strongly connected (for any $x, y \in \overrightarrow{\Lambda}$, there is a (correctly oriented) path from x to y in Λ), then again p_T, p_H are independent of x.

17. Bond vs Site: Theorem 5: Let Λ be a locally finite, oriented multigraph; $x \in \overrightarrow{\Lambda}$. i) $p_H^b(\overrightarrow{\Lambda}; x) \le p_H^s(\overrightarrow{\Lambda}; x)$, and the same for p_T ii) Suppose Δ_{in}, the max in-degree of Λ, is finite (theorem is actually valid otherwise, but a bit silly). Then $p_H^s(\overrightarrow{\Lambda}; x) \le 1 - (1 - p_H^b(\overrightarrow{\Lambda}; x))^{\Delta_{\text{in}}}$: i) take $0 < p < 1$. It suffices to show $\mathbb{P}_p^s(|C_x| \ge n) \le \mathbb{P}_p^b(|C_x| \ge n) \forall n$, as then by taking the limit as $n \to \infty$ we have the result for p_H, and since $\chi_x^S(\overrightarrow{\Lambda}) = \sum_n \mathbb{P}_p^S(|C_x| \ge n) \ge \sum_n \mathbb{P}_p^b(|C_x| \ge n) = \chi_x^b$, we have the result for p_T. In fact we shall prove a little more, that $\mathbb{P}_p^s(|C_x| \ge n) \le p\mathbb{P}_p^b(|C_x| \ge n)\forall n$, i.e. $\mathbb{P}_p^s(|C_x| \ge n \mid x \text{ is open}) \le \mathbb{P}_p^b(|C_x| \ge n)$. We may wlog treat $\overrightarrow{\Lambda}$ as finite (e.g. by considering $\overrightarrow{\Lambda}_n$, the ball of radius n about x). We explore C_x by considering a random sequence of tripartitions of $V(\overrightarrow{\Lambda})$; first, for site: $T = (T_t) = (R_t, D_t, U_t)_{t=1}^l$ - respectively, the "reached", "dead" and "untested" vertices of $\overrightarrow{\Lambda}$. $R_1 = \{x\}, D_1 = \emptyset, U_1 = V \setminus \{x\}$.

Given a tripartition $T = (R, D, U)$ of V, if there is no $\overrightarrow{ru}$ for $r \in R, u \in U$, set $f(T) = 0$. Otherwise pick a bond $\overrightarrow{ru}$ for $r \in R, u \in U$ and set $f(T) = \overrightarrow{ru}$. We shall use the MTU (Mene Tekel Upharsim) algorithm, an exploration process.

For site: we'll define a random sequence $\mathcal{T} = (T_t)_{t=1}^{l} = (R_t, D_t, U_t)$ of tripartitions of V: $R_1 = \{x\}, D_1 = \emptyset, U_1 = V \setminus \{x\}$. Suppose we've reached $T_t = (R_t, D_t, U_t)$. If $f(T_t) = \emptyset$, end the search; $l = t$. Otherwise if $f(T_t) = \overrightarrow{ru}$, test u; it is open or closed. If u is closed, $R_{t+1} = R_t, D_{t+1} = D_t \cup \{u\}, U_{t+1} = U_t \setminus \{u\}$. If u is open, $R_{t+1} = R_t \cup \{u\}, D_{t+1} = D_t, U_{t+1} = U_t \setminus \{u\}$. Clearly, $R_l = C_X$.

For bond: the condition on x being open is irrelevant. $\mathcal{T}' = (T'_t) = (R'_t, D'_t, U'_t)$; $R'_1 = \{x\}, D'_1 = \emptyset, U'_1 = V \setminus \{x\}$. Suppose we've reached T'_t; if $f(T'_t) = \emptyset$, finish. Otherwise, if $f(T'_t) = \overrightarrow{ru}$ test $\overrightarrow{ru}$; if it is closed set $R'_{t+1} = R'_t, D'_{t+1} = D'_t \cup \{u\}, U'_{t+1} = U'_t \setminus \{u\}$. If $\overrightarrow{ru}$ is open we set $R'_{t+1} = R'_t \cup \{u\}, D'_{t+1} = D'_t, U'_{t+1} = U'_t \setminus \{u\}$. Clearly $R'_l \subset C^b_x$. But R_l, R'_l have the same resolution. So we have part i) of theorem 5.

For part ii), that $p^s \leq 1 - (1 - p^b)^{\Delta_{\text{in}}}$ where Δ_{in} is the maximum in-degree, consider bond percolation on $\overrightarrow{\Lambda}$ with probability p, $0 < p < 1$; declare x open with probability p. Declare $z \neq x$ open if at least one of the bonds into z is open ($\exists \overrightarrow{yz}$ open). $r_z = \mathbb{P}(z \text{ is open}) = 1 - (1-p)^{iu(z)} \leq r = 1 - (1-p)^{\Delta_{\text{in}}}$. We've defined a site percolation measure with probability $\boldsymbol{r} = (r_z)$; $r_z \leq r \forall z$. Hence if $p > p^b_H$ then $r > p^s_H$.

Corollary 6: For Λ a connected, locally finite infinite multigraph, $p^b_H(\Lambda) \leq p^s_H(\Lambda)$, and if $\Delta < \infty$, $p^s_H(\Lambda) \leq 1 - (1 - p^b_H(\Lambda))^{\Delta}$. This is instant from theorem 5 - form a directed graph by replacing each edge in Λ with two edges in opposite directions to form $\overrightarrow{\Lambda}$, and $\Delta_{\text{in}}(\overrightarrow{\Lambda}) = \Delta(\Lambda)$.

18. $\mathbb{Z}^2$ and $\mathbb{Z}^d$: Lemma 7: $\frac{1}{3} \leq p^b_T(\mathbb{Z}^2) \leq \cdots \leq p^s_H(\overrightarrow{\mathbb{Z}^2}) \leq \frac{80}{81}$. We have the first inequality already; clearly $p^b_T(\mathbb{Z}^2) \leq p^b_T(\overrightarrow{\mathbb{Z}^2}) \leq p^b_H(\overrightarrow{\mathbb{Z}^2})$, and this is $\leq p^s_H(\overrightarrow{\mathbb{Z}^2})$ by the previous theorem. We have $\Lambda = \overrightarrow{\mathbb{Z}^2}, \Lambda^\star \cong \mathbb{Z}^2$; consider percolation with probability p. Call a cycle in $\Lambda^\star$, taken anticlockwise (here considering $\Lambda^\star$ to be unoriented), <u>blocking</u> if it surrounds 0 and for any $\overrightarrow{xy} \in \overrightarrow{\mathbb{Z}^2}$, if $\overrightarrow{xy}^\star$ is in the cycle then y is closed. If $\overrightarrow{C}_0$ is the component of $\mathbf{0}$ (in our oriented percolation) then its outer boundary is a blocking cycle. If B is a cycle of length $2l$ in $\Lambda^\star$ (around 0), $\mathbb{P}_p(B \text{ is blocking}) \leq (1-p)^{\frac{l}{2}}$, since l of the edges of B are going up or left, and if we count the vertices to their right and above them (i.e. the vertices which we need to be closed) we count each at most twice (since a vertex can be above one edge of B and to the right of another, but no more), so there are at least $\frac{l}{2}$ distinct vertices which must be closed. Let $L_k = \{(0,0), \ldots, (k,0)\}, A_k$ the event that all the sites in L_k are open, B_k the event that there is no blocking cycle surrounding L_k. $A_k \cap B_k \subset$ the event that $\overrightarrow{C}_0$ is infinite; A_k, B_k depend on disjoint sets of vertices so $\mathbb{P}(A_k \cap B_k) = \mathbb{P}(A_k)\mathbb{P}(B_k) = p^{k+1}\mathbb{P}(B_k)$. We want that this is ¿0. Write Y_l for the number of blocking cycles of length $2l$; $\mathbb{P}(\overline{B_k}) \leq \mathbb{P}(\sum_{l>k} Y_l > 0) \leq \sum_{l>k} \mathbb{P}(Y_l > 0) \leq \sum_{l>k} \mathbb{E}(Y_l) \leq \sum l 3^{2l-1}(1-p)^{\frac{l}{2}} \leq \sum_{l>k} l(81(1-p))^{\frac{l}{2}}$. Hence if $p > \frac{80}{81}$, so that $81(1-p) < 1$, and k is large enough, $\mathbb{P}_p(\overline{B_k}) < 1$ and we have the result.

Let $(e_i)_1^d$ be the standard basis, $k = \lfloor \frac{d-1}{2} \rfloor$; we will consider $\overrightarrow{\mathbb{Z}^{2k}}$. For $\overrightarrow{uv} \in \overrightarrow{\mathbb{Z}^d}$, $v = u + e_i$ for some i; we say v is an x-neighbour of u if $1 \leq i \leq k$

and a y-neighbour if $k+1 \leq i \leq 2k$. Let $\varphi(\boldsymbol{u}) = (\sum_1^k u_i, \sum_{k+1}^{2k} u_j)$. If P is an infinite oriented path in $\overrightarrow{\mathbb{Z}^d}$ starting at $\mathbf{0}$, then $\varphi(P)$ is an (infinite) oriented path in $\overrightarrow{\mathbb{Z}^2}$; however, any naively stated converse is false.

Theorem 8: $\frac{1}{2d-1} \leq p_T^b(\mathbb{Z}^d) \leq p_H^s(\overrightarrow{\mathbb{Z}^d}) = O(\frac{1}{d})$: the first inequality by maximal degree being $2d$, the middle by each of the 3 differences between the two terms increasing the critical probability. For the final part, consider percolation on $\overrightarrow{\mathbb{Z}^d}$ with probability p. Let $k = \lfloor \frac{d-1}{2} \rfloor$, $2k \leq d$; consider $\overrightarrow{\mathbb{Z}^{2k}}$. We'll use the MTU algorithm: consider tripartitions of $\mathbb{Z}^2$ as $(T_t) = (R_t, D_t, U_t)_{t=0}^\infty$ and a sequence of subsets $(\tilde{R}_t)_0^\infty \subset \mathbb{Z}^d, \varphi(\tilde{R}_t) = R_t$. We have $R_t \subset C_0 \subset \mathbb{Z}^2$ [under percolation with some probability] and $\tilde{R}_t \subset \overrightarrow{C}_0$ in $\overrightarrow{\mathbb{Z}^d}$. Condition on the event that $\mathbf{0}$ is open; $R_0 = \{\mathbf{0}\}, D_0 = \emptyset, U_0 = \mathbb{Z}^2 \setminus \{0\}; \tilde{R}_0 = \{\mathbf{0}\}$. Update as follows: given $(R_t, D_t, U_t), \tilde{R}_t$, we ask: is there an oriented bond from R_t to U_t? If so, pick one, say $\overrightarrow{ru}$. Suppose $\overrightarrow{ru}$ goes in the x direction; pick $\tilde{r} \in \tilde{R}_t$ with $\varphi(\tilde{r}) = r$. Does $\tilde{r}$ has an open x-neighbour? If not, $R_{t+1} = R_t, D_{t+1} = D_t \cup \{u\}, U_{t+1} = U_t \setminus \{u\}, \tilde{R}_{t+1} = \tilde{R}_t$. If there is such an open site, [pick one] $\tilde{u}$, then $R_{t+1} = R_t \cup \{u\}, D_{t+1} = D_t, U_{t+1} = U_t \setminus \{u\}, \tilde{R}_{t+1} = \tilde{R}_t \cup \{\tilde{u}\}$. [If $\overrightarrow{ru}$ goes in the y direction, similar]. If there is no $\overrightarrow{ru}$, finish.

If $|\bigcup R_t| = \infty$ then $|\bigcup \tilde{R}_t| = \infty$. The exploration process (R_t, D_t, U_t) is just an exploration process of $\overrightarrow{C}_0$ in $\to Z^2$; $\bigcup R_t = C_0$. In $\to Z^2$, each site is taken (added to R_t) with probability $1-(1-p)^k$. Hence, if $1-(1-p)^k > \frac{80}{81}$, $\mathbb{P}(|\bigcup R_t| = \infty) > 0$; in this case, we have percolation on $\overrightarrow{\mathbb{Z}^d}$. Thus if $1-(1-p)^k > \frac{80}{81}$ (in fact if $1-(1-p)^k > p_T^s(\overrightarrow{\mathbb{Z}^2})$) then $p \geq p_H^s(\overrightarrow{\mathbb{Z}^d})$. For this to occur it suffices that $(1-p)^k < \frac{1}{81}$; since $(1-p)^k < e^{-kp}$ it suffices that $e^{-kp} < \frac{1}{81}$, i.e. that $pk > \log 81$, i.e. $p > \frac{\log 81}{k}$. So it suffices that $p > \frac{2\log 81}{d-1}$; in particular $p_H^s(\mathbb{Z}^d) < \frac{10}{d}$ for d sufficiently large.

Probabilistic Tools

We have already seen Kolmogrov's 0-1 law.

Lemma 1 (Tekete's Lemma): Let $(a_n)_1^\infty$ be a non-negative sequence of reals which is subadditive $a_{n+m} \leq a_n + a_m$. Then $\lim \frac{a_n}{n}$ exists (and is $< \infty$; if we relax the conditions and allow the a_i to be negative, the limit still exists but may be $-\infty$): let $a = \underline{\lim} \frac{a_n}{n}$ (or just $\inf \frac{a_n}{n}$), then $\forall \epsilon > 0 \exists k$ such that $\frac{a_k}{k} < a + \epsilon$. Let $c = \max_{1 \leq i \leq k-1} a_i$. Then for n, write $n = kq + r, 0 \leq r \leq k-1$, and $a_n \leq q a_k + c$. So $\frac{a_n}{n} \leq \frac{ak}{k} + \frac{c}{n} \leq a + \epsilon + \frac{c}{n}$. Therefore $\overline{\lim}\, a_n \leq a + \epsilon$; this is true $\forall \epsilon > 0$. so we are done.

We shall be working in the weighted cube $Q_{\boldsymbol{p}}^n$; $Q^n \cong \{0,1\}^{[n]} \cong \mathcal{P}([n])$: for $A \subset [n]$, $A \leftrightarrow \chi_A =$ a binary sequence e.g. $(0,1,\ldots)$. $\boldsymbol{p} = (p_i)_1^n$; for $A \subset Q^n$, $\mathbb{P}_{\boldsymbol{p}}(A) = \sum_{a \in A} \prod_{a_i=1} p_i \prod_{a_i=0} (1-p_i)$. If $p_i = p \forall i$ we write Q_p^n.

Although "officially" we are interested in infinite graphs, in practice knowing about finite subgraphs will tell us everything - e.g. for $\mathbb{Z}^2$ we only really need to know what happens in finite rectangles. If such a rectangle contains N bonds, the probability space relevant to bond percolation is Q_p^N.

An event or property $A \subset Q_p^n$ is monotone increasing or an up-set if whenever $\boldsymbol{a} = (a_i) \in A, \boldsymbol{b} = (b_i) \in Q^n$ and $\boldsymbol{a} \leq \boldsymbol{b}$ (i.e. $a_i \leq b_i \forall i$) then $\boldsymbol{b} \in A$; the obvious analagous definition exists for a monotone decreasing event or down-set.

Lemma 2 (Harris' Lemma): If A, B are up-sets in Q_p^n then $\mathbb{P}_{\boldsymbol{p}}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$ $(\star)$; if both are down-sets, we have the same result while if one is up and the other down, $\mathbb{P}(A \cap B) \leq \mathbb{P}(A)\mathbb{P}(B)$: suppose we have both up or both down. We shall prove $(\star)$ by induction on n; the base case $n = 1$ is trivial or $n = 0$ even more so. Let $A_0 = \{\boldsymbol{a} \in Q^{n-1} : (a_1, \ldots, a_{n-1}, 0) \in A\}, A_1 = \{\boldsymbol{a} \in Q^{n-1} : (a_1, \ldots, a_{n-1}, 1) \in A\}$; we have $\mathbb{P}_{\boldsymbol{p}}(A) = (1 - p_n)\mathbb{P}_{\boldsymbol{p'}}(A_0) + p_n\mathbb{P}_{\boldsymbol{p'}}(A_1)$; similarly for B. $\mathbb{P}(A \cap B) = (1 - p_n)\mathbb{P}_{\boldsymbol{p'}}(A_0 \cap B_0) + p_n\mathbb{P}_{\boldsymbol{p'}}(A_1 \cap B_1)$, which by induction is $\geq (1 - p_n)\mathbb{P}_{\boldsymbol{p'}}(A_0)\mathbb{P}(_{\boldsymbol{p'}}(B_0) + p_n\mathbb{P}(A_1)\mathbb{P}(B_1)$. We want that this is $\geq ((1 - p_n)\mathbb{P}(A_0) + p_n\mathbb{P}(A_1))((1 - p_n)\mathbb{P}(B_0) + p_n\mathbb{P}(B_1))$; subtracting these gives $p(1-p)\mathbb{P}(A_0)\mathbb{P}(B_0) + p(1-p)\mathbb{P}(A_1)\mathbb{P}(B_1) - p(1-p)\mathbb{P}(A_0)\mathbb{P}(B_1) - p(1-p)\mathbb{P}(A_1)\mathbb{P}(B_0) = p(1-p)(\mathbb{P}(A_0) - \mathbb{P}(A_1))(\mathbb{P}(B_0) - \mathbb{P}(B_1)) \geq 0$ as required. For A up and B down, $\mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}(A \cap B^c) \leq \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B^c) = \mathbb{P}(A) - \mathbb{P}(A)(1 - \mathbb{P}(B)) = \mathbb{P}(A)\mathbb{P}(B)$ [other cases similar].

For $\mathcal{A}, \mathcal{B}$ up-sets in $\mathcal{P}(n)$, define $\mathcal{A}\Box\mathcal{B} = \{C \subset [n] : \exists A \in \mathcal{A}, B \in \mathcal{B} : A \cap B = \emptyset, C \supset A \cup B\}$, i.e. the set of elements of Q^n which have disjoint "certificates" for belonging to $\mathcal{A}$ and to $\mathcal{B}$. E.g. if A is the set of elements of Q^n containing four consecutive 1s and B the set of elements of Q^n containing 3 1s where there are at least 2 elements in between the first and second and the second and third, these are both up-sets, and $A\Box B$ is the set of elements of Q_n with 3 1s separated by 2 elements in between, and disjoint from these, four consecutive 1s. Clearly we always have $A\Box B \subset A \cap B$.

Theorem 3 (van den Berg - Kesten): If A, B are up-sets, $\mathbb{P}(A\Box B) \leq \mathbb{P}(A)\mathbb{P}(B)$; induct on n, the $n = 1$ case being trivial. Define A_0, A_1, B_0, B_1 as before; $C_0 = A_0\Box B_0 \subset (A_0\Box B_1) \cap (A_1\Box B_0), C_1 = A_0\Box B_1 \cup A_1\Box B_0 \subset A_1\Box B_1$. $\mathbb{P}(C_0) + \mathbb{P}(C_1) \leq \mathbb{P}((A_0\Box B_1) \cap (A_1\Box B_0)) + \mathbb{P}((A_0\Box B_1) \cup (A_1\Box B_0)) = \mathbb{P}(A_0\Box B_1) + \mathbb{P}(A_1\Box B_0)) \leq \mathbb{P}(A_0)\mathbb{P}(B_1) + \mathbb{P}(A_1)\mathbb{P}(B_0)$. Multiplying this by $p_n(1 - p_n)$ and adding $(1 - p_n)^2\times\ \mathbb{P}(C_0) \leq \mathbb{P}(A_0)\mathbb{P}(B_0)$ and $p_n^2\times\ \mathbb{P}(C_1) \leq \mathbb{P}(A_1)\mathbb{P}(B_1)$ (both these inequalities being true by the induction hypothesis), $(1-p)\mathbb{P}(C_0) + p\mathbb{P}(C_1) \leq ((1-p)\mathbb{P}(A_0) + p\mathbb{P}(A_1))((1-p)\mathbb{P}(B_0) + p\mathbb{P}(B_1))$ i.e. $\mathbb{P}(C) \leq \mathbb{P}(A)\mathbb{P}(B)$ as required.

1. Suppose $A_1, A_2 \subset Q_{\boldsymbol{p}}^n, \mathbb{P}(A_1 \cup A_2) \geq 1 - \frac{1}{100} \Rightarrow \max_i \mathbb{P}(A_i) \geq \frac{1-\frac{1}{100}}{2}$. If the A_i are increasing, then this becomes $\geq 1 - \frac{1}{10}$: suppose A_i are increasing so $\overline{A_i}$ are decreasing. $\mathbb{P}(A_1 \cup A_2) \geq 1 - \epsilon \Rightarrow \mathbb{P}(\overline{A_1} \cap \overline{A_2}) \leq \epsilon \therefore \mathbb{P}(\overline{A_1})\mathbb{P}(\overline{A_2}) \leq \epsilon \therefore \min \mathbb{P}(\overline{A_i}) \leq \sqrt{\epsilon} \therefore \max \mathbb{P}(A_i) \geq 1 - \sqrt{\epsilon}$. Slightly more generally, for $\mathbb{P}(\bigcup_{i=1}^n A_i) \geq 1 - \epsilon$, A_i increasing, $\max_i \mathbb{P}(A_i) \geq 1 - \epsilon^{\frac{1}{n}}$.

2. Consider $G_{n,p}$, and A the event that there is a cycle of length $\frac{n}{2}$, B the event that there exists a Hamiltonian cycle; both these are increasing, so we have $\mathbb{P}(A\Box B) \leq \mathbb{P}(A)\mathbb{P}(B) \leq \mathbb{P}(A \cap B)$. ($A\Box B$ is of course the event that there are, edge-disjoint, a Hamiltonian cycle and a cycle of length $\frac{n}{2}$).

3. $\mathbb{P}_p(A)$, for A increasing, as a function of p, will be an increasing function; its graph will always have positive gradient. E.g. in the simplest possible case, $A = \{S \subset [n] : 1 \in S\}$, $\mathbb{P}_p(A) = p$ and the graph is simply a straight line of gradient 1. But in fact in all "interesting" cases, the graph has a "sharp threshold"; it is close to 0 until p gets close to some threshold probability, then "shoots up" to close to 1. We aim to show that this occurs by showing that (for A always assumed increasing) if $\mathbb{P}_p(A)$ is neither very small nor very large, then $\frac{d}{dp}\mathbb{P}_p(A)$ is large.

4. The influence of a RV: For $A \subset Q_{\boldsymbol{p}}^n, \omega \in Q_{\boldsymbol{p}}^n$, ω_i is <u>pivotal</u> at ω for A if precisely one of $(\omega_1, \ldots, \omega_{i-1}, 0, \omega_{i+1}, \ldots, \omega_n), (\omega_1, \ldots, \omega_{i-1}, 1, \omega_{i+1}, \ldots, \omega_n)$

belongs to A. Wlog consider ω_n rather than ω_i in the following: set $A_i = \{\omega' = (\omega_j)_{j=1}^{n-1} \in Q_{\boldsymbol{p'}}^{n-1} : (\omega_1, \dots, \omega_{n-1}, i) \in A\}$ where $\boldsymbol{p'} = (p_1, \dots, p_{n-1})$, for $i = 0, 1$. Set $A_+ = A_1 \setminus A_0, A_- = A_0 \setminus A_1, A_b = A_0 \cap A_1$. We have $\mathbb{P}_{\boldsymbol{p}}(A) = p_n \mathbb{P}_{\boldsymbol{p'}}(A_+) + (1-p_n)\mathbb{P}_{\boldsymbol{p'}}(A_-) + \mathbb{P}_{\boldsymbol{p'}}(A_b)$. The (signed) influence of ω_n on A is defined by $\beta_n(A) := \mathbb{P}_{\boldsymbol{p'}}(A_+) - \mathbb{P}_{\boldsymbol{p'}}(A_-)$; the (absolute) influence of ω_n is $\overline{\beta_n}(A) := \mathbb{P}_{\boldsymbol{p'}}(A_+) + \mathbb{P}_{\boldsymbol{p'}}(A_-)$; note that if A is increasing these are the same (since A_- is empty).

Margulis (and Russo)

Lemma 4: $\frac{\partial}{\partial p_n}\mathbb{P}_{\boldsymbol{p}}(A) = \beta_n(A)$; in particular, if $A \subset \mathbb{Q}_{\boldsymbol{p}}^n$ and $\boldsymbol{p} = (p, \dots, p)$ then $\frac{\partial}{\partial p}\mathbb{P}_p(A) = \sum_{i=1}^n \beta_i(A)$ (this follows by the chain rule): $\frac{\partial}{\partial p_n}\mathbb{P}_{\boldsymbol{p}}(A) = \frac{\partial}{\partial p_n}(p_n \mathbb{P}_{\boldsymbol{p'}}(A_+) + (1-p_n)\mathbb{P}_{\boldsymbol{p'}}(A) + \mathbb{P}_{\boldsymbol{p'}}(A_b)) = \mathbb{P}_{\boldsymbol{p'}}(A_+) - \mathbb{P}_{\boldsymbol{p'}}(A_-)$.

Consider $A \subset Q^n$ $(= Q_{\frac{1}{2}}^n)$; $\overline{\beta_n}(A) = \frac{|\{\text{edge boundary of } A \text{ in direction } n\}|}{2^{n-1}}$ so $\sum_{i=1}^n \overline{\beta_i}(A) = \frac{|\partial_e(A)|}{2^{n-1}}$. If $|A| = 2^{n-1}$, $|\partial_e(A)| \geq 2^{n-1}$; more generally if $|A| = 2^x$, $|\partial_e(A)| \geq 2^x(n-x)$. If $|A| = t2^n$ (i.e. $\mathbb{P}_{\frac{1}{2}}(A) = t$) then $\sum_{i=1}^n \overline{\beta_i}(A) \geq t2^n \frac{n - \log(t2^n)}{2^{n-1}} = 2t \log \frac{1}{t}$. This is "already looking good", since this quantity is nonnegligible precisely for t not near 0 or 1, but we can improve it a lot. If $\mathbb{P}_{\frac{1}{2}}(A) = \frac{1}{2}$, the extremal example is a half-cube, for which $\sum_1^n \overline{\beta_i}(A) = \overline{\beta_n}(A)$. This suggests we can do better if the β_i are close to each other:

Ben-Or and Linial

Theorem 5 (Kahm, Kalai and Linial '88): $\sum_1^n \overline{\beta_i}(A)^2 \geq ct^2(1-t)^2 \frac{(\log n)^2}{n}$, where $t = \mathbb{P}_{\frac{1}{2}}(A)$ and c is an absolute constant. Bourgain, Kahn, Kalai, Katznelson and Linial found a similar result for the solid cube $[0,1]^n$.

Theorem 6: For every $p > 0$, there is an absolute constant c such that $\max_i \overline{\beta_i}(A) \geq ct(1-t)\frac{\log n}{n}$.

$A \subset Q^n$ is called symmetric if $\forall i, j \in [n] \exists$ a permutation $\pi : [n] \to [n]$ such that $\pi(i) = j$ and $\omega \in A \Rightarrow \omega_\pi := (\omega_{\pi(1)}, \dots, \omega_{\pi(n)}) \in A$.

E.g. $[n] \times [n]$, a grid, and consider bond percolation with probability p, A the event that the resulting graph is connected; this is not symmetric. However, if we identify opposite sides to form a torus, it becomes symmetric, and indeed is symmetric for any other event e.g. $A = \exists$ a long path.

Maximal influence: For $A \subset Q_{\frac{1}{2}}^n$, II implies $\max \overline{\beta_i}(A) \geq ct\frac{\log \frac{1}{t}}{n}$, where $t = \mathbb{P}_{\frac{1}{2}}(A)$; in fact we may find $c = 2$. BKKKL implies for $A \subset Q_p^n$, $\max \overline{\beta_i}(A) \geq ct(1-t)\frac{\log n}{n}$.

Thresholds (increase of $\mathbb{P}_p(A)$: Suppose $A \subset Q_p^n$ is a symmetric up-set; using the Margulis-Russo lemma (which we haven't the time to prove) we get:

Theorem 7: Suppose $A \subset Q_p^n$ is a symmetric up-set, $\mathbb{P}_p(A) > \epsilon > 0$, $\epsilon < \frac{1}{2}$. Then $\mathbb{P}_q(A) \geq 1 - \epsilon$ provided $q - p \geq c\frac{\log \frac{1}{2\epsilon}}{\log n}$. This is a very good result, but says little if p is very small; fortunately for that case we have:

Theorem 8: The same result holds provided $q - p \geq cp \log \frac{1}{p} \frac{\log \frac{1}{2\epsilon}}{\log n}$.

Note that these results do not hold for A almost given by a "junta", e.g. $A = \{(\omega_i)_1^n : \omega_1 + \omega_2 + \omega_3 \geq 2\}$, which has $\mathbb{P}_{\frac{1}{2}}(A) = \frac{1}{2}$ [this A is very much not symmetric].

Bond percolation on $\mathbb{Z}^2$, the square lattice

Our aim in this section is to proove the celebrated results of Haris and Kosten, which give values for $p_T^b(\mathbb{Z}^2)$ and $p_H^b(\mathbb{Z}^2)$.

1. Crossing rectangles: a rectangle $R = [m] \times [n]$ has mn sites and $2mn - m - n$ bonds. For $\Lambda = \mathbb{Z}^2$, we can simply say $\Lambda^\star = \mathbb{Z}^2$, but for a rectangle it is not clear how we should treat the boundary, so we make some more definitions: the horizontal dual R^h is an $(m-1) \times (n+1)$ rectangle, having vertices inside each face of R and extra rows along the top and bottom, but no extra columns along the edges. For each horizontal bond f of R, $f^\star$ is a vertical bond of R^h.

The vertical dual of R, R^v, is $(m+1) \times (n-1)$; for $m, n \geq 2$, $(R^h)^v = R = (R^v)^h$; if R is $(n+1) \times n$ then we consider $R^\star$ to be R^h, $n \times (n+1)$. Let $H(R)$ be the event that there is an open crossing of R from left to right; $V(R)$ the same for top to bottom.

Consider bond percolation with probability p on $\Lambda = \mathbb{Z}^2$; this corresponds to bond percolation with probability $1-p$ on $\Lambda^\star = \mathbb{Z}$, by: for any configuration ω on Λ, $\omega^\star$ is the configuration on $\Lambda^\star$ such that f is open iff $f^\star$ is closed.

Lemma 1: For R an $m \times n$ rectangle, $m, n \geq 2$, for every configuration ω on R exactly one of $\omega \in H(R), \omega^\star \in V(R^h)$ holds: draw R. Draw a square in the centre of each bond of R, i.e. around the intersection of each bond f with its corresponding $f^\star$. Then connect the corners of these squares with diagonal lines, so that we now have a tiling by squares and octagons, with an octagon around each vertex of R and each vertex of R^h.

Colour the octagons black if they correspond to vertices of R, white for vertices of R^h, and colour the squares according to our percolation: black if f is open, white if $f^\star$ is open (and the squares on the left and right edges for which there is no $f^\star$ always black). Consider the lines separating black and white regions to form the interface graph $I(w)$; orient every edge therof such that the black region is on the right and the white on the left.

For every vertex v in the interior of the tiling, there is exactly one bond going into v and one leaving it; the only vertices for which this does not hold are the four corners. At the top left we have a bond going into the graph, at the top right one coming out, at the bottom left coming out and at the bottom right going in. Thus $I(\omega)$ is some oriented cycles and two oriented paths. Each of these paths gives either an open horizontal crossing of R or an open vertical crossing of R^h; in fact we obtain either the topmost and bottommost horizontal crossings in R or the leftmost and rightmost vertical crossings in R^h. (Note that this proof shows e.g. the leftmost vertical crossing depends only on bonds in it and to its left (since we can find it via a "hand-on-left-wall method" in the coloured tiling); thus it will be independent of an event defined depending only on the bonds to its right). We cannot have both events as if so, we have two vertices top and bottom joined to each other, and two left and right joined to each other, and neither path crossing each other or the outer square; then add a vertex outside the square and join it to the four previously mentioned ones, and we have a planar drawing of K_5.

Corollary 2: i) For $R = [m+1] \times [n]$ a rectangle, $\forall p$, $\mathbb{P}_p(H(R)) + \mathbb{P}_{1-p}(V(R^h)) = 1$ ii) IF $R = [n+1] \times [n]$ then $\mathbb{P}_{\frac{1}{2}}(H(R)) = \frac{1}{2}$ iii) If S is a square, $\mathbb{P}_{\frac{1}{2}}(H(S)) > \frac{1}{2}$: i) If ω is distributed as a percolation with probability p then $\omega^\star$ has the distribution of a percolation with probability $1-p$; $\{\omega \in H(R)\}$ and $\{\omega^\star \in V(R^h)\}$ partition [the state space] Ω. ii) For $R^h n \times (n+1)$, clearly $\mathbb{P}_{\frac{1}{2}}(H(R)) = \mathbb{P}_{\frac{1}{2}}(V(R^h))$; combine this with the previous result. iii) When we enlarge S from an $n \times n$ square to an $(n+1) \times n$ rectangle, the probability becomes $\frac{1}{2}$ by the previous result.

Write $h_p(m,n) := \mathbb{P}_p(H(R))$ where R is an $m \times n$ rectangle, and $h(m,n) = h_{\frac{1}{2}}(m,n)$. We want to show that for $p > \frac{1}{2}$, the probability of a horizontal crossing of a rectangle of any "aspect ratio" is large, provided n is large enough; we shall get their slowly.

Lemma 3: Let $R = [m] \times [2n]$, S the $[r] \times n]$ rectangle in the "bottom left" corner of R, $X(R,S)$ the event that there are open paths P_1, P_2 such that P_1 is a vertical crossing of S and P_2 joins P_1 to the RHS of R. Then $\mathbb{P}_p(X(R,S)) \geq \frac{1}{2}\mathbb{P}_p(H(R))\mathbb{P}_p(V(S))$.

The proof is conceptually simple: consider P_1 and its reflection P_1'; any P_2 which is a horizontal crossing of R must hit one of these. But we need to be able to use Harris' lemma, and P_1' doesn't form an up-set, so we shall need some technical manouvers. Write $LV(S) = P_1$ for P_1 the leftmost vertical crossing of S, $\emptyset$ if there is no such crossing. Let P_1' be the reflection of P_1, $\tilde{P}_1 = P_1 \cup P_1' \cup$ the single bond in the middle joining them, $R(P_1) =$ the part of R to the right of $\tilde{P}_1$. Let $Y(P_1)$ be the event that $R(P_1)$ has an open path from $\tilde{P}_1$ to the RHS of R, $Z(P_1)$ be that that $R(P_1)$ has an open path from P_1 to the RHS of R. We clearly have $\mathbb{P}(Z(P_1)) \geq \frac{1}{2}\mathbb{P}(Y(P_1)) \geq \frac{1}{2}\mathbb{P}(H(R))$. The event $LV(S) = P_1$ is independent of all the bonds of $R(P_1)$; in particular it is independent of $Z(P_1)$. $\{LV(S) = P_1\} \cap Z(P_1) \subset X(R,S)$; thus $\bigcup_{P_1}\{LV(S) = P_1\} \cap Z(P_1) \subset X(R,S) \therefore \mathbb{P}(X(R,S)) \geq \sum_{P_1} \mathbb{P}(Z(P_1) \mid LV(S) = P_1)\mathbb{P}(LV(S)) = P_1 \geq \sum_{P_1} \frac{1}{2}\mathbb{P}(H(R))\mathbb{P}(LV(S) = P_1) = \frac{1}{2}\mathbb{P}(H(R))\mathbb{P}(V(S))$, as required.

Corollary 4: $h(m_1 + m_2 - r, 2n) \geq \frac{1}{4}h(m_1, 2n)h(m_2, 2n)h(r,n)^2 h(n,r)$, since if we consider a $m_1 + m_2 - r \times 2n$ rectangle as being an $m_1 \times 2n$ rectangle overlapping an $m_2 \times 2n$ rectangle, with the overlap being two $r \times n$ rectangles, say one of them is S, if we have an open vertical crossing of S and an open "horizontal" path from this crossing to the LHS of the left rectangle, another open vertical crossing of S and an open "horizontal" path from this to the RHS of the right rectangle, and an open horizontal crossing of S, then by their powers combined we have an open horizontal crossing of the entire rectangle.

For example, applying this $m_1 = m_2 = 2n+1, r = n-1$ gives $h(3n+3, 2n) \geq 2^{-7}$; feeding this result back in we have $h(5n+7, 2n) \geq 2^{-19}$ and similarly $h(6n+9, 2n) \geq 2^{-25}$, which we shall use.

Theorem 4 (Harris, 1960): For bond percolation on $\mathbb{Z}^2$, $\vartheta(\frac{1}{2} = 0$: write $r(C_0)$ for the l_∞-radius of C_0. We claim $\mathbb{P}_{\frac{1}{2}}(r(C_0) \geq n) \leq n^{-c}$ $(\star)$, where $c > 0$ is an absolute constant: consider $\Lambda = \mathbb{Z}^2, \Lambda^\star \cong \mathbb{Z}^2$, and probability $\frac{1}{2}$; couple $f \leftrightarrow f^\star$ in the usual fashion. Consider a square annulus in $\Lambda^\star$; if we have an open path going around it, we are done. Consider the four "long" versions of the "side" rectangles; if we have lengthwise open crossings of each, they join to give a cycle as required - and we can use Harris freely for this.

More formally, let A_k be the square annulus in $\Lambda^\star$ with centre $(\frac{1}{2}, \frac{1}{2})$, inner radius 4^k and outer radius 3×4^k. Let E_k be the event that there is an open cycle in A_k surrounding 0; A_k is made up of four $(3 \times 4^k + 1) \times (4^k + 1)$ rectangles (overlapping), and the probability that there is a crossing of such a rectangle "the long way" is, as we saw, $\geq 2^{-25}$, so $\mathbb{P}(E_k) \geq 2^{-100}$ by Harris.

Also $E_k \subset \{r(C_0) \leq 3 \times 4^k < 4^{k+1}\} \therefore \mathbb{P}(r(C_0) \geq n) \leq (1-\epsilon)^l \leq e^{-\epsilon l}$, if $4^{l+1} \leq n$. Indeed, $(E_k)_{k=1}^{\infty}$ are independent since the annuli are disjoint; $\mathbb{P}(|C_0| = \infty) \leq \mathbb{P}(r(C_0) \geq n) \forall n \therefore \mathbb{P}(|C_0| = \infty) = 0$.

We showed that $\mathbb{P}_{\frac{1}{2}}(r(C_0) \geq n) \leq n^{-C}$; we have also that it is $\geq \frac{1}{2n}$ by considering an $n \times n$ square: the probability that it has an open crossing is $\frac{1}{2}$, so there is at least one vertex on the left hand edge for which there is an open crossing starting at that vertex with probability $\geq \frac{1}{2n}$; position the square such that that vertex is the origin, then done.

Thus $0 < c \leq \liminf \frac{-\log \mathbb{P}_{\frac{1}{2}}(r(C_0) \geq n)}{\log n} \leq \limsup \frac{-\log \mathbb{P}_{\frac{1}{2}}(r(C_0) \geq n)}{\log n} \leq 1$. Clearly the limit exists, but proving this would guarantee the reader a fellowship at their college of choice. An even simpler open question, is to prove that e.g. $h(10n, n) \to$ some h_n.

Harris tells us that $p_H^b(\mathbb{Z}^2) \geq \frac{1}{2}$.

2. A sharp transition. We saw $h_{\frac{1}{2}}(4n, n) \geq c_4 > 0$. We aim to show that for $p > \frac{1}{2}$, $h_p(\rho n, n) \to 1$ as $n \to \infty$, for any fixed $\rho > 1$ e.g. $\rho = 100$.

Lemma 5: Let $p > \frac{1}{2}$ and $\rho \geq 1$. Then $\exists \gamma = \gamma(p) > 0, n_0 = n_0(p, \rho)$ such that if $n \geq n_0$ then $h_p(\rho_n, n) \geq 1 - n^{-\gamma}$: our main weapon here is that if $A \subset Q_p^N$ is a symmetric upset with $\mathbb{P}_p(A) \geq \epsilon$ ($< \frac{1}{2}$) then $\mathbb{P}_q(A) \geq 1 - \epsilon$ provided $q - p \geq c_0 \frac{\log \frac{1}{2\epsilon}}{\log N}$. It suffices to show the result for $h_p(3n, 2n)$, since everything remains exponential if we "glue" multiple rectangles to make larger ones (we shall see this more formally later).

Define $\mathbb{T}_{5n}$ the $5n \times 5n$ torus, with $25n^2$ sites and $50n^2$ bonds. Let A be the event that there is a $4n \times n$ or $n \times 4n$ rectangle in $\mathbb{T}_{5n}$ with a crossing "the long way"; this is clearly a symmetric upset in Q_p^N, where $N = 50n^2$. $\mathbb{P}_{\frac{1}{2}}(A) \geq c_4 > 0$ (by just considering some particularl fixed $4n \times n$ rectangle). Let $\delta = \frac{p - \frac{1}{2}}{25C_0}, \epsilon = n^{-50\delta}$ (thus $\delta = \frac{\log \frac{1}{\epsilon}}{50 \log n}$. $p - \frac{1}{2} = 25C_0 \frac{\log \frac{1}{\epsilon}}{50 \log n}$. Take n large enough to have $\epsilon < c_4$, then $\mathbb{P}_{\frac{1}{2}}(A) > \epsilon$. Then $\mathbb{P}_p(A) \geq 1 - \epsilon = 1 - n^{-50\delta}$.

Let $R_1, \ldots, R_{50}$ be the canonical $3n \times 2n$ and $2n \times 3n$ rectangles in $\mathbb{T}_{25n^2}$; the bottom-left vertices are (in, jn). Let F_i be the event that R_i is crossed "the long way". Then $A \subset \bigcup_1^{50} F_i$; each F_i is an upset so $\exists F_i$ such taht $\mathbb{P}_p(F_i) \geq 1 - \epsilon^{\frac{1}{50}} = 1 - n^{-\delta}$.

For general p, e.g. taking $\gamma = \frac{\delta}{2}$ works: $1 - \mathbb{P}(\text{open crossing}) \leq (2\rho - 5)n^{-\delta} < n^{-\frac{\delta}{2}}$ for ρ fixed and n large enough, and we can cross a $\rho n \times 2n$ rectangle using $2\rho - 5$ crossings of $3n \times 2n$ rectangles - divide it into $n \times n$ blocks, then consider horizontal crossings of each first $3n \times 2n$ rectangle and vertical crossings of each $2n \times 2n$ square (other than those at the end); these will combine to give a horizontal crossing of the entire rectangle.

Corollary 6: $\forall \rho \geq 1, p > \frac{1}{2}, h_p(\rho n, n) \to 1$.

Theorem 7 (Kesten, 1980): For $p > \frac{1}{2}$, $\vartheta(p) > 0$ (i.e. $\mathbb{P}_p(E_\infty) = 1$); $p_H^b(\mathbb{Z}^2) \leq \frac{1}{2}$): Let R_k be $[2^k n] \times [2^{k+1} n]$ if k is even, $[2^{k+1} n] \times [2^k n]$ for k odd; place all the rectangles with their bottom left corner at the origin (then a vertical crossing of R_1 and a horizontal crossing of R_2 must meet, and this must meet a vertical crossing of R_3, and so on). Let E_k be the event that R_k has an open crossing the long way; $\bigcap_{k=0}^{\infty} R_k \subset E_\infty$ [yes, technically false, but only in a sense that doesn't matter]. $\mathbb{P}(\bigcap R_k) = 1 - \mathbb{P}(\bigcup R_k^c) \geq 1 - \sum \mathbb{P}(R_k^c) \geq 1 - \sum (2^k n)^{-\gamma}$, which is > 0 if n is sufficiently large.

4. Exponential decay: Usually (as we have just seen), above the "threshold"

probability, the probability of an event tents to 1 polynomially; however, usually below the threshold the probability tents to 0 exponentially.

Consider site percolation on Λ; our state space is $\Omega = \{0,1\}^{V(\Lambda)}$. A cylindrical set is a subset $E_F \subset \Omega$ depending only on the states of the sites in some finite set F. Write $\mathcal{C}$ for the algebra of cylindrical sets; we define a site percolation measure on Λ to be the completion of a finitely additive probability measure on $\mathcal{C}$.

A SPM $\mathbb{P}$ on Λ is called k-independent if any two cylindrical events $E_F, E_{F'}$ with $d(F, F') \geq k$ the events are independent (i.e. for finite sets $U, W \subset V(\Lambda)$ at distance $\geq k$, the states in U, W are independent); for $k = 1$ this of course becomes independence in the usual sense, for $k \geq 2$ we "get funnier measures"

Examples: 1. Consider $\mathbb{P}$ an (independent) bond percolation measure on Λ; define $\tilde{\mathbb{P}}$ on $V(\Lambda)$ by: a site $v \in V(\Lambda)$ is open if there is an open path (in $\mathbb{P}$) of length 3 starting at v. This measure $\tilde{\mathbb{P}}$ is 6-independent. 2. Pick 3×3 squares independently with probability p and declare a site open if it is in one of these squares; this is 5-independent.

Lemma 8: Let $\Delta \geq 2$. Then $\exists a = a(\Delta, k) > 0$ and $p_1(\Delta, k)$ such that if $\Delta(\Lambda) \leq \Delta$ and $\mathbb{P}$ is a k-independent site percolation measure on $V(\Lambda)$ with $\mathbb{P}(v \text{ open}) \leq p_1(\Delta, k) \forall v \in V(\Lambda)$ then $\tilde{\mathbb{P}}(|C_v| \geq n) \leq e^{-an} \forall n \geq 1, v \in V(\Lambda)$: Fix $p > 0, v \in V(\Lambda)$. 1) The number of n-sets $U \subset V(\Lambda)$ containing v such that $\Lambda[U]$ is connected is $\leq (e\Delta)^{n-1}$ (exercise; we can also obtain a slightly stronger result using trees rather than connected graphs). 2) Write $b(r, \Delta) = 1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{r-1}$; $\forall w \in V(\Lambda)$ there are at most $b(r, \Delta)$ vertices within distance r of w. $\forall U \subset V(\Lambda), |U| = n \exists W \subset U, |W| \geq \frac{n}{b(k-1,\Delta)}$ such that $d(w, w') \geq k \forall w, w' \in W, w \neq w'$ (by just picking vertices greedily). Hence, given $U \subset V(\Lambda), |U| = n$, the probability that every site in $\Lambda_p[U]$ is open is $\leq p^{\frac{n}{b(k-1,\Delta)}}$. 3) $\mathbb{P}(|C_v| \geq n) \leq (e\Delta)^{n-1} p^{\frac{n}{b(k-1,\Delta)}} < (e\Delta p^{\frac{1}{b(n-1,\Delta)}})^n$; hence choosing $p_1 = p$ such that $e^{-a} = e\delta p^{\frac{1}{b(k-1,\Delta)}} < 1$ and this value of a we have the result.

Theorem 9: Let $p < \frac{1}{2}$, then $\exists a(p) > 0$ such that in bond percolation on $\mathbb{Z}^2$ with probability p, $\mathbb{P}(|C_0| \geq n) \leq e^{-an} \forall n \geq 2$: $\mathbb{P}$ is an independent bond percolation on $\Lambda = \mathbb{Z}^2$. We'll define $\tilde{\mathbb{P}}$ a 5-independent site percolation on $\mathbb{Z}^2$ such that the probability of any site being open is small and large open clusters in $\mathbb{P}$ correspond to lange open clusters in $\tilde{\mathbb{P}}$ (This is an important general technique, but one which must be used with care; if applied crudely it gives terrible bounds). Let $s = m + 1$, to be defined later; consider a tiling of $\mathbb{Z}^2$ by $s \times s$ squares $S_{ij} = \{is, is + 1, \ldots, is + m\} \times \{js, js + 1, \ldots, js + m\}$: $(i, j) \in \mathbb{Z}^2$. We have $\Lambda \leftrightarrow \Lambda^\star, p \leftrightarrow 1 - p$ as usual. Declare (i, j) to be closed (in our new percolation $\tilde{\mathbb{P}}$) if in the square annulus of outer diameter $3m$ and inner diameter m around S_{ij} in $\Lambda^\star$, there is an open cycle surrounding S_{ij}; $\tilde{\mathbb{P}}((i, j) \text{ is closed}) \geq (h_{1-p}(3m, m))^4 \therefore \tilde{\mathbb{P}}((i, j) \text{ open}) \leq 1 - h_{1-p}(3m, n)^4 < p_1(5, 4)$ (our site percolation $\tilde{\mathbb{P}}$ is 5-independent). So $\tilde{\mathbb{P}}(|\tilde{C_0}| \geq n) \leq e^{-an}$ where $a = a(5, 4)$. If $|C_0| \geq (3s)^2$ ther every "square" S_{ij} that C_0 meets is open (i.e. the corresponding (i, j) is open, because C_0 meets the square in Λ so there cannot be a cycle in $\Lambda^\star$ surrounding the square). Hence $\mathbb{P}(|C_0| \geq n) \leq \mathbb{P}(|\tilde{C_0}| \geq \frac{n}{(3s)^2}) \leq e^{-\frac{a}{3s^2} n}$ (strictly speaking this is only true for $n \geq 3s^2$, but we can find a bound for $n < 3s^2$ and incorporate this into our exponent); this gives the result.

Theorem 10 (Kesten's theorem): $p_T^b(\mathbb{Z}^2) = p_H^b(\mathbb{Z}^2) = \frac{1}{2}$: We know $p_T^b(\mathbb{Z}^2) \leq p_H^b(\mathbb{Z}^2) \leq \frac{1}{2}$ (and in fact we know this second $\leq$ is an $=$, by Harris). If $p < \frac{1}{2}$,

$\mathbb{E}_p^b(|C_0|) \leq 1 + \sum_{k=2}^{\infty} ke^{-ka} < \infty$, so we are done.

Lemma 11: For $k \geq 1 \exists p_k < 1$ such that if P is a k-independent bond percolation measure on $\mathbb{Z}^2$ (or $\mathbb{Z}^n$) with [each] bond open with probability $> p_k$ then $\mathbb{P}(E_\infty) = 1$: the proof is by Peierl's argument. If we have a cycle around 0, we can find many bounds in this which are all far apart from each other. (Proving that e.g. p_1 is in fact quite small is a far trickier matter).

Another proof that $p_H^b(\mathbb{Z}^2) \leq \frac{1}{2}$: consider $P > \frac{1}{2}$, take a "grid" of $n \times n$ squares in our original Λ, and "shade them" such that every square whose x and y coordinates are both odd is shaded, otherwise left blank. Consider these shaded squares as forming the sites of a new percolation [on $\mathbb{Z}^2$] (in fact the argument also works if we consider every square as a site in the new percolation, but it "looks more like a lattice" this way). In this new percolation we consider a horizontal bond from a to b to be open if we have an open horizontal crossing of the $3n \times n$ rectangle with a and b as its two ends in the original percolation, and also a vertical crossing of the leftmost of the two, a (this last condition so that if we have bonds from a to b to c in the new percolation, we really do have an open crossing from a to c in the original percolation); similarly a vertical bond is considered open if we have a vertical crossing of the $n \times 3n$ rectangle it corresponds to and also a horizontal crossing of the lowest $n \times n$ square of this. This defines a new 1-independent (note that k-independence is defined in terms of sites, so this is really true; it is not the case that two bonds joined to the same site are independent in the new percolation, but it does not need to be) percolation measure $\tilde{\mathbb{P}}$ where the probability of a bond being open is $\geq h_p(3n, n)h_p(n, n)$, which will be $> p_1$ for n large enough.

The Aizenmann-Kesten-Newmann Theorem and Critical Probabilities

1. The AKN Theorem. Consider Λ (with the usual conditions e.g. locally finite). A subgroup $\Phi \subset \mathrm{Aut}\Lambda$ is a group of translations if $\forall$ finite $F \subset V(\Lambda)$, $\exists \varphi \in \Phi, \varphi(F) \cap F = \emptyset$ (equivalently, $\forall x \in V(\Lambda), n \geq 1 \exists \varphi \in \Phi : d(x, \varphi(x) \geq n)$). Consider site percolation, as we shall do throughout thes section unless otherwise stated. A site percolation measure on Λ is translation invariant if $\forall F \subset V(\Lambda)$ finite, $\mathbb{P}(E_F) = \mathbb{P}(\varphi^\star(E_F)) \forall \varphi \in \Phi$, and translation independent if $\forall F \subset V(\Lambda)$ finite $\exists \varphi \in \Phi$ such that $E_F, \varphi^\star(E_F)$ are independent, for E_F any cylindrical event depending only on F and $\varphi^\star$ denoting "the translation of the event under φ".

Theorem 1 (0-1 law for translation invariant events): Let Λ be as usual, $\mathbb{P}$ translation invariant and translation indepedent. Let E be a translation invariant event. Then $\mathbb{P}(E) = 0$ or 1: let $\epsilon > 0$. Then $\exists$ finite F such that $\mathbb{P}(E \Delta E_F) < \epsilon$. We may assume $\mathbb{P}(E_F) \leq \mathbb{P}(E)$ (otherwise replace E by E^c). We want to bound $\mathbb{P}(E) - \mathbb{P}(E)^2$: Let $\varphi \in Phi$ be such that $E_f, \varphi^\star(E_F)$ are independent. $\mathbb{P}(E) - \mathbb{P}(E)^2 \leq \mathbb{P}(E) - \mathbb{P}(E_F)^2 = \mathbb{P}(E) - \mathbb{P}(E_F \cap \varphi^\star(E_F)) \leq \mathbb{P}(E \setminus E_F) + \mathbb{P}(E \setminus \varphi^\star(E_F)) < \epsilon + \mathbb{P}(\varphi^\star(E)\varphi^\star(E_F)) < 2\epsilon$ (since the second term is the same as $\mathbb{P}(E \setminus E_F)$). Since this holds for all ϵ, it must hold for one of E, E^c for arbitrarily small ϵ; thus one of these has $\mathbb{P}(E) \leq \mathbb{P}(E^2)$ and so $\mathbb{P}(E) = 0$ or 1.

Write I_k for the event that there are exactly k infinite open clusters.

Theorem 2: For Λ, Φ as usual, $\mathbb{P}$ an independent translation invariant site percolation probability, one of I_0, I_1, I_∞ has probability 1: We have $\mathbb{P}(I_k) = 1$ for exactly one of the I_k. Suppose $2 \leq k < \infty$. Pick $x_0 \in V$; write A_n for the event that $\Lambda \setminus B_n(x_0)$ has at least one infinite open cluster and every such meets $S_{n+1}(x_0)$. We have $I_k \subset \bigcup_{n=1}^\infty A_n$, so $\mathbb{P}(A_n) > 0$ for some n. I_1 contains the intersection of the event that every site in $B_n(x_0)$ is open with A_n, so $\mathbb{P}(I_1) \geq (\prod_{x \in B_n} p_x)\mathbb{P}(A_n) > 0$ (we take percolation measures to always be positive, $p_x > 0 \forall x$ - otherwise our assumption that Λ is connected would be meaningless). So $\mathbb{P}(I_1) > 0$, a contradiction.

Using K's earlier 0-1 law alone would only give us that one of $I_0, \bigcup_{1 \leq k < \infty} I_k, I_\infty$ has probability 1, so this is a big improvement.

Technical lemma: Lemma 3: Let G be a finite graph with k components, $L, C \subset V(G), L \cap C = \emptyset$ and every component of G contains at least one vertex of C. Write G_c for the component of G containing $c \in C$; we have $\bigcup_{c \in C} G_c = G$. Suppose [for each $c \in C$] $G_c - c$ has $m_c \geq 3$ components containing vertices of L. Then $|L| \geq 2k + \sum_{c \in C}(m_c - 2)$: since this is linear in components we may take $k = 1$, and we may assume G is a minimal connected graph containing $L \cup C$ (as any additional edges or vertices can only help us). This means G is a tree and all leaves belong to L ($d(c) \geq m_c \geq 3$); also $|L| \geq 3$. But then $|L|$ is at least the number of leaves, $\geq 2 + \sum_{c \in C}(m_c - 2)$.

We want to rule out the case $\mathbb{P}(I_\infty) = 1$, but clearly some graphs (e.g. the Bethe lattice where each vertex has degree 3, with $p > \frac{1}{2}$) may have this. So we define: a graph Λ is <u>amenable</u> if $\forall x \in V(\Lambda), \frac{|S_n(x)|}{|B_n(x)|} \to 0$, and <u>uniformly amenable</u> if this convergence is uniform (i.e. $\forall \epsilon > 0 \exists n_0 : \forall x \in V(\Lambda) \forall n > n_0 \frac{|S_n(x)|}{|B_n(x)|} < \epsilon$). Λ is of <u>finite type</u> if $V(\Lambda) = \bigcup_1^k V_i$ with $\forall x, y \in V_i \exists \varphi \in \mathrm{Aut}\Lambda : \varphi(x) = y$. Note that any amenable graph of finite type is uniformly amenable.

Theorem 4 (AKN): Let Λ be an amenable graph of finite type, $\mathbb{P}$ an <u>independent</u> site percolation measure on Λ with $p_x = p_y > 0$ if x, y have the same type. Then I_0 or I_1 has probability 1 (this is one of the two "pillars", our main tools for studying critical probabilities): suppose not, then $\mathbb{P}(I_\infty) = 1$. Let $A_r(x)$ be the event that $\Lambda - B_r(x)$ has ≥ 3 infinite open clusters meeting $S_{r+1}(x)$. $\mathbb{P}(A_r(x)) \to 1$ as $r \to \infty$; pick an r such taht $\mathbb{P}(A_r(x)) \geq a > 0 \forall x \in V(\Lambda$. Fix x_0, then take W a maximal subset of $V(B_n(x_0))$ such that if $w, w' \in W, w \neq w'$ then $d(w, w') \geq 2r + 2$. We have $|W| \geq \frac{|B_n(x)|}{b_{2r+1}(\Delta)}$ (where $b_n(\Delta) = 1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{n-1}$ and $\Delta = \Delta(\Lambda)$, which exists by finite type). Hence, if n is large enough, $|W| > a^{-1}|S_{n+r+1}(x_0)|$.

Call $B_r(x)$ a <u>cut-ball</u> if $A_r(x)$ holds. The expected number of cut-balls is $\geq a|W| > |S_{n+r+1}(x_0)|$; pick a configuration ω for which there are $s > |S_{n+r+1}(x_0)|$ cut-balls, say $C_1, \ldots, C_s$. There are some components in the open subgraph given by $\omega - \bigcup_1^s C_i$ that meet the spheres $S_{r+1}(w)$ about the centres of the cut-balls; let $L_1, \ldots, L_t$ be the infinite components and $F_1, \ldots, F_u$ the finite ones. Considre $\Lambda[\bigcup_1^s C_i \cup \bigcup_1^t L_j \cup \bigcup_1^u F_k]$; contract this to form G by [each] $C_i \to$ [a single vertex] c_i, $L_j \to l_j, F_k \to f_k$. Then $G, C = \{c_1, \ldots, c_s\}, L = \{l_1, \ldots, l_t\}$ satisfy the conditions of Lemma 3; hence $t \geq s + 2$ and we have a contradiction, since the L_j are disjoint and every L_j must meet $S_{n+r+1}(x_0)$.

2. The Harris-Kesten Theorem, Once Again

Theorem 5 (Harris, '60; Zhang's proof, '98): For bond percolation on $\mathbb{Z}^2$,

$\vartheta(\frac{1}{2}) = 0$: set $p = \frac{1}{2}$ and suppose $\vartheta(\frac{1}{2}) > 0$. Then $\mathbb{P}(I_1) > 0$, so by theorem 4, $\mathbb{P}(I_1) = 1$. Let $S_n = [n] \times [n]$. For any infinite component, a sufficiently large square will meet it, so there is an n_0 such that if $n \geq n_0 - 1$ then $\mathbb{P}(S_n$ meets an infinite component$) \geq 1 - 10^{-4}$; call this probability $\mathbb{P}(F)$. Write E_1 for the event that there is an infinite open path P_1 leaving S_n "upwards" (i.e. from the top side), and analagously E_2 to the right, E_3 down, E_4 left. $\bigcup_1^4 E_i \supset F \therefore \mathbb{P}(\bigcup_1^4 E_i) \geq 1 - 10^{-4} \therefore \mathbb{P}(E_i) \geq 1 - \frac{1}{10}$ for some i, and by symmetry this holds for all i. Let S' be the $(n-1) \times (n-1)$ square in the dual $\Lambda^\star \cong \mathbb{Z}^2$ contained in S; define $E_1', \ldots, E_4'$ in the obvious way; we have $\mathbb{P}(E_i') \geq 1 - \frac{1}{10} \forall i$. Then let $E = E_1 \cap E_2' \cap E_3 \cap E_4'$; we have $\mathbb{P}(E) \geq \frac{3}{5} \therefore$ the probability of E and all the bonds of S_n being open is > 0. Then let P_1 be the path in E_1, and analagously; P_1 and P_3 can be joined within S_n. So these either meet (outside S_n), forming a cycle surrounding the left or right side of S_n, but this contradicts that P_2', P_4' are infinite, or they form an open two-way infinite path, dividing the plane into two, so I_1' cannot hold (since we have P_2', P_4' infinite on both sides of the plane, a contradiction. This proof is nice since it is quite general, in that it does not rely on any local properties of $\mathbb{Z}^2$.

The second "pillar" is Aisenmann-Newman (though a slightly weaker form of the result, proven by Menshikov, contains all the critical features): Suppose Λ under the usual conditions is of finite type and $|B_r(x)| \leq r^{\frac{\log r}{100}} \forall x$ for sufficiently large r. Then for $p < p_H^s(\Lambda)$, $\mathbb{P}(|C_x| \geq n) \leq e^{-an}$, where $a = a(p, \Lambda) > 0$; the proof is on one of the example sheets. This gives us:

Theorem 6 (Kesten): $p_T^b(\mathbb{Z}^2) = p_H^b(\mathbb{Z}^2) = \frac{1}{2}$ (on the example sheet we shall see $p_T^b = p_H^b =: p_C^b$ for a large class of lattices): Suppose $p_H^b(\mathbb{Z}^2) > \frac{1}{2}$. Then at $p = \frac{1}{2}$, $\mathbb{P}(|C_0| \geq n+1) \leq e^{-an}$. But $\mathbb{P}(H(R)) = \frac{1}{2}$ for any $(n+1) \times n$ rectangle R, so $\mathbb{P}(|C_x| \geq n+1) \geq \frac{1}{2n}$, which gives a contradiction if n is large enough.

3. Site percolation on the Triangular Lattice

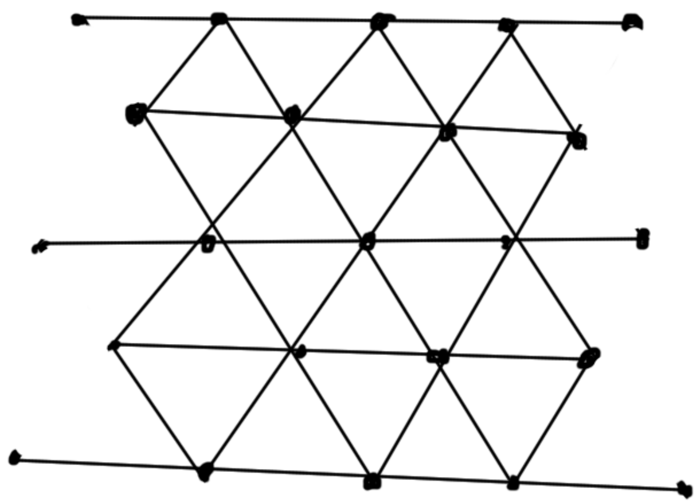
The triangular lattice T is closely associated with the hexagonal or honeycomb lattice H; site percolation on T corresponds to face percolation on H.

Lemma 7: Let R be an $m \times n$ "parallelogram" in T. Then for any p, $\mathbb{P}_p(H(R)) + \mathbb{P}_{1-p}(V(R)) = 1$: more is true: we claim that for all configurations (i.e. assignments of open and closed states [to sites]), there is either an open horizontal crossing or a closed vertical crossing, but not both: pass to the "interface graph", a parallelogram in H, and colour hexagons black if they correspond to open sites, white for closed; the space above and below the parallelogram is white and that to its left and right black. Direct edges so that they have black on their right and white on their left, then the inward-pointing edge at the top left must connect to one of the outgoing edges and so on as before. We cannot have both crossings as this gives (with a very small amount of work) a planar drawing of K_5.

Theorem 8: $p_T^s(T) = p_H^s(T) = \frac{1}{2}$. i) Suppose $p_C^s(T) > \frac{1}{2}$. Consider site percolation with probability $p = \frac{1}{2}$; by AN we have exponential decay: $\exists a > 0$ such that $\mathbb{P}(|C_0| \geq n) \leq e^{-an}$. On the other hand, considering an $n \times n$ parallelogram P, $\mathbb{P}(|C_x| \geq n) \geq \frac{1}{2n}$, a contradiction for n large. ii) Suppose $p_C^s(T) < \frac{1}{2}$. Take percolation with probability $\frac{1}{2}$; consider the "dual" obtained by exchanging open bonds for closed. Then $\mathbb{P}(I_1) = \mathbb{P}'(I_1') = 1$. Let H_n be the regular hexagon in T with $6n$ sites on the perimeter; write E_i for the event that there is an [infinite] open path with exactly one site in H_n, on side i [defining the sides so that they do not overlap]. We may take $\mathbb{P}(\bigcup_1^6 E_i) \geq 1 - 10^{-6}$ (by n sufficiently large) so $\mathbb{P}(E_i) \geq 1 - \frac{1}{10} \forall i$; thus $\mathbb{P}(E_1 \cap E_2' \cap E_4 \cap E_5') \geq 1 - \frac{4}{10}$; then the probability that this occurs and every site in the interior of H_n is open is ¿0, since the E_i, E_i' are independent of the sites inside H_n. However, we then as before have P_1, P_4 open leaving the hexagon from top and bottom and P_2, P_5 closed leaving it from left and right; as the sites inside are all open we can join P_1, P_4 by some path, so there is a two-way infinite open path separating P_2', P_5' and so $\mathbb{P}(I'_{\geq 2}) > 0$, a contradiction.

4. Bond percolation on T and H

Lemma: $p_c^b(T) + p_c^b(H) = 1$: let R be a "rectangle" in T, where the "last zigzag" of edges on the "vertical" sides is not present, so that e.g. the right hand



edge looks like a "stack" of Σs:
Dualise, then $\mathbb{P}_p(H(R)) + \mathbb{P}_{1-p}(V(R^\star)) = 1$ by the usual "interface graph" proof. i) Suppose $p_c^b(T) + p_c^b(H) < 1$; pick p with $p > p_c^b(T), 1 - p > p_c^b(H)$. Consider bond percolation with probability p on T, coupled with bond percolation on H with probability $1 - p$, and consider semi-infinite paths away from a hexagon as usual. ii) If $p_c^b(T) + p_c^b(H) > 1$, pick p, $p < p_c^b(T), 1 - p < p_c^b(H)$, then we have exponential decay, but since we always have either a horizontal crossing of a rectangle or a vertical crossing in the dual the decay can be at most $\frac{c}{n}$, so we have a contradiction, again as usual.

Star-triangle transformation: consider replacing a triangle xyz with a "star" where each of x, y, z is connected to a central vertex w. If we could find probability distributions such that the probability of each possible "connectedness combination" is the same for both, then we could use this to dualise T. Observe that if the probability of bond percolation on the triangle is p and that on the star r, the case where all vertices are connected means we need $p^3 + 3p^2(1-p) = r^3$, two connected (e.g. $\{x, y\}$, or symetrically any other pair) gives $p(1-p)^2 = r^2(1-r)$, and none connected $(1-p)^3 = (1-r)^3 + 3r(1-r)^2$. If we set $r = 1 - p$ these reduce to $p^3 + 3p^2 - 3p^3 - 1 + 3p - 3p^2 + p^3 = 0$ i.e. $p^3 - 3p + 1 = 0$; the solution (in $(0,1)$) to this is $p_0 := 2\sin\frac{\pi}{18} = 0.3472\ldots$.

Theorem 9: $p_c^b(T) = 2\sin\frac{\pi}{18}; p_c^b(H) = 1 - 2\sin\frac{\pi}{18}$: dualise the triangular lattice by replacing each "upward" pointing triangle with a star, giving a hexagonal lattice. Consider bond percolation on T with probability p_0; couple it with an independent percolation on H by chosing the bonds in each triangular "domain" (star) with probability $1 - p_0$ such that the set of sites connected in a given domain in T is precisely that connected in H (we can do this by the above). Conisder C_0, C_0'; $C_0 \subset C_0'$ as for any open path in T, the corresponding path in H is open. $|C_0| \leq |C_0'| \leq 4|C_0|$: being very crude, the sites of C_0' are at most the sites of C_0 and the three neighbours of each. So $\mathbb{P}_{p_0}(|C_0| \geq n) \leq \mathbb{P}(|C_0'| \geq n) \leq \mathbb{P}(|C_0| \geq \frac{n}{4}) \therefore \vartheta_{p_0}(T,0) = \vartheta_{1-p_0}(H,0)$.

Suppose $p_c(T) < p_0$; then $p_c(H) > 1 - p_0$, but then $\vartheta(T, p_0) > 0, \vartheta(H, 1 - p_0) = 0$, a contradiction; the converse is similar.

This appears to be the end of the course. The lecturer wished to emphasise that this is not a reflection of the state of the art in the subject; while there are still physicists attempting to find better bounds on the critical probabilities of various lattices, most mathematical work in the field is now on deeper results.