Category Theory

Richard Garner (rhgg2@cam.ac.uk)

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Books

There are three books mentioned in the schedules, but none are perfect for this cours; MacLane's "Categories for the Working Mathematician" is quite cheap, Awodey's "Category Theory" is nice but slow paced and thus very expensive. Borceux's "Handbook of Categorical Algebra" is probably the best choice; volume 1 is the only one which will be needed for this course.

Course Information

There will be 4 examples sheets for this course, hopefully supervised. The exam will consist of six questions chosen from ten (all of equal weight); the examples sheets questions should be representative of the exam questions, since unlike many courses in which one simply memorizes what the lecturer says, in category theory it is possible to get real work done in the space of an exam.

The departmental category theory seminar meets at 2:15pm on Tuesdays in term time in MR9; students here may be interested in attending.

What is category theory?

1) The mathematics of mathematics: where group theory, ring theory, geometry etc. abstract away from the real world, category theory abstracts away from mathematics itself.

2) A language for mathematics: notation is important, e.g. writing $\frac{d}{dx}$ already implies some of the properties a differential should have. Category theory is a kind of "subject-agnostic notation" for mathematics.

3) A way of making mathematical intuitions precise: e.g. we can formalize in what sense "the compactification of a topological space", "the abelianisation of a group" and "the sheafification of a presheaf" are "the same construction".

4) Implementation-free mathematics: A set theorist says "the cartesian product of two sets A, B is the set $\{\{\{a\}, \{a, b\}\} : a \in A, b \in B\}$". A category theorist says "the product of A and B is any set $A \times B$ satisfying the properties: there are projections $\pi_1 : A \times B, \to A, \pi_2 : A \times B \to B$, and for any maps $f : C \to A, g : C \to B$ there is a unique pairing $(f, g) : C \to A \times B$ such that $\pi_1 \circ (f, g) = f, \pi_2 \circ (f, g) = g$."

5) Mathematics in alternate mathematical realities: e.g. in domain theory we can have an object X satisfying $X \cong X^X$. Category theory allows us to make sense of "doing mathematics" in this setting.

1 Categories, Functors and Natural Transformations

1.1 Definition

A category C is given by: a collection ob$C$ of objects; for each $X, Y \in \text{ob}C$, a collection $C(X, Y)$ of morphisms from X to Y; for each $X \in \text{ob}C$, an identity morphism $1_X \in C(X, X)$, for each $X, Y, Z \in C$, an operation of composition $C(Y, Z) \times C(X, Y) \to C(X, Z)$ $(g, f) \mapsto g \circ f$ satisfying axioms of: unitality: for every $f \in C(X, Y)$ we have $\text{id}_Y \circ f = f = f \circ \text{id}_X$ (this is really two axioms, left and right unitality; id_X is notation for 1_X), and associativity: for $f \in C(W, X), g \in C(X, Y), h \in C(Y, Z)$, $h \circ (g \circ f) = (h \circ g) \circ f$.

1.2 Remarks

a) We write $f : X \to Y$ to indicate that $f \in C(X, Y)$. We call X the source (or domain) of f and Y the target (or co-domain).

b) We also call morphisms arrows or maps.

c) We write mor$C$ for the disjoint union of the $C(X, Y)$s and $s, t : \text{mor}C \to \text{ob}C$ for the source and target assignations.

d) By associativity we may write e.g. $khgf$ and this is well defined ($= k \circ (h \circ (g \circ f))$ etc.)

e) We don't assume that either of ob$C$ or $C(X, Y)$ are sets; we do this because we want to talk about things like "the category of all sets".

1.3 Definition

A category C is locally small if each $C(X, Y)$ is a set. It is small if it is locally small and ob$C$ is a set; otherwise it is large. Note that a large category may still be locally small; in fact this is the most common case.

1.4 Examples

Large, but locally small, categories of mathematical objects:

Set - objects are sets, morphisms $X \to Y$ are functions

Set_{inj} - objects are sets, morphisms $X \to Y$ are injective functions

Set_{p} - objects are sets, morphisms $X \to Y$ are partial functions

Poset - objects are posets, morphisms $X \to Y$ are order-preserving functions

Grp - objects are groups, morphisms are group homomorphisms

Ab - objects are abelian groups, morphisms are group homomorphisms

CRng - objects are commutative rings, morphisms are ring homomorphisms

Vect_k - objects are k-vector spaces, morphisms are linear maps

Top - objects are topological spaces, morphisms are continuous maps

$\mathcal{H}$(Top) - objects are topological spaces, morphisms are homotopy equivalence classes of continuous maps

$\text{Top}_\star$ - objects are topological spaces equipped with a base point, morphisms are continuous maps preserving the base point.

Small categories, which are themselves mathematical objects:

A poset $(P, \leq)$ can be viewed as a category $\mathcal{P}$ with ob$\mathcal{P} = P$ and there is a unique morphism $x \to y$ just when $x \leq y$.

In particular, any set X can be viewed as a category, with only identity morphisms.

A monoid (recall a monoid or semigroup is a group without inverse) $(M, \cdot, 1)$ can be viweed as a category $\mathcal{M}$, with one (formal) object $\star$ and $\mathcal{M}(\star, \star) = M$; the composition law is monoid multiplication. (So any group can be viewed as a category).

There is a category Δ, the simplicial category: objects are non-empty finite ordinals $[n] = \{0 \leq 1 \leq \cdots \leq n\}$, maps are order-preserving functions.

There is a category $\mathbb{B}$: objects are finite cardinals $\ltimes = \{0, 1, \ldots, n-1\}$, maps are bijections between them.

We can present categories by a directed multigraph G. Given such a G, we can form a category whose objects are the vertices of G and morphisms are paths through the graph. We can then impose equations identifying certain paths with the same sources and targets.

Example: the category 0 has no objects and no arrows, and is presented by the null graph. The category 1 has one object and one arrow; it is presented by the graph consisting of a single vertex. The category 2 has two objects and three arrows; it is presented by a graph of two vertices and a directed edge between them (the other two morphisms are identity morphisms, given by paths of one vertex).

The category presented by the graph with a single vertex and an edge from it to itself is isomorphic to the additive monoid $(\mathbb{N}, +)$ viewed as a one-object category. The category generated by a "figure 8" graph with one vertex and two edges g, f from it to itself subject to the equations $fg = \mathrm{id}, gf = \mathrm{id}$ is isomorphic to $(\mathbb{Z}, +)$ viewed as a one-object category.

Some constructions which build new categories from old ones:

1.5 Definition

Let C be a category. A subcategory of C is given by a subcollection ob$\mathcal{D} \subset$ ob$C$ and subcollections $\mathcal{D}(X, Y) \subset C(X, Y)$ for $X, Y \in$ ob$\mathcal{D}$ such that id$X \in \mathcal{D}(X, X) \forall X \in$ ob$\mathcal{D}$ and for any $f \in \mathcal{D}(X, Y), g \in \mathcal{D}(Y, Z)$, $g \circ f \in \mathcal{D}(X, Z)$. Any subcategory $\mathcal{D} \subset C$ can be viewed as a category in its own right, under the induced composition law. E.g. $\underline{\mathrm{Set}}_{\mathrm{inj}} \subset \mathrm{Set}$.

1.6 Definition

Given C a category and $I \in C$ (by which of course we "really" mean $I \in$ ob$C$, we can form the slice category $\frac{C}{I}$: objects are pairs $(X \in C, x : X \to I)$, morphisms $(X, x) \to (Y, y)$ are morphisms $f : X \to Y$ in C such that $x = yf$. Dually, we define $\frac{I}{C}$, the coslice category, with objects pairs $(X \in C, x : I \to X)$, and morphisms $(X, x) \to (Y, y)$ being maps $f : X \to Y$ such that $fx = y$.

1.7 Definition

Given $C, \mathcal{D}$ categories, the product category $C \times \mathcal{D}$ has objects ob$(C \times \mathcal{D}) =$ ob$C \times$ ob$\mathcal{D}$, morphisms $(C \times \mathcal{D})((X_1, X_2), (Y_1, Y_2)) = C(X_1, Y_1) \times \mathcal{D}(X_2, Y_2)$; composition and identities are given componentwise.

1.8 Definition

Given C a category, we obtain the dual category C^{op} by "reversing the arrows of C": $\mathrm{ob}(C^{\mathrm{op}}) = \mathrm{ob}C, C^{\mathrm{op}}(X, Y) = \{f^{\mathrm{op}} : f \in C(Y, X)\}$ (f^{op} is just a meaningless label). Identities are $\mathrm{id}_X^{\mathrm{op}} \in C^{\mathrm{op}}(X, X)$, composition $g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$.

Morphisms in Categories

1.9 Definition

Let $f : X \to Y$ in C. An inverse for f is a morphism $g : Y \to X$ such that $gf = \mathrm{id}_X, fg = \mathrm{id}_Y$. We say that f is an isomorphism if it has an inverse; we say $X, Y \in C$ are isomorphic if there is an isomorphism $f : X \to Y$.

1.10 Examples

The isomorphism in Set are bijective functions, those in Grp are group isomorphism and those in Top are homeomorphisms.

1.11 Proposition

If g, g' are inverses for $f : X \to Y$ then $g = g'$: we have $g = g(\mathrm{id}_Y) = g(fg') = (gf)g' = (\mathrm{id}_X)g' = g'$.

1.12 Proposition

The composition of two isomorphisms is an isomorphism, as are identity maps and the inverse of an isomorphism (exercise).

1.13 Definition

A morphism $f : X \to Y$ in C is said to be an epimorphism if for all $Z \in C, g, h : Y \to Z$, we have $gf = hf \Rightarrow g = h$. f is said to be a monomorphism if $f \forall W \in C, g, h : W \to X, fg = fh \Rightarrow g = h$.

1.14 Examples

In Set, the epimorphisms are the surjective functions and the monomorphisms are the injective functions. In Grp, the epimorphisms are the surjective group homomorphisms and the monomorphisms the injective group homomorphisms. In Top, the epimorphisms are injective continuous maps and the monomorphisms surjective maps. However, beware; it's not always this simple: e.g. in CRng, the canonical inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.

1.15 Proposition

If $f : X \to Y, g : Y \to X$ have $fg = \mathrm{id}_Y$, then f is an epimorphism: suppose $Z \in \mathcal{C}, h, k : Y \to Z$ with $hf = kf$, then $h = h(\mathrm{id}_Y) = h(fg) = k(fg) = k(\mathrm{id}_Y) = k$.

1.16 Proposition

Given $f : X \to Y, g : Y \to X$ with $fg = \mathrm{id}_Y$, then g is a monomorphism. We shall use a longer proof which shows a useful general technique: (1) g is a monomorphism in $\mathcal{C} \Leftrightarrow g^{\mathrm{op}}$ is an epimorphism in $\mathcal{C}^{\mathrm{op}}$ (2), (3) $fg = \mathrm{id}_Y$ in $\mathcal{C} \Leftrightarrow g^{\mathrm{op}} f^{\mathrm{op}} = \mathrm{id}_Y^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$ (4). Since (4)$\Rightarrow$(2), we deduce also that (3)$\Rightarrow$(1).

1.17 Remark

This is an example of the <u>principle of duality</u>: any definition, proposition or proof can be dualised by replacing $\mathcal{C}$ by $\mathcal{C}^{\mathrm{op}}$. We frequently denote the dual of a notion by the prefix "co-".

Functors

1.18 Definition

Let $\mathcal{C}, \mathcal{D}$ be categories. A <u>functor</u> $F : \mathcal{C} \to \mathcal{D}$ is given by: an assignation $\mathrm{ob}\mathcal{C} \to \mathrm{ob}\mathcal{D}$ $X \mapsto FX$, assignations $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ $f \mapsto Ff$ such that: $\forall X \in \mathrm{ob}\mathcal{C}, F(\mathrm{id}_X) = \mathrm{id}_{FX}$, and $\forall f : X \to Y, g : Y \to Z, \in \mathcal{C}, F(g \circ f) = Fg \circ Ff$.

A functor F is full, faithful, or full and faithful if each assignation $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is respectively surjective, injective, bijective.

1.19 Definition

The category <u>Cat</u> has as objects <u>small</u> categories, and as morphisms, functors. The identity functor on $\mathcal{C}$ is $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} : X \mapsto X, f \mapsto f$. Composition is similarly simple. The category <u>CAT</u> is the category of all categories.

1.20 Remark

<u>Cat</u> is a large category, but is locally small; <u>CAT</u> is not even locally small.

1.21 Examples

Forgetful functors, e.g. U : <u>Grp</u> $\to$ <u>Set</u> : $(G, \cdot, e) \mapsto G, f : (G, \cdot, e) \to (H, \cdot, e) \mapsto f : G \to H$, or similarly <u>Top</u> $\to$ <u>Set</u>, <u>CRng</u> $\to$ <u>Ab</u> (mapping to the underlying addative group), $\underline{\mathrm{Top}}_\star \to$ <u>Top</u>

"Free" functors, e.g. F : <u>Set</u> $\to$ <u>Grp</u> $X \mapsto$ the free group on X. Or <u>Grp</u> $\to$ <u>Ab</u> $G \mapsto \frac{G}{[G,G]}$, or <u>Set</u> $\to$ <u>Top</u> $X \mapsto$ the discrete space on X. (For these last few examples we have only given the mapping of objects; this is because the map of morphisms is usually obvious. Where it is not, we will give it explicitly).

Given a commutative ring R, there is a functor <u>Grp</u> $\to$ <u>CRng</u> $G \mapsto R[G]$.

There is a functor $\underline{\mathrm{Top}}_\star \to \underline{\mathrm{Grp}}(X, x) \mapsto \pi_1(X, x)$.

If $\mathcal{M}, \mathcal{N}$ are monoids (or groups) seen as one-object categories, then a functor $\mathcal{M} \to \mathcal{N}$ is a monoid (or group) homomorphism.

If $\mathcal{P}, \mathcal{Q}$ are posets, seen as categories, then a functor $\mathcal{P} \to \mathcal{Q}$ is an order-preserving map.

For any categories $\mathcal{C}, \mathcal{D}$ and any $I \in \mathcal{D}$, we have a <u>constant functor</u> $\Delta_I : \mathcal{C} \to \mathcal{D}$ by $X \mapsto I, f \mapsto \mathrm{id}_I$.

For categories $\mathcal{C}, \mathcal{D}$ we have the projection functors $\pi_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ $(X_1, X_2) \mapsto X_1, (f_1, f_2) \mapsto f_1$, similarly π_2.

Any subcategory $\mathcal{D} \subset \mathcal{C}$ induces an inclusion functor $\mathcal{D} \hookrightarrow \mathcal{C}$.

1.22 Definition

A <u>contravariant functor</u> from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ (i.e. an assignation $\mathrm{ob}\mathcal{C} \to \mathrm{ob}\mathcal{D}$ $X \mapsto FX$ and assignations $\mathcal{C}(X, Y) \to \mathcal{D}(FY, FX)$ $f \mapsto Ff$ such that $F(\mathrm{id}_X) = \mathrm{id}_{FX}, F(g \circ f) = Ff \circ Fg$.

1.23 Examples

There is a functor $\underline{\mathrm{Vect}}_k^{\mathrm{op}} \to \underline{\mathrm{Vect}}_k$ $V \mapsto V^\star, f : V \to W \mapsto f^\star : W^\star \to V^\star$.

There is a functor $\mathcal{O} : \underline{\mathrm{Top}}^{\mathrm{op}} \to \underline{\mathrm{Set}}$ by $X \mapsto$ the set of open sets in X, $\mathcal{O}(X)$, and $f : X \to Y \mapsto \mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X)$ defined by $A \subset Y \mapsto f^{-1}(A) \subset X$.

There is a functor $\underline{\mathrm{Top}}^{\mathrm{op}} \to \mathbb{C} - \mathrm{Alg}$: $X \mapsto$ the set of continuous functions $X \to \mathbb{C}, \mathbb{C}^X, f : X \to Y \mapsto$ a map $\mathbb{C}^Y \to \mathbb{C}^X : g : y \to \mathbb{C} \mapsto g \circ f : X \to \mathbb{C}$

1.24 Remark

A functor $F : \mathcal{C} \to \mathcal{D}$ sends commutative diagrams in $\mathcal{C}$ to commutative diagrams in $\mathcal{D}$. Consequently, any property defined purely in terms of commutative diagrams is preserved by functors, e.g. if $f : X \to Y$ is an isomorphism in $\mathbb{C}$ then so is $Ff : FX \to FY$ in $\mathcal{D}$.

1.25 Definition

Let $\mathcal{C}$ be a locally small category (this definition is essentially the reason we're interested mostly in locally small categories). Let $V \in \mathcal{C}$. We have a functor $\mathcal{C}(V, -) : \mathcal{C} \to \mathrm{Set}$ by $X \mapsto \mathcal{C}(V, X), f : X \to Y \mapsto \mathcal{C}(V, y) : \mathcal{C}(V, X) \to \mathcal{C}(V, y)$ $g : V \to X \mapsto f \circ g : V \to Y$. We sometimes write H^V for $\mathcal{C}(V, -)$. Dually, we have a functor $\mathcal{C}(V, -) : \mathcal{C} \to \mathrm{Set}$ by $X \mapsto \mathcal{C}(V, X), f : X \to Y \mapsto \mathcal{C}(f, V) : \mathcal{C}(Y, V) \to \mathcal{C}(X, V)$ $g : Y \to V \mapsto g \circ f : X \to V$. We may write H_V for $\mathcal{C}(-, V)$.

Natural Transformations

We can think of a functor $F : \mathcal{C} \to \mathcal{D}$ as providing an "image of $\mathcal{C}$ inside $\mathcal{D}$". A natural transformation is a way of "translating" between two such images.

1.26 Definition

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Then a natural transformation $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ is given by an $(\mathrm{ob}\mathcal{C})$-indexed collection of morphisms $\alpha_X : FX \to GX$ in $\mathcal{D}$, called the components of α, such that, for every $f : X \to Y$ in C, the obvious square commutes: $Gf \circ \alpha_X : FX \to GX \to GY = \alpha_Y \circ Ff : FX \to FY \to GY$. This is called the naturality condition.

1.27 Remark

If we say that some family of maps $\alpha_X : FX \to GX$ is natural in X we mean that they form the components of a natural transformation $\alpha : F \Rightarrow G$.

1.28 Definition

Let $C, \mathcal{D}$ be categories. The functor category $[C, \mathcal{D}]$ or $\mathcal{D}^C$ has objects functors $C \to \mathcal{D}$, morphisms from F to G are natural transformations $\alpha : F \Rightarrow G$. The identity of $F \in \text{ob}[C, \mathcal{D}]$ is $\text{id}_F : F \Rightarrow F$ with components $(\text{id}F)_X = \text{id}_{FX} : FX \to FX$. Composition of $\alpha : F \Rightarrow G, \beta : G \Rightarrow H$ is $\beta \circ \alpha : F \Rightarrow H$ with components $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X : FX \to HX$; checking these really are natural is left as an exercise.

1.29 Remark

If $C, \mathcal{D}$ are both small then so is $[C, \mathcal{D}]$; if C is small and $\mathcal{D}$ is locally small then $[C, \mathcal{D}]$ is locally small.

1.30 Example

Let $\mathcal{G}$ be a group (or monoid) viewed as a one-object category. The functor category $[\mathcal{G}, \text{Vect}_k]$ has: objects are functors $F : \mathcal{G} \to \text{Vect}_k$. Recall $\mathcal{G}$ has only one object $\star$, so we must have $\star \mapsto F\star = \text{some } V$, and $g : \star \to \star \mapsto F_p = \rho_g : V \to V$, i.e. our objects are linear representations of G. Morphisms $F \Rightarrow H$ are natural transformations. So, supposing $F\star = V, H\star = W$, such a natural transformation $\alpha : F \Rightarrow H$ is given by a component $f = \alpha_\star : F\star = V \to H\star = W$ such that $\forall g : \star \to \star$ in $\mathcal{G}$ (i.e. elements of the group), there is a commuting square: $Hg \circ \alpha_\star : F\star \to H\star \to H\star = \alpha_\star \circ Fg : F\star \to F\star \to H\star$, or $\rho_g \circ f = f \circ \rho_g$, i.e. this is a map of G-representations, or an equvariant map or intertwiner.

More generally, we can think of a functor $C \to \text{Vect}_k$ as a "linear representation of C", and of natural transformations between them as "equivariant maps".

1.31 Definition

A natural isomorphism is an invertible natural transformation. $F, G : C \to \mathcal{D}$ are naturally isomorphic if there is a natural isomorphism $F \Rightarrow G$.

1.32 Example

The identity functor $\text{id}_{\text{Set}} : \text{Set} \to \text{Set}$ is naturally isomorphic to the functor $F : \text{Set} \to \text{Set}\ X \mapsto \amalg_{x \in X} \{\star\}$ (i.e. the disjoint union of copies of a single element indexed by X).

1.33 Proposition

A natural transformation $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ is invertible iff each component $\alpha_X : FX \to GX$ is invertible in $\mathcal{D}$: If α is invertible with inverse $\beta : G \Rightarrow F$, then for each $X \in \mathcal{X}$ we have $\alpha_X \circ \beta_X = (\alpha \circ \beta)_X = (\mathrm{id}_G)_X = \mathrm{id}_{GX} : GX \to GX$; similarly $\beta_X \circ \alpha_X = \mathrm{id}_{FX}$, so α_X is invertible. Conversely, suppose each α_X is invertible in $\mathcal{D}$ with inverse $\beta_X : GX \to FX$. We claim the β_Xs are natural in X, i.e. that a square commutes: $Ff \circ \beta_X : GX \to FX \to FY = \beta_Y \circ Gf : GX \to GY \to FY$. But we have $Ff \circ \beta_X = \beta_Y \circ \alpha_Y \circ Ff \circ \beta_X$ (since α_Y, β_Y are inverses) $= \beta_y \circ G_f \circ \alpha_X \circ \beta_Y$ (by naturality of α) $= \beta_Y \circ Gf$.

1.34 Remark

We showed that if $Gf \circ \alpha_X = \alpha_Y \circ Ff$ (naturality of α) then $Ff \circ \alpha_X^{-1} = \alpha_Y^{-1} \circ Gf$; in general, if a diagram $\mathcal{D}$ commutes, then so does any $\mathcal{D}'$ obtained from it by "turning around some isomorphisms".

1.35 Definition

An <u>equivalence of categories</u> between $\mathcal{C}$ and $\mathcal{D}$ is given by functors $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF, \epsilon : FG \to \mathrm{id}_{\mathcal{D}}$. Note that any isomorphism of categories is an equivalence of categories; for more substantiative examples, see the first example sheet.

As well as composing natural transformations with each other, we can also precompose them with functors - given $F : \mathcal{C} \to \mathcal{D}, \beta : G \Rightarrow H : \mathcal{D} \to \mathcal{E}$, we have $\beta F : GF \Rightarrow HF : \mathcal{C} \to \mathcal{E}$ with components $(\beta F)_X = \beta_{FX} : GFX \to HFX$. Naturality for βF follows from that for β. Dually, we can postcompose a natural transformation with a functor: given $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{D} \to \mathcal{E}$, we have $H\alpha : HF \Rightarrow HG : \mathcal{C} \to \mathcal{E}$ with components $(H\alpha)_X = H(\alpha_X) : HFX \to HGX$. For naturality we must show that for all $f : X \to X$ in $\mathcal{C}$, $HGf \circ H\alpha_X : HFX \to HGX \to HGY = H\alpha_Y \circ HFf : HFX \to HFY \to HGY$, but this is just $H(-)$ of a naturality square for α, and functors preserve commuting diagrams.

1.36 Proposition

Consider a diagram where we have $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}, \beta : H \Rightarrow K : \mathcal{D} \to \mathcal{E}$. Then we have a commuting diagram in the functor category $[\mathcal{C}, \mathcal{E}]$: $\beta G \circ H\alpha : HF \to HG \to KG = K\alpha \circ \beta F : HK \to KF \to KG$: it suffices to show that for all $X \in \mathcal{C}$ we have a commuting square in $\mathcal{E}$: $\beta_{GX} \circ H\alpha_X : HFX \to HGX \to KGX = K\alpha_X \circ \beta_{FX} : HFX \to KFX \to KGX$. But this is just a naturality square for β at the map $\alpha_X : FX \to GX$ in $\mathcal{D}$. So we are done.

2 Presheaves, Representations and the Yoneda Lemma

This is a section on universal properties - the heart of category theory.

2.1 Definition

A covariant presheaf on a category C is a functor $C \to \mathrm{Set}$. A contravariant presheaf on a category C is a functor $C^{\mathrm{op}} \to \mathrm{Set}$.

2.2 Examples

Presheaves on large categories equip each object of the category with a set of "attributes" which can be "transported" along morphisms.

There is a presheaf $\mathrm{ob} : \mathrm{Cat} \to \mathrm{Set}$ assigning to each C its set of objects

There is a presheaf $\mathrm{mor} : \mathrm{Cat} \to \mathrm{Set}$ assigning to each C its set of morphisms

There is a presheaf $\mathrm{iso} : \mathrm{Cat} \to \mathrm{Set}$ assigning to each C its set of isomorphisms

There is a presheaf $\mathrm{el} : \mathrm{Grp} \to \mathrm{Set}$ assigning to each group G its set of elements

There is a presheaf $\mathrm{el}_k : \mathrm{Cat} \to \mathrm{Set}$ assigning to each group G its set of elements with order dividing $k \in \mathbb{N}$

There is a presheaf $\mathcal{P} : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ assigning to each set X its set of subsets

There is a presheaf $\mathcal{O} : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$ assigning to each space T its set of open sets

There is a presheaf $\mathrm{Sp} : \mathrm{CRng}^{\mathrm{op}} \to \mathrm{Set}$ assigning to each ring R its set prime ideals

Presheaves on small categories are themselves mathematical objects, e.g. A presheaf $(\cdot \rightrightarrows \cdot) \to \mathrm{Set}$ is a directed multigraph, the two arrows going to the source and target maps $E \to V$. A presheaf $\Delta^{\mathrm{op}} \to \mathrm{Set}$ is a simplicial set. A presheaf $\mathcal{G} \to \mathrm{Set}$ (where $\mathcal{G}$ is as always a group seen as a one-object category) is a set equipped with a left $\mathcal{G}$-action.

Recall also that for each $V \in C$, C locally small, we have presheaves $H^V = C(V, -) : C \to \mathrm{Set}, H_v = C(-, V) : C^{\mathrm{op}} \to \mathrm{Set}$. In fact $V \mapsto H_V$ gives a functor $C \to [C^{\mathrm{op}}, \mathrm{Set}]$.

2.3 Definition

Let C be locally small. The Yoneda embedding is the functor $H_\bullet : C \to [C^{\mathrm{op}}, \mathrm{Set}]$ $V \mapsto H_V, f : V \to W \mapsto H_f : H_V \Rightarrow H_W$ where H_f has components $(H_f)_X : H_V X \to H_W X$ (i.e. $C(X, V) \to C(X, W)$) $g \mapsto f \circ g$. We must check this is a well defined natural transformation, i.e. that given $g : X \to Y$ in C, $(H_W g = (-) \circ g) \circ ((H_f)_Y = f \circ (-)) : (C(Y, V) = H_V Y) \to (C(Y, W) = H_W Y) \to (C(X, W) = H_W X) = ((H_f)_X = f \circ (-)) \circ ((-) \circ g = H_v g) : (C(Y, V) = H_V Y) \to (C(X, V) = H_V X) \to (C(X, W) = H_W X)$. So on elements we need that $h \mapsto h \circ g \mapsto f \circ (h \circ g)$ and $h \mapsto f \circ h \mapsto (f \circ h) \circ g$ are equal, but this is just associativity of composition.

Dually we have another Yoneda embedding $H^\bullet : C^{\mathrm{op}} \to [C, \mathrm{Set}]$ $V \mapsto H^V$, $f : V \to W \mapsto H^f : H^W \to H^V$ where $(H^f)_X : H^W_X = C(W, X) \to H(V, X) = C(V, X)$ is given by $g \mapsto g \circ f$, and the reader may check that this is valid.

2.4 Definition

A presheaf $F : C \to \mathrm{Set}$ is representable if it is naturally isomorphic to H^V for some $V \in C$. Dually, $G : C^{\mathrm{op}} \to \mathrm{Set}$ is representable if it is naturally isomorphic to H_V for some $V \in \mathcal{C}$.

A representation of $F : \mathcal{C} \to \mathrm{Set}$ is a choice of isomorphism $\theta : H^V \to F$ (similarly, a representation of $G : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a choice of isomorphism $\theta : H_V \to G$)

2.5 Examples

Recall that presheaves on large categories equip each object with a set of "attributes". A representation for such a presheaf is an "object containing a generic example of that attribute".

The presheaf $\mathrm{ob} : \mathrm{Cat} \to \mathrm{Set}$ is represented by the category $\cdot = 1$ ("the generic category containing an object")

The presheaf $\mathrm{mor} : \mathrm{Cat} \to \mathrm{Set}$ is represented by the category $\cdot \to \cdot = 2$ ("the generic category containing a morphism")

The presheaf $\mathrm{iso} : \mathrm{Cat} \to \mathrm{Set}$ is represented by the category $\cdot_a \leftrightarrows \cdot_b$ where the arrows are f, g, subject to $fg = \mathrm{id}_b, gf = \mathrm{id}_a$ ("the generic category containing an isomorphism")

The presheaf $\mathrm{el} : \mathrm{Grp} \to \mathrm{Set}$ is represented by the group $\mathbb{Z}$ ("the generic group containing an element")

The presheaf $\mathrm{el}_k : \mathrm{Grp} \to \mathrm{Set}$ is represented by the group C_k ("the generic group containing an element of order k")

Suppose $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ has a representation $\theta : H_V \Rightarrow F$. This says that there are bijections $\theta_X : \mathcal{C}(X, V) \cong FX$ natural in X (†). In particular, we have $\theta_V : \mathcal{C}(V, V) \to FV$, so we have an element $u \in FV$ corresponding to the identity map $\mathrm{id}_V : V \to V$. In fact, this element u completely determines all the θ_X. Why is this? Naturality in (†) says in particular that for $f : X \to V$, $Ff \circ \theta_V : \mathcal{C}(V, V) \to FV \to FX = \theta_X \circ ((-) \circ f = \mathcal{C}(f, V)) : \mathcal{C}(V, V) \to \mathcal{C}(X, V) \to FX$. In particular, $\mathrm{id}_V \mapsto \theta_V(\mathrm{id}_V) = u \mapsto (Ff)(u)$ and $\mathrm{id}_V \mapsto \mathrm{id}_V \circ f = f \mapsto \theta_X(f)$ must be equal, i.e. $\theta_X : \mathcal{C}(X, V) \to FX$ is given by $f \mapsto (Ff)(u)$. In fact, if we are given any element $u \in FV$, this determines a natural transformation $\widehat{u} : H_V \to F$ where $\widehat{u_X} : \mathcal{C}(X, V) \to FXf \mapsto (Ff)(u)$.

2.6 Theorem (The Yoneda Lemma)

Let $\mathcal{C}$ be a locally small category, $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. Then there is a bijection $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](H_X, F) \cong FX$ for all $X \in \mathcal{C}$. This bijection is natural in X and F, i.e. for $\alpha : F \Rightarrow G : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ we have $\alpha_X \circ$ our bijection: $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](H_X, F) \to FX \to GX =$ our bijection $\circ\, (\alpha \circ (-)) : [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](H_X, F) \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](H_X, G) \to GX$ (naturality in F), and for $f : X \to Y$, $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](H_Y, F) \to FY \xrightarrow{Ff} FX = [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](H_y, F) \xrightarrow{(-)\circ H_f} [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](H_X, F) \to FX$ (naturality in X).

The proof here is notationally complicated, but requires little thought - at every stage we do "the only thing we can".

a) Given $\alpha : H_V \Rightarrow F$, define $\widehat{\alpha} \in FV$ by $\widehat{\alpha} = \alpha_V(\mathrm{id}_V)$

b) Given $a \in FV$ define $\widehat{a} : H_V \Rightarrow F$ with components $(\widehat{a})_X : H_V X = \mathcal{C}(X, V) \to FX\ f \mapsto (Ff)(a)$. We must check naturality of $\widehat{a}$: given $f : X \to Y$ we need that

$$\begin{array}{ccc} \mathcal{C}(Y, V) & \xrightarrow{\widehat{a}_Y} & FY \\ H_V f = (-) \circ f \downarrow & & \downarrow Ff \\ \mathcal{C}(X, V) & \xrightarrow{\widehat{a}_X} & FX \end{array}$$

commutes. On elements, going right and then down we have $g \mapsto (Fg)(a) \mapsto Ff(Fg(a))$, and going down and then right $g \mapsto g \circ f \mapsto F^{(}g \circ f)(a)$. But $Ff(Fg(a)) = (Ff \circ Fg)(a) = F(g \circ f)(a)$ by functoriality of F.

c) We must check that $\left(\widehat{\widehat{-}}\right) = (-)$. Given $a \in FV$, $\widehat{\widehat{a}} = \widehat{a}_V(\mathrm{id}_V) = (F\mathrm{id}_V)(a) = (\mathrm{id}_F V)(a) = a$. Given $\alpha : H_V \Rightarrow F$, we have $\widehat{\widehat{\alpha}}_X : C(X,V) \to FX$ $f \mapsto (Ff)(\widehat{\alpha}) = (Ff)(\alpha_V(\mathrm{id}_V))$. So we need to check $(Ff)(\alpha_V(\mathrm{id}_V)) = \alpha_X(f)$. By naturality of α the square

$$\begin{array}{ccc} C(V,V) & \xrightarrow{\alpha_V} & FV \\ H_V f = (-) \circ f \downarrow & & \downarrow Ff \\ C(X,V) & \xrightarrow{\alpha_X} & FX \end{array}$$

commutes. In particular, applying this to id_V, going right and then down $\mathrm{id}_V \mapsto \alpha_V(\mathrm{id}_V) \mapsto Ff(\alpha_V(\mathrm{id}_V))$ and going down then right $\mathrm{id}_V \mapsto \mathrm{id}_V \circ f = f \mapsto \alpha_X(f)$, i.e. $\alpha_X = \widehat{\widehat{\alpha}}_X$, so $\alpha = \widehat{\widehat{\alpha}}$ as required.

d) Check naturality in F, i.e. that given $\beta : F \Rightarrow G : C^{\mathrm{op}} \to \mathrm{Set}$, the square

$$\begin{array}{ccc} [C^{\mathrm{op}}, \mathrm{Set}]\,(H_V, F) & \xrightarrow{\widehat{(-)}} & FV \\ \beta \circ (-) \downarrow & & \downarrow \beta_V \\ [C^{\mathrm{op}}, \mathrm{Set}]\,(H_V, G) & \xrightarrow{\widehat{(-)}} & GV \end{array}$$

commutes. On elements these maps are $\alpha \mapsto \alpha_V(\mathrm{id}_V) \mapsto \beta_V(\alpha_V(\mathrm{id}_V))$ and $\alpha \mapsto \beta \circ \alpha \mapsto (\beta \circ \alpha)_V(\mathrm{id}_V)$, but $\beta_V(\alpha_V(\mathrm{id}_V)) = (\beta_V \circ \alpha_V)(\mathrm{id}_V) = (\beta \circ \alpha)_V(\mathrm{id}_V)$ by the definition of $\beta \circ \alpha$.

e) Check naturality in V, i.e. given $h : V \to W$ in C, the square

$$\begin{array}{ccc} [C^{\mathrm{op}}, \mathrm{Set}]\,(H_W, F) & \xrightarrow{\widehat{(-)}} & FW \\ (-) \circ H_h \downarrow & & \downarrow Fh \\ [C^{\mathrm{op}}, \mathrm{Set}]\,(H_V, F) & \xrightarrow{\widehat{(-)}} & FV \end{array}$$

commutes. It is easier to work with the square

$$\begin{array}{ccc} [C^{\mathrm{op}}, \mathrm{Set}]\,(H_W, F) & \xleftarrow{\widehat{(-)}} & FW \\ (-) \circ H_h \downarrow & & \downarrow Fh \\ [C^{\mathrm{op}}, \mathrm{Set}]\,(H_V, F) & \xleftarrow{\widehat{(-)}} & FV \end{array},$$

which we may do since we know we can "turn around" isomorphisms in commuting diagrams. Applied to elements, going left and then down gives $a \mapsto \widehat{a} \mapsto \widehat{a} \circ H_h$, while going down and then left gives $a \mapsto (Fh)(a) \mapsto \widehat{Fh(a)}$. These natural transformations have components $(\widehat{a} \circ H_h)_X : C(X,V) \xrightarrow{H_h(X) = h \circ (-)} C(X,W) \xrightarrow{\widehat{a}} FX$ $f \mapsto h \circ h \mapsto F(h \circ f)(a)$ and $(\widehat{Fh(a)})_X : C(X,V) \to FX$ $f \mapsto Ff(Fh(a))$, but as above, $Ff(Fh(a)) = F(h \circ f)(a)$.

2.6′ Theorem (The dual Yoneda lemma)

For C locally small, $F : C \to \mathrm{Set}$, there is an isomorphism $[C, \mathrm{Set}](H^V, F) \cong FV$ naturally in F, V.

2.7 Corollary

The Yoneda embedding $H_\bullet : C \to [C^{\mathrm{op}}, \mathrm{Set}]$ is full and faithful: we need to show that $(\star)$ $C(V,W) \to [C^{\mathrm{op}}, \mathrm{Set}](H_V, H_W)$ $f \mapsto H_f$ are isomorphisms. By the Yoneda lemma, we have isomorphisms $(\dagger)$ $[C^{\mathrm{op}}, \mathrm{Set}](H_V, H_{)} \xrightarrow{\widehat{(-)}} H_W V = C(V,W)$; we need to check that $(\star), (\dagger)$ are inverse to each other (i.e. are the same isomorphism), i.e. that $\widehat{f} = H_f$ or equally well that $\widehat{H_f}/f$. But $\widehat{H_f} = (H_f)_V(\mathrm{id}_V)$ where $(H_f)_V : C(V,V) \to C(V,W)$ is given by $g \mapsto f \circ g$. So in particular $(H_f)_V(\mathrm{id}_V) = f \circ \mathrm{id}_V = f$ as required.

2.8 Corollary

We have $V \cong W$ in $\mathcal{C} \Leftarrow \mathcal{C}(X,V) \cong \mathcal{C}(X,W)$ naturally in X $\forall X \Leftarrow \mathcal{C}(V,X \cong \mathcal{C}(W,X)$ naturally in X $\forall$X: $H_\bullet$ is full and faithful so $V \cong W$ iff $H_V \cong H_W$ (see question 1 on the first example sheet) i.e. $\mathcal{C}(X,V) \cong \mathcal{C}(X,W)$ naturally in X; likewise $H^\bullet$ is full and faithful so $V \cong W$ iff $H^V \cong H^W$ i.e. $\mathcal{C}(V,X) \cong \mathcal{C}(W,X)$ naturally in X.

Suppose now we are given a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{I} \to \mathrm{Set}$. For each $I \in \mathcal{I}$ we have a functor $F(-,I) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ by $X \mapsto F(X,I)$, $f : X \to Y \mapsto F(f,\mathrm{id}_I) : F(Y,I) \to F(X,I)$. By a paramaterised representation of F, we mean: for each $I \in \mathcal{I}$ an object $V_I \in \mathcal{C}$ and an isomorphism $\theta_I : H_{V_I} \to F(-,I)$ (i.e. an ordinary representation for each $F(-,I)$ such that the assignation $I \mapsto V_I$ is a functor $\mathcal{I} \to \mathcal{C}$ in a nice way:

2.9 Proposition (Paramaterised Representability)

Suppose we have $\mathcal{C}$ locally small, $\mathcal{I}$ a category, $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{I} \to \mathrm{Set}$, and are given a paramaterised representatin for F, i.e. for each $I \in \mathcal{I}$ a representation $\theta_I : H_{V_I} \xrightarrow{\sim} F(-,I) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. Then there is a unique way of extending the assignation $I \mapsto V_I$ to a functor $V : \mathcal{I} \to \mathcal{C}$ such that the isomorphisms $(\theta_I)_X : \mathcal{C}(X,V_i) \to F(X,I)$ are natural in I as well as X. Naturality means that, for $f : I \to V$ in $\mathcal{I}$,
$$\begin{array}{ccc} H_{V_I} & \overset{\theta_I}{\Rightarrow} & F(-,I) \\ H_{Vf} \Downarrow & & \Downarrow F(-,f) \\ H_{V_J} & \overset{\theta_J}{\Rightarrow} & F(-,J) \end{array}$$
commutes. In fact, this determines what $Vf : V_I \to V_J$ must be. Note that the horizontal arrows are isomorphisms, so given the arrow on the right hand side there is a unique map on the left hand side $H_{V_I} \to H_{V_J}$ making the square commute. But because the Yoneda embedding $H_\bullet$ is full and faithful, this map is induced by a unique map $V_I \to V_J$; call this map V_f. It remains to check that $f \mapsto V_f$ is functorial: a) $V(\mathrm{id}_I) = \mathrm{id}_{V_I}$ (for $I \in \mathcal{I}$): consider the diagram
$$\begin{array}{ccc} H_{V_I} & \overset{\theta_I}{\Rightarrow} & F(-,I) \\ \mathrm{id} = H_{\mathrm{id}_{VI}} \Downarrow & & \Downarrow F(-,\mathrm{id}_I) = \mathrm{id} \\ H_{VI} & \Rightarrow & F(-,I) \end{array}.$$
This commutes, but by definition $H_{V(\mathrm{id}_I)}$ is the unique map which when placed on the left makes this square commute. So $H_{V(\mathrm{id}_I)} = H_{\mathrm{id}_{VI}}$. But because $H_\bullet$ is full and faithful this implies $V(\mathrm{id}_I) = \mathrm{id}_{VI}$. b) We need to check $V(g \circ f) = Vg \circ Vf$. Consider the diagram
$$\begin{array}{ccc} H_{V_I} & \overset{\theta_I}{\to} & F(-,I) \\ H_{Vf} \downarrow & & \downarrow F(-,f) \\ H_{V_J} & \overset{\theta_J}{\to} & F(-,J) \\ H_{Vg} \downarrow & & \downarrow F(-,g) \\ H_{V_K} & \overset{\theta_K}{\to} & F(-,K) \end{array}.$$
The two small squares commute, so the large rectangle commutes. The composite down the right is $F(-,g) \circ F(-,f) = F(-,g \circ f)$, so by definition $H_{V(g \circ f)}$ is the unique map down the left which makes the rectangle commute, i.e. $H_{V(g \circ f)} = H_{Vg} \circ H_{Vf} = H_{Vg \circ Vf}$. Since $H_\bullet$ is full and faithful, this implies $V(g \circ f) = Vg \circ Vf$ as required.

3 Limits and Colimits

3.1 Definition

Let $F : \mathcal{I} \to \mathcal{C}$ be a functor. A <u>cone over F</u> is given by: an object $X \in \mathcal{C}$, and for every $I \in \mathcal{I}$, a map $\alpha_I : X \to FI$ such that for all $f : I \to J$ in $\mathcal{I}$, we have a commuting triangle $Ff \circ \alpha_I : X \to FI \to FJ = \alpha_J : X \to FJ$. The picture is that we have arrows from $X \in \mathcal{C}$ down to all points of the image $F(\mathcal{I})$ in $\mathcal{C}$, hence the term cone. We call X the <u>vertex</u> of the cone.

3.2 Remark

A cone over F is really a special kind of natural transformation: given $X \in \mathcal{C}$, we have the constant functor $\Delta_X : \mathcal{I} \to \mathcal{C}\ I \mapsto X\ f \mapsto \mathrm{id}_X$. A cone over F with vertex X is then the same as a natural transformation $\alpha : \Delta_X \Rightarrow F$ (exercise).

3.3 Definition

This is the heart of this chapter: Given $F : \mathcal{I} \to \mathcal{C}$, a <u>limit</u> for F is a "universal cone over F", i.e. a cone $(p_I : V \to FI)_{I \in \mathcal{I}}$ such that for any other cone $(\alpha_I : X \to FI)_{I \in \mathcal{I}}$, there is a unique map $k : X \to V$ such that $\rho_I k = \alpha_I \forall I \in \mathcal{I}$.

3.4 Example

Let $\mathcal{I}$ be the discrete category with two objects. A functor $F : \mathcal{I} \to \mathcal{C}$ is a pair of objects $A, B \in \mathcal{C}$. A cone over F is any X and pair of arrows $f : X \to A, g : X \to B$. A limit (universal cone overe F) is an object we shall call $A \times X$ and a pair of maps $\pi_1 : A \times B \to A, \pi_2 : A \times B \to B$ such that given X and a pair of arrows $f : X \to A, g : X \to B$, there is a unique $(f, g) : X \to A \times B$ such that things commute: $f = \pi_1 \circ (f, g), g = \pi_2 \circ (f, g)$.

E.g. in <u>Set</u>, the product of sets A, B is their cartesian product. (Our requirement that the map is unique is equivalent to the fact that if $A \times B = C \times D$ as sets then $A = C, B = D$ or $A = D, B = C$). In <u>Grp</u>, the product of groups A, B is their cartesian product with pointwise operations. In <u>Top</u>, the product of spaces A, B is their cartesian product with product topology.

More generally, let $\mathcal{I}$ be any small discrete category. Then a functor $\mathcal{I} \to \mathcal{C}$ is an $(\mathrm{ob}\mathcal{I})$-indexed set of objects of $\mathcal{C}$. A limit for this functor is a product $\prod_{i \in \mathcal{I}} FI$ and projections $\pi_I, \pi_J, \pi_K, \ldots$. In particular, if $\mathcal{I}$ is the empty category there is a unique functor $\mathcal{I} \to \mathcal{C}$; a cone over this functor is just an object X of $\mathcal{C}$. A limit is an object $1 \in \mathcal{C}$ such that for any other $X \in \mathcal{C}$ there is a unique map $X \to 1$. We call such a 1 a <u>terminal object</u> of $\mathcal{C}$.

E.g. any 1-element set is terminal in Set, any 1-element group is terminal in Grp, any 1-element space is termital in Top. In the slice category $\frac{\mathcal{C}}{I}$, the terminal object is $(I, \mathrm{id}_I : I \to I)$.

Recall we are studying limits - "universal cones" over functors.

3.5 Example

Suppose $\mathcal{I}$ is the category $\bullet \rightrightarrows \bullet$. A functor $F : \mathcal{I} \to \mathcal{C}$ is a diagram $A\bullet \rightrightarrows^f_g \bullet B$ in $\mathcal{C}$; a cone over f is an X with arrows $e : X \to A, z : X \to B$ such that $z = fe = ge$. Note that this condition completely determines z, so more simply a cone over F is X and $k : X \to A$ such that $fk = gk$. A limit for F is $Eq(f, g) \xrightarrow{e} A \rightrightarrows B$ such that given X and $k : X \to A$, there is a unique map $l : X \to Eq(f, g)$ such that $k = el$. We call $Eq(f, g)$ the equalizer of f and g. E.g. in Set, the equalizer of $A \rightrightarrows^f_g B$ is the set $\{a \in A : f(a) = g(a)\}$ (with the map e being inclusion into A). We will only describe what these limits mean in Set, but this should give the correct intuition to apply to other categories.

3.6 Example

Let $\mathcal{I}$ be the category $\bullet \to \bullet \leftarrow \bullet$. A functor $F : \mathcal{I} \to \mathcal{C}$ is a diagram $A \xrightarrow{f} C \xleftarrow{g} B$. A cone over F is X and maps $h : X \to B, k : X \to A$; really there is also a map $l : X \to C$, but $l = gk = fh$ so we can omit it. A limit cone is a V and $p_1 : V \to A, p_2 : V \to B$ such that for any other cone there is a unique map $u : X \to V$ such that $p_1 \circ u = h, p_2 \circ u = k$. We call V the pullback of f along g (or of course equivalently of g along f). E.g. in Set, the pullback of $A \xrightarrow{f} C \xleftarrow{g} B$ is $\{(a, b) \in A \times B : f(a) = g(b) \text{ in } C\}$. This is also called the fibre product of A and B over C - the reason for this terminology will be seen on the second example sheet.

3.7 Proposition

Let $F : \mathcal{I} \to \mathcal{C}$. Suppose that $(p_I : V \to F_I)_{I\in\mathcal{I}}$ and $(q_I : W \to FI)_{I\in\mathcal{I}}$ are both limit cones for F, then there is a unique isomorphism $\phi : V \to W$ which "commutes with the limit cones", i.e. $q_I\phi = p_I \forall I \in \mathcal{I}$: by the universal property of $(q_I : W \to FI)$, there is a unique map $\phi : V \to W$ such that $q_I\phi = p_I$. But by the same property of $(p_I : V \to FI)$ there is a unique map $\psi : W \to V$ such that $p_I\psi = q_I$. But now by the universal property of $(q_I : W \to FI)$ there is a unique map $\theta : W \to W$ such that $q_I\theta = q_I \forall I \in \mathcal{I}$. But both $\theta = \mathrm{id}_W$ and $\theta = \phi\psi$ have this property, so by uniqueness $\phi\psi = \mathrm{id}_W$; dually $\psi\phi = \mathrm{id}_V$ so ϕ is an isomorphism as required.

3.8 Notation

Consequently, we speak of the limit of $F : \mathcal{I} \to \mathcal{C}$ rather than a limit. We write it as $(\lim F \xrightarrow{p_I} FI)$ or $(\int_J F_J \xrightarrow{p_I} FI)$.

3.9 Definition

Given $\mathcal{I}, \mathcal{C}$ we say that $\mathcal{C}$ has limits of shape $\mathcal{I}$ if every functor $F : \mathcal{I} \to \mathcal{C}$ has a limit. We say that $\mathcal{C}$ is complete if it has limits of shape $\mathcal{I}$ for all small categories $\mathcal{I}$, and finitely complete if it has limits of shape $\mathcal{I}$ for all finite categories $\mathcal{I}$.

Limits as Representations

3.10 Definition

Let $F : \mathcal{I} \to \mathcal{C}$. The <u>cone functor</u> $\mathrm{Cone}(-, F)$ associated to F is $\mathrm{Cone}(-, F) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ given by $X \mapsto$ the set of cones $(X \xrightarrow{\alpha_I} FI)_{I \in \mathcal{I}}$ over F with vertex X, $f : X \to Y \mapsto$ the map $\mathrm{Cone}(Y, F) \to \mathrm{Cone}(X, F)$ given by $(Y \xrightarrow{\beta_I} FI) \mapsto (X \xrightarrow{\beta_I f} FI)$.

3.11 Proposition

Limits for $F : \mathcal{I} \to \mathcal{C}$ are the same as representations for $\mathrm{Cone}(-, F)$: recall a representation for $\mathrm{Cone}(-, F)$ is an object $V \in \mathcal{C}$ and a natural isomorphism $\theta : H_V \Rightarrow \mathrm{Cone}(-, F)$. By the Yoneda lemma we have $\widehat{\theta} \in \mathrm{Cone}(V, F)$ (i.e. a cone $(V \xrightarrow{p_I} FI)$) such that the natural isomorphism θ has components $\theta_X : \mathcal{C}(X, V) \to \mathrm{Cone}(X, F)$ $f \mapsto \mathrm{Cone}(f, F)(\widehat{\theta}) = (X \xrightarrow{p_I f} FI)$ for a unique $f : X \to V$ in $\mathcal{C}$ - i.e., $(V \xrightarrow{p_I} FI)$ is a limit cone for F. The converse is similar.

Note: we could use this result to show that H_V exists even when $\mathcal{C}$ is not necessarily locally small. But it is easier to just assume all categories are locally small; most of the interesting ones are.

3.12 Corollary (Paramaterised limits)

Suppose we are given $F : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ such that each functor $F(-, J) : \mathcal{I} \to \mathcal{C}$ has a limit $\int_I F(I, J)$ for each $J \in \mathcal{J}$. Then there is a unique way of extending the assignation $J \mapsto \int_I F(I, J)$ to a functor $\mathcal{J} \to \mathcal{C}$ so that the isomorphisms $\mathcal{C})X, \int_I F(I, J) \cong \mathrm{Cone}(X, F(-, J))$ are natural in both X and J: we just apply paramaterised representability (Proposition 2.9).

3.13 Application

Suppose $\mathcal{C}$ is locally small and has all limits of shape $\mathcal{I}$. Then the assignation $(F : \mathcal{I} \to \mathcal{C}) \mapsto \lim F$ underlies (i.e. can be extended to) a functor $\lim : [\mathcal{I}, \mathcal{C}] \to \mathcal{C}$: consider the functor $\mathrm{ev} : \mathcal{I} \times [\mathcal{I}, \mathcal{C}] \to \mathcal{C}$ $(I, F) \mapsto FI$. For each $F \in [\mathcal{I}, \mathcal{C}]$, the functor $\mathrm{ev}(-, F) = F : \mathcal{I} \to \mathcal{C}$ has a limit. Now apply Corollary 3.12.

3.14 Proposition

<u>Set</u> is complete (i.e. has all small limits): suppose we are given $F : \mathcal{I} \to \mathcal{C} = \mathrm{Set}$ (for $\mathcal{I}$ a small category). Recall that a limit for F is an object $\lim F \in \mathrm{Set}$ and a bijection $\mathrm{Set}(X, \lim F) \cong \mathrm{Cone}(X, F)$ natural in X. In particular $\mathrm{Set}(1, \lim F) \cong \mathrm{Cone}(1, F)$, but our LHS here is just $\cong \lim F$. So let's try and make the set $\mathrm{Cone}(1, F)$ into a limit for F: set $\lim F := \mathrm{Cone}(1, F) = \{(x_I)_{I \in \mathcal{I}} : x_I \in FI \forall I \in \mathrm{ob}\mathcal{I}, (Ff)(x_I) = x_J \forall f : I \to J \text{ in } \mathcal{I}\}$. We need a cone from $\lim F$ to F: take $(\lim F \xrightarrow{p_J} F_J)_{J \in \mathcal{I}}$ defined by $(x_I)_{I \in \mathcal{I}} \mapsto x_J$.

We claim this is a cone, i.e. that for any $f : J \to K$ in $\mathcal{I}$, the square

$$\begin{tikzcd} & \lim F \arrow[dl, "p_J"'] \arrow[dr, "p_K"] & \\ F_J \arrow[rr, "Ff"] & & FK \end{tikzcd}$$

commutes. On elements, $(x_I)_{I \in \mathcal{I}} \mapsto x_J \mapsto (Ff)(k_J)$ if we go left and right or $(x_I)_{I\in\mathcal{I}} \mapsto x_K$ if we go right immediately, but these are equal by the definition of elements of $\lim F$.

We claim this cone is universal, i.e. given another cone $(X \xrightarrow{\alpha_I} FI)_{I\in\mathcal{I}}$, there is a unique may $X \xrightarrow{k} \lim F$ such that $\alpha_I = p_I k(\star)$. It remains to show that this gives a well defined map $X \xrightarrow{k} \lim F$¡ i.e. that for $f : I \to J$ in $\mathcal{I}$¡ $(Ff)(\alpha_I(x)) = \alpha_J(x) \forall x \in X$. But this (just) says that

$$\begin{tikzcd} & X \arrow[dl, "\alpha_I"'] \arrow[dr, "\alpha_J"] & \\ FI \arrow[rr, "Ff"] & & FJ \end{tikzcd}$$

should commute, which it does because $(\alpha_I)_{I\in\mathcal{I}}$ is a cone.

3.15 Proposition

$\mathcal{C}$ has all products and equalizers if and only if $\mathcal{C}$ is complete; the reverse implication is trivial. For the forward, suppose we are given $F : \mathcal{I} \to \mathcal{C}$. We form the product $(P = \prod_{I\in\mathrm{ob}\mathcal{I}} FI \xrightarrow{p_I} FI)_{I\in\mathcal{I}}$. Now we form the product $(Q = \prod_{g\in\mathrm{mor}\mathcal{I}} F(\mathrm{cod} g) \xrightarrow{q_f} F(\mathrm{cod} f))_{f\in\mathrm{mor}\mathcal{I}}$ ($\mathrm{cod} g$ being the codomain of g). These are two families of projections. We'll construct two maps $P \rightrightarrows^{\phi}_{\psi} Q$ of which $\lim F$ will be the equalizer.

For ϕ, we have a family of maps $(P = \prod_{I\in\mathcal{I}} FI \xrightarrow{p_{\mathrm{cod} f}} F(\mathrm{cod} f))_{f\in\mathrm{mor}\mathcal{I}}$. So by the universal property of Q, we induce a unique map $\phi : P \to Q$ such that (1) $q_f \phi = p_J \forall f : I \to J$ in $\mathcal{I}$. For ψ, we have a family of maps $(P = \prod_{I\in\mathcal{I}} FI \xrightarrow{p_{\mathrm{dom} f}} F(\mathrm{dom} f) \xrightarrow{Ff} F(\mathrm{cod} f))_{f\in\mathrm{mor}\mathcal{I}}$. So by the universal property of Q we induce a unique map $\psi : P \to Q$ such that (2) $q_f \circ \psi = Ff \circ p_I \forall f : I \to J$ in $\mathcal{I}$. Now we form the equalizer $E \xrightarrow{e} P \rightrightarrows^{\phi}_{\psi} Q$ of ϕ and ψ. Claim: we can make E into a limit for $F : \mathcal{I} \to \mathcal{C}$, i.e. we can find bijections natural in X $\mathcal{C}(X, E) \cong \mathrm{Cone}(X, F)$: note that by the universal property of the equalizer (E, e) we have bijections $\mathcal{C}(X, E) \to$ the set of maps $X \xrightarrow{k} P$ such that $\phi k = \psi k$, given by $f \mapsto ef$, natural in X. So it suffices to prove that there are bijections natural in X between the set of maps $X \xrightarrow{k} P$ with $\phi k = \psi k$ and $\mathrm{Cone}(X, F)$.

By the universal property of P, we have bijections, natural in X, between sets of maps $X \xrightarrow{k} P$ and sets of families of maps $(X \xrightarrow{\alpha_I} FI)_{I\in\mathcal{I}}$ by $k \mapsto (p_I k)_{I\in\mathcal{I}}$. So enough to prove that $k : X \to P$ has $\phi k = \psi k$ if and only if the corresponding $(X \xrightarrow{p_I k} FI)_{I\in\mathcal{I}}$ is a cone over F.

Suppose we are given $k : X \to P$. Observe, by the universal property of Q, there's a bijection between the set of maps $X \to Q$ and the set of families $(X \xrightarrow{\beta f} F(\mathrm{cod} f))_{f\in\mathrm{mor}\mathcal{I}}$ by $\iota \mapsto (q_f \iota)_{f\in\mathrm{mor}\mathcal{I}}$. Therefore $\phi k = \psi k : X \to Q$ if and only if $q_f \phi k = q_f \psi k$ for all $f \in \mathrm{mor}\mathcal{I}$. But by (1), $q_f \phi k = p_J k$ and my (2), $q_f \psi k = Ff \circ p_I k$, i.e. $\phi k = \psi k$ if and only if $\forall f : I \to J$ in $\mathcal{I}$, $p_J k = Ff(p_I k)$, i.e. the diagram

$$\begin{tikzcd} & X \arrow[dl, "p_I k"'] \arrow[dr, "p_J k"] & \\ FI \arrow[rr, "Ff"] & & FJ \end{tikzcd}$$

commutes, i.e. if and only if $(p_I k : X \to FI)_{I\in\mathcal{I}}$ is a cone over F, as required.

Colimits

Colimits in C are limits in C^{op}. In place of products we have coproducts $A + B$ (the disjoint union), in place of a pullback we have a pushout (for maps $C \to A, C \to B$ we have D and maps $A \to D, B \to D$), and in place of an equalizer we have a coequalizer (for $A \rightrightarrows B$, the coequalizer goes $B \to C$). Corresponding to a terminal object is an initial object, in place of $\lim F$ we write $\mathrm{colim} F$, and rather than $\int_I FI$ we write $\int^I FI$

Limits in Functor Categories

Suppose we're given some $F : \mathcal{I} \to [C, \mathcal{D}]$. For each $X \in C$, we have a functor $F(-)X : \mathcal{I} \to \mathcal{D}$ by $I \mapsto (FI)(X)$, and for each $f : X \to Y$ in C we have a natural transformation $F(-)f : F(-)X \Rightarrow F(-)Y$.

3.16 Proposition

Given $F : \mathcal{I} \to [C, \mathcal{D}]$ as above, if each $F(-)X : \mathcal{I} \to \mathcal{D}$ has a limit $(\int_I (FI)(X) \xrightarrow{(p^X)_I} (FI)(X))_{I \in \mathcal{I}}$ for all $X \in C$ then also $F : \mathcal{I} \to [C, \mathcal{D}]$ has a limit $(\int_I FI \overset{\phi_I}{\Rightarrow} FI)_{I \in \mathcal{I}}$ in $[C, \mathcal{D}]$ and this limit is computed pointwise i.e. $(\int_I FI)(X) = \int_I (FI)(X)$, $(\phi_I)_X : \int_I (FI)(X) \to (FI)(X) = p_I^X$: by paramaterised limits, the assignation $X \mapsto \int_I FI(X)$ induces a functor $\int_I (FI)()$: $C \to \mathcal{D}$ such that the maps $p_I^X : \int_I (FI)(X) \to (FI)(X)$ are natural in X, i.e. we have a family of natural transformations in $[C, \mathcal{D}]$ $(\int_I FI(-) \overset{p_I^{(-)}}{\Rightarrow} FI)_{I \in \mathcal{I}} (\star)$. We claim $(\star)$ is a cone over F in $[C, \mathcal{D}]$, i.e. that

$$\begin{tikzcd} & \int_I FI(-) \arrow[dl, "p_I^{(-)}"'] \arrow[dr, "p_J^{(-)}"] & \\ F_I \arrow[rr, Rightarrow, "Ff"] & & FJ \end{tikzcd}$$

(where all the arrows are natural transformations)

commutes in $[C, \mathcal{D}] \forall f : I \to J$ in $\mathcal{I}$, i.e. that

$$\begin{tikzcd} & \int_I (FI)(X) \arrow[dl, "p_I^X"'] \arrow[dr, "p_J^X"] & \\ FIX \arrow[rr, "(Ff)_X"] & & FJX \end{tikzcd} \quad (\dagger)$$

commutes $\forall f : I \to J$ in $\mathcal{I}$, $X \in C$. But $(\int_I (FI)(X) \xrightarrow{p_I^X} (FI)X)_{I \in \mathcal{I}}$ is a cone by assumption, so $(\dagger)$ commutes. We claim further than $(\star)$ is a universal cone: suppose we are given another cone $(K \overset{\alpha_I}{\Rightarrow} FI)_{I \in \mathcal{I}}$ in $[C, \mathcal{D}]$. Then for each $X \in C$, we obtain a cone $(KX \xrightarrow{(\alpha_I)_X} (FI)(X))_{I \in \mathcal{I}}$ in $\mathcal{D}$ over $F(-)(X)$. By the universal property of $\int_I (FX)(X)$ we induce a map $KX \xrightarrow{k_X} \int_I (FI)(X)$ such that $p_I^X \circ k_X = (\alpha_I)_X$ (1). We claim the maps $k_X : KX \to \int_I FI(X)$ are the components of a natural transformation $ku : K \Rightarrow \int_I FI(-)$. Note that then we have $p_I^{(-)} \circ k = \alpha_I : K \Rightarrow FI$ (2) by (1), and by the uniqueness of the factorizations (1), k is necessarily unique such that (2) holds. To prove the claim, we must show that squares like $\begin{array}{ccc} KX & \xrightarrow{k_X} & \int_I (FI)(X) \\ K_g \downarrow & & \downarrow \int_I (FI)(g) \\ KY & \xrightarrow{k_Y} & \int_I (FI)(Y) \end{array}$ (‡) commute $\forall g : X \to Y$ in C.

Now, $(\int_I (FI)(Y) \xrightarrow{p_I} (FI)(Y))_{I \in \mathcal{I}}$ is a limit for $F(-)Y$¡ so two maps $KX \rightrightarrows \int_I (FI)Y$ are the same iff their composites with p_I^Y are the same for all $I \in \mathcal{I}$, i.e. to check (‡) commutes, it suffices ot show that it does so when postcomposed with $p_I^Y : \int_I (FI)(X) \to (FI)(Y) \forall I \in \mathcal{I}$.

Now consider the diagram $\begin{array}{ccccc} KX & & \xrightarrow{(\alpha_I)_X} & & (FI)X \\ & \searrow^{k_X} & & \nearrow^{p_I^X} & \\ K_g \downarrow & & \int_I (FI)(X) & & \downarrow (FI)(g) \\ KY & & \xrightarrow{(\alpha_I)_Y} & & (FI)Y \\ & \searrow^{k_Y} & \downarrow \int_I (FI)(g) & \nearrow^{p_I^Y} & \\ & & \int_I (FI)(Y) & & \end{array}$ (visualize this as a 3d triangular-based prism). The back commutes by naturality of $\alpha_I : K \Rightarrow IF$, the top and bottom commute by (1), and the right commutes by naturality of $p_I^{(-)} : \int_I FI(-) \Rightarrow FI$. Hence going southeast, south, northeast gives the same result as going south, southeast, northeast, as required (southeast, south, notheast = southeast, northeast, south = east, south = south, east = south, southeast, northeast).

Interchange of Limits

Suppose we are given $F : \mathcal{I} \times \mathcal{J} \to C$. For each $J \in \mathcal{J}$ we have $F(-, J) : \mathcal{I} \to C$ and for each $f : J \to J'$ in $\mathcal{J}$ we have a natural transformation $F(-, f) : F(-, J) \Rightarrow F(-, J')$, so we have a functor $F(-, ?) : \mathcal{J} \to [\mathcal{I}, C]$ $J \mapsto F(-, J)$. By proposition 3.16, if each $F(I, ?) : \mathcal{J} \to C$ has a limit $\int_J F(I, J)$ then so does $F(-, ?)$ given by $\int_J F(-, J) : \mathcal{I} \to C$ $I \mapsto \int_J F(I, J)$. If this functor $\mathcal{I} \to C$ in turn has a limit, then we have an object $\int_I \int_J F(I, J) \in C$.

3.17 Proposition

With the above notation, $\int_I \int_J F(I, J)$ is a limit for $F : \mathcal{I} \times \mathcal{J} \to C$: we must exhibit a bijection $\text{Cone}(X, F) \cong C(X, \int_I \int_J F(I, J))$ natural in X. We calculate $\text{Cone}(X, F) = [\mathcal{I} \times \mathcal{J}, C](\Delta_X, F)$, the set of natural transformations from $\Delta_X : \mathcal{I} \times \mathcal{J} \to C$ $\bullet \mapsto X$ to F (by remark 3.2). This is $\cong [\mathcal{J}, [\mathcal{I}, C]](\Delta_{\Delta_X}, F(-, ?))$ (by calculation) $= \text{Cone}(\Delta_X, F(-, ?))$ the set of cones over $F(-, L) : \mathcal{J} \to [\mathcal{I}, C]$ (by remark 3.2) $\cong [\mathcal{I}, C](\Delta_X, \int_J F(-, J))$ (universal property of $\int_J F(-, J)$) $= \text{Cone}(X, \int_J F(-, J))$ (remark 3.2) $= CG(X, \int_I \int_J F(I, J))$ (universal property of $\int_I \int_J F(I, J)$). And everything is natural in X.

Recall proposition 3.17: if $f : \mathcal{I} \times \mathcal{J} \to C$ is such that $\int_J F(-, J) : \mathcal{I} \to C$ exists and also $\int_I \int_J F(i, J)$ exists, then in fact $\int_I \int_J F(I, J)$ is a limit for $F : \mathcal{I} \times \mathcal{J} \to C$.

3.18 Corollary (Fubini)

If $F : \mathcal{I} \times \mathcal{J} \to C$ is such that both $\int_J F(-, J) : \mathcal{I} \to C$ and $\int_I F(I, -) : \mathcal{J} \to C$ then we have $\int_I \int_J F(I, J) \equiv \int_J \int_I F(I, J)$, in the sense that one exists iff the other does and they have corresponding limit cones.

3.19 Proposition

Every presheaf $F : C^{\text{op}} \to \text{Set}$ is a colimit of a representable functor: we define the <u>category of elements of F</u> el$F$ with objects pairs $(X \in C, x \in FX)$ and morphisms $(X, x) \to (Y, y)$ maps $f : X \to Y$ in C such that $(Ff)(y) = x$.

Consider the functor $\text{el}F \xrightarrow{U} C \xrightarrow{H_\bullet} [C^{\text{op}}, \text{Set}]$ $(X, x) \mapsto X \mapsto H_X$. We will show that F is a colimit for $H_\bullet \circ U$. Recall the Yoneda isomorphisms $(\star)$ $[C^{\text{op}}, \text{Set}](H_V, F) \cong FV$ naturally in V and F. For every $(X \in C, x \in FX)$ we have a map $\widehat{x} : H_X \to F$ in $[C^{\text{op}}, \text{Set}]$. Claim: $(\widehat{x} : H_X \to F)_{(X,x) \in \text{el}F}$ is a cocone from $H_\bullet \circ U$ to F: we need to show that for $f : (X, x) \to (Y, y)$ in el$F$,
$$\begin{tikzcd} H_X \arrow[rr, "H_f"] \arrow[dr, "\widehat{x}"'] & & H_y \arrow[dl, "\widehat{y}"] \\ & F & \end{tikzcd}$$
commutes. But $\widehat{y} \circ H_f = \widehat{(Ff)(y)}$ (by naturality of $(\star)$ in V) $= \widehat{x}$ (because $f : (X, x) \to (Y, y)$ is a map in el$F$). Claim: $(\widehat{x} : H_X \to F)_{(X,x) \in \text{el}F}$ is a universal cocone under $H_\bullet \circ U$: suppose we are given another cocone, i.e. $(\alpha_{(X,x)} : H_X \to G)_{(X,x) \in \text{el}F}$ such that (‡)
$$\begin{tikzcd} H_X \arrow[rr, "H_f"] \arrow[dr, "\alpha_{(X,x)}"'] & & H_Y \arrow[dl, "\alpha_{(Y,y)}"] \\ & G & \end{tikzcd}$$
commutes $\forall f : (X, x) \to (Y, y)$ in el$F$. By naturality of $(\star)$ in F, if we are given $\widehat{x} : H_X \to F$ then $k \circ \widehat{x} : H_X \to G = \widehat{k_X(x)} : H_X \to G$. Hence $k_X(x) = \widehat{\widehat{k_X(x)}} = \widehat{\alpha_X(x)}$. So k, if it exists, must have $k_X : FX \to GX$ given by $x \mapsto \widehat{\alpha_{(X,x)}}$ (†). It remains only to check that this definition works, i.e. that (†) defines a natural transformation, i.e.
$$\begin{tikzcd} FY \arrow[r, "k_Y"] \arrow[d, "Ff"'] & GY \arrow[d, "Gf"] \\ FX \arrow[r, "k_X"] & GX \end{tikzcd}$$
commutes for all $f : X \to Y$ in C. On elements, going right and then down $y \mapsto \widehat{\alpha_{(Y,y)}} \mapsto (Gf)(\widehat{\alpha_{(Y,y)}})$ and going down then right $y \mapsto (Ff)(y) \mapsto \widehat{\alpha_{(X,(Ff)(y))}}$. But $(Gf)(\widehat{\alpha_{(Y,y)}}) = \widehat{\alpha_{(Y,y)} \circ H_f}$ by naturality of $(\star)$ in V. We have a map $f : (X, (Ff)(y)) \to (Y, y)$ in el$F$, so by (‡) $\alpha_{(Y,y)} \circ H_f = \alpha_{(X,(Ff)(y))}$ $\therefore$ $(Gf)(\widehat{\alpha_{(Y,y)}}) = \widehat{\alpha_{(Y,y)} \circ H_f} = \widehat{\alpha_{(X,(Ff)(y)}}$ as required.

3.20 Definition

Let $F : \mathcal{I} \to C, G : C \to \mathcal{D}$. We say that:

G <u>preserves the limit of F</u> if whenever $(V \xrightarrow{p_I} FI)_{I \in \mathcal{I}}$ is a limit for F, also $(GV \xrightarrow{Gp_I} GFI)_{I \in \mathcal{I}}$ is a limit for GF; we have the obvious definitions for "G preserves limits of shape $\mathcal{I}$" and "G preserves (all) limits".

G <u>reflects the limit of F</u> if, whenever $(X \xrightarrow{\alpha_I} FI)_{I \in \mathcal{I}}$ is a cone over F and $(GX \xrightarrow{G\alpha_I}$

$GFI)_{I\in\mathcal{I}}$ is a limit cone for GF, then also $(X \xrightarrow{\alpha_I} FI)$ is a limit for F.

G creates the limit of F if (GF has a limit) $\Rightarrow$ (F has a limit which is preserved and reflected by G) (previous, uglier definition: it reflects the limit of F and also, whenever $(V \xrightarrow{p_I} GFI)_{I\in\mathcal{I}}$ is a limit cone for GF, we can find a cone $(W \xrightarrow{q_I} FI)_{I\in\mathcal{I}}$ which is a limit cone and an isomorphism $\phi : GW \cong V$ commuting with the projections, i.e.

$$\begin{tikzcd} GW \arrow[d, "\phi"'] \arrow[dr, "Gq_I"] & \\ V \arrow[r, "p_I"'] & GFI \end{tikzcd}$$

commutes $\forall I \in \mathcal{I}$). This definition is weaker than that in McClean, but that definition is morally wrong.

3.21 Proposition

Representable functors preserve limits: suppose we are given $F : \mathcal{I} \to C$, $C(X,-) : C \to \mathrm{Set}$. We need to show that if $(\int_I FI \xrightarrow{p_I} FI)_{i\in\mathcal{I}}$ is a limit cone for F, then also $(\star)$ $(C(X, \int_I FI) \xrightarrow{p_I\circ(-)} C(X, FI))_{I\in\mathcal{I}}$ is a limit for $C(X, F-) : \mathcal{I} \to \mathrm{Set}$. We know Set is complete, and so $C(X, F-)$ has a limit $(\int_I C(X, FI) \xrightarrow{q_I} C(X, FI))_{I\in\mathcal{I}}$, and so to show $(\star)$ is a limit it suffices to find an isomorphism $\phi : C(X, \int_I FI) \to \int_I C(X, FI)$ such that $(\dagger)$

$$\begin{tikzcd} C(X, \int_I FI) \arrow[rr, "\phi"] \arrow[dr, "p_I\circ(-)"'] & & \int_I C(X, FI) \arrow[dl, "q_I"] \\ & C(X, FI) & \end{tikzcd}$$

commutes $\forall I \in \mathcal{I}$.

Now $\int_I C(X, FI) = \{(\alpha_I)_{I\in\mathcal{I}} | \alpha_I \in C(X, FI) \forall I \in \mathcal{I}, C(X, Ff)(\alpha_I) = \alpha_J \forall f : I \to J \in \mathcal{I}$ i.e. $Ff \circ \alpha_I = \alpha_J\} = \mathrm{Cone}(X, F)$, and $q_I : \int_I C(X, FI) \to C(X, IF)$ $(\alpha_I)_{I\in\mathcal{I}} \mapsto \alpha_I$. So now, because $\int_I FI$ is a limit, the map $\phi : C(X, \int_I FI) \to \int_I C(X, FI) = \mathrm{Cone}(X, F)$ $k \mapsto (p_I k)_{I\in\mathcal{I}}$ is an isomorphism, and moreover it makes $(\dagger)$ commute, because on elements, going right and then down $k \mapsto (p_I k)_{I\in\mathcal{I}} \mapsto p_I k$, while going immediately down we just have $k \mapsto p_I k$.

3.22 Corollary

Full and faithful functors reflect limits: let $F : \mathcal{I} \to C$, (and let) $G : C \to \mathcal{D}$ be full and faithful. Suppose we are given a cone $(V \xrightarrow{p_I} FI)_{I\in\mathcal{I}}$ over F such that $(GV \xrightarrow{Gp_I} GFI)$ is a limit cone. We must show that $(p_I)_{I\in\mathcal{I}}$ is also a limit cone, i.e. that $C(X, V) \to \mathrm{Cone}(X, F)$ $k \mapsto (p_I k)$ is an isomorphism $\forall X \in C$. Consider

$$\begin{tikzcd} C(X,V) \arrow[r, "p_I\circ(-)"] \arrow[d, "G\ (1)"'] & \mathrm{Cone}(X,F) \arrow[d, "(3)\ G"] \\ \mathcal{D}(GX,GV) \arrow[r, "Gp_I\circ(-)"', "(2)"] & \mathrm{Cone}(GX,GF) \end{tikzcd}$$

; this commutes, because on elements, going right and then down $k \mapsto (p_I \circ k)_{I\in\mathcal{I}} \mapsto (G(p_I \circ k))_{I\in\mathcal{I}}$, while going down and then right $k \mapsto Gk \mapsto (Gp_I \circ G_k)_{I\in\mathcal{I}}$, and these are equal by functoriality of G. Note (1) is an isomorphism because G is full and faithful, (2) is an isomorphism because $(GV \xrightarrow{Gp_I} GFI)_{I\in\mathcal{I}}$ is a limit cone, and (3) is an isomorphism because we have $\mathcal{C}(X, FI) \cong \mathcal{D}(GX, GFI)$ naturally in I, i.e. $\mathcal{C}(X, F-) \cong \mathcal{D}(GX, GF-)$ in $[\mathcal{I}, \mathrm{Set}]$ so $\mathrm{Cone}(X, F) = \int_I \mathcal{C}(X, FI) \cong \int_I \mathcal{D}(GX, GFI) = \mathrm{Cone}(GX, GF)$, and (3) is an isomorphism as required.

3.23 Corollary

The Yoneda embedding $H_\bullet : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ preserves limits: let $F : \mathcal{I} \to \mathcal{C}$ and suppose $(V \xrightarrow{p_I} FI)_{I \in \mathcal{I}}$ is a limit cone for F. We need to prove that $(H_V \xrightarrow{H_{p_I}} H_{FI})_{I \in \mathcal{I}}$ is a limit cone for $H_{F(-)} : \mathcal{I} \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$. Because Set is complete, limits in $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ are computed pointwise, so it suffices to show that $(H_V X \xrightarrow{(H_{p_I})_X} H_{FI}X)_{I \in \mathcal{I}}$ is a limit cone in Set $\forall X \in \mathcal{C}$, i.e. that $(\mathcal{C}(X, V) \xrightarrow{p_I \circ (-)} \mathcal{C}(X, FI))_{I \in \mathcal{I}}$ is a limit cone. But $\mathcal{C}(X, -)$ preserves limits, so we have the result.

3.24 Remark

In proposition 3.21, we saw that $\mathcal{C}(X, -) : \mathcal{C} \to \mathrm{Set}$ preserves limits. The dual of this statement is less obvious than dualisations usually are: it says that $\mathcal{C}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ preserves limits, i.e. sends limits in $\mathcal{C}^{\mathrm{op}}$ (colimits in $\mathcal{C}$) to limits in Set, that is $\mathcal{C}(\int^I FI, X) \cong \int_I \mathcal{C}(FI, X)$.

4 Adjunctions

This is the fun part of the course. Where I use "fun" in a particularly specialised sense.

4.1 Definition

Let $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$. An adjunction $F \dashv G$ is given by isomorphisms $\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY)$ natural in X and Y. We say that F is left adjoint to G or that G is right adjoint to F. A common case is if G forgets some structure and F freely adds it back in:

4.2 Examples

Free $\dashv$ Forgetful:

$U : \mathrm{Alg}_k \to \mathrm{Vect}_k$ forgetting the multiplicative structure has a left adjoint $F : \mathrm{Vect}_k \to \mathrm{Alg}_k$ by $V \mapsto$ the tensor algebra on V.

$U : \mathrm{Grp} \to \mathrm{Set}$ forgetful has a left adjoint $X \mapsto$ the free group on X.

$U : \mathrm{Ab} \to \mathrm{Grp}$ has a left adjoint "abelianisation" $G \mapsto \frac{G}{[G,G]}$.

$U : \mathrm{Fld} \to \mathrm{Dom}_i$ from the category of fields to that of integral domains and injective homomorphisms has a left adjoint $R \mapsto$ the field of fractions of R.

$U : \mathrm{Alg}_k \to \mathrm{Lie}_k$ has a left adjoint "universal enveloping algebra".

Let $\mathcal{H} \subset \mathcal{G}$ be groups viewed as one-object categories. Recall that $[\mathcal{G}, \mathrm{Vect}_k]$ is the category of k-linear representations of $\mathcal{G}$. We have a functor "restriction" $[\mathcal{G}, \mathrm{Vect}_k] \to [\mathcal{H}, \mathrm{Vect}_k]$ $L \mapsto L I_{\mathcal{H}}$. This has a left adjoint "induction".

$\mathrm{ob} : \mathrm{Cat} \to \mathrm{Set}$ has a left adjoint $\mathrm{Set} \to \mathrm{Cat}$ $X \mapsto$ the discrete category on X.

$U : \mathrm{Top} \to \mathrm{Set}$ has a left adjoint $\mathrm{Set} \to \mathrm{Top}$ $X \mapsto X$ with the discrete topology.

"Forgetful $\dashv$ cofree":

(The cofree structure can be seen as the "most restricted" structure we can put on an object to make it another kind of object, in the same way the free structure is the "least restricted")

$U : \mathrm{Top} \to \mathrm{Set}$ has a right adjoint $\mathrm{Set} \to \mathrm{Top}$ by $X \mapsto X$ with the indiscrete topology.

$U : \mathrm{Cat} \to \mathrm{Set}$ has a right adjoint $\mathrm{Set} \to \mathrm{Cat}$ $X \mapsto$ the indiscrete category on X - the object set is X and there is exactly one morphism between any two objects (This is also called the chaotic category on X).

$U : \mathrm{Grp} \to \mathrm{Mon}$ has a right adjoint $\mathrm{Mon} \to \mathrm{Grp}$ $M \mapsto \{m \in M : m \text{ invertible}\}$.

"(Tensor) Product $\dashv$ function space" (we'll see more of this later)

For $A \in \mathrm{Set}$, $(-) \times A : \mathrm{Set} \to \mathrm{Set}$ has a right adjoint $(-)^A : \mathrm{Set} \to \mathrm{Set}$ $X \mapsto X^A$ (recall this is the set of functions $A \to X$) (i.e. $\mathrm{Set}(X \times A, Y) \cong \mathrm{Set}(X, Y^A)$).

For $A \in \mathrm{Top}$ locally compact, $(-) \times A \dashv (-)^A : \mathrm{Top} \to \mathrm{Top}$ $X \mapsto X^A$ with the compact open topology (this definition doesn't work for general A)

For $C \in \mathrm{Cat}$, $(-) \times C \dashv [C, -] : \mathrm{Cat} \to \mathrm{Cat}$

For $V \in \mathrm{Vect}_k$, $(-) \otimes V \dashv \mathrm{Hom}_k(V, -) : \mathrm{Vect}_k \to \mathrm{Vect}_k$.

4.3 Example

Consider $\mathcal{C}(V, -) : \mathcal{C} \to \mathrm{Set}$. Suppose $\mathcal{C}$ has coproducts. Then $\mathcal{C}(V, -)$ has a left adjoint $\mathrm{Set} \to \mathcal{C}$ $X \mapsto \amalg_{x \in X} V$ (where $\amalg$ denotes the coproduct). Indeed, we have $\mathcal{C}(\amalg_{x \in X} V, Y) \cong \prod_{x \in X} \mathcal{C}(V, Y) \cong \mathcal{C}(V, Y)^X = \mathrm{Set}(X, \mathcal{C}(V, Y))$. Similarly $\mathcal{C}(-, V) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ has a left adjoint whenever $\mathcal{C}$ has products, given by $\mathrm{Set} \to \mathcal{C}^{\mathrm{op}}$ $X \mapsto \prod_{x \in X} V$ (this is the product in $\mathcal{C}$, i.e. the coproduct in $\mathcal{C}^{\mathrm{op}}$.

I realised sometime after this lecture that I have in fact not been following this course for some time; however, the next definition in the course is especially confusing. Therefore, given that I am no longer attending lectures, this seems the correct place at which to end these notes. My apologies to any readers hoping to find out how it ends.