Algebraic Topology

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December 22, 2008

1 Introduction

We shall be somewhat informal in this lecture, but anything not rigorous will be revisited later.

Algebraic topology is the study of the "connectivity" properties of topological spaces.

Recall that a topological space is a set X with a preferred collection of subsets, the open sets, such that arbitrary unions of opens sets are open, finite intersections of open sets are open, and $\emptyset, X$ are open. This gives a notion of closeness without requiring a notion of distance - points are close if they tend to be in the same open sets.

A map of spaces $f : X \to Y$ is continuous (cts) if $f^{-1}(U) \subset X$ is open for every $U \subset Y$ open. We assume the reader knows the meanings of the terms compact, connected, and Hausdorff.

Recall that X is connected if we cannot write $X = U \cup V$ for U, V disjoint, open, and both nonempty. For example $\mathbb{R}$ is connected, but $\mathbb{R} \setminus \{0\}$ is not ($\mathbb{R}$ and any similar spaces have the usual Euclidean topology unless otherwise stated).

Corollary: Intermediate Value Theorem: if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) > 0, f(y) < 0$ then f vanishes somewhere: otherwise $f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$ would disconnect $\mathbb{R}$.

In this course we will concentrate on "reasonable" spaces; there is a subject, analytic topology, which works by finding spaces which are counterexamples to any obvious nice property a topological space should have, but this is a peversion and will not be discussed here. For reasonable spaces, connectedness is equivalent to path-connectedness; X is path-connected if $\forall x, y \in X \exists$ a continuous $\gamma : [0, 1] \to X$ with $\gamma(0) = x, \gamma(1) = y$. An equivalent, but philisophically important, way to see this, is that every two continuous maps from a point into X can be continuously deformed into one another.

To generalize this, we consider maps from other spaces into X.

Definition: X is simply connected if any two maps $S^1 \to X$ can be continuously deformed into one another.

Example: $\mathbb{R}^2$ is simply connected, but $\mathbb{R}^2 \setminus \{0\}$ is not. A map $\gamma : S^1 \to \mathbb{R}^2 \setminus \{0\}$ has a winding number $\deg(\gamma) \in \mathbb{Z}$, which we can define e.g. using complex analysis as $\deg(\gamma) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z}$. This is invariant under continuous deformation of γ, and the loop $\gamma_n : t \mapsto (\cos 2\pi nt, \sin 2\pi nt) = e^{2\pi int}$ has $\deg(\gamma_n) = n \in \mathbb{Z}$.

Corollary (Fundamental Theorem of Algebra)

If $f(z) = z^n + a_1z^{n-1} + \cdots + a_n$ is a non-constant polynomial, then it has a root: suppose in fact $f(z) \neq 0 \forall z$. Then let $\gamma_R(t) = f(Re^{2\pi it}) : S^1 \to \mathbb{R}^2 \setminus \{0\}$. γ_0 is a constant map, $\deg(\gamma_0) = 0$. Now taking $R >> |a_1| + \cdots + |a_n|$, consider $\gamma_{R,s}(t) = (z^n + s(a_1z^{n-1} + \cdots + a_n)) |_{z=Re^{2\pi it}}$ for $0 \leq s \leq 1$. The condition on R implies $\gamma_{R,s}(t) \in \mathbb{R}^2 \setminus \{0\} \forall s, t$. The original map $\gamma_R(t) = \gamma_{R,1}(t)$, so we have $\deg(\gamma_R) = \deg(\gamma_0) = 0$ but also $\deg(\gamma_R) = \deg(\gamma_{R,1}) = \deg(\gamma_{R,0}) = \det(t \mapsto e^{2\pi int}) = n$, a contradiction.

Generalizing further, we have: Fact: All maps $S^n \to \mathbb{R}^{n+1}$ can be continuously deformed into one another, but maps $S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ have an integer-valued degree, with the degree of the constant map being 0 and that of the standard inclusion being 1.

Corollary (Brouwer fixed point theorem): If $D^n = \{x \in \mathbb{R}^n : ||x|| < 1\}$ is the closed disc, every continuous $f : D^n \to D^n$ has a fixed point, i.e. an x such that $f(x) = x$: suppose not (for contradiction). If $0 \leq R \leq 1$, let $\gamma_R : S^{n-1} \to S^{n-1}$ be defined by $v \mapsto \frac{Rv - f(Rv)}{||Rv - f(Rv)||}$; this is well defined. By construction, γ_0 is constant so of degree 0.

Also consider $\gamma_{1,s}(v) = \frac{v - sf(v)}{||v - sf(v)||}$, for $0 \leq s \leq 1$. This makes sense, since if $s < 1$, $|v| = 1 > |sf(v)|$. For $0 \leq s \leq 1$ this is again a continuous family of maps $S^{n-1} \to S^{n-1}$. But now at $s = 0$ we get $v \mapsto \frac{v}{||v||}$ i.e. the inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n$, of degree 1, a contradiction.

This next is the key definition of the course: Definition: two continuous maps $f, g : X \to Y$ are homotopic if $\exists$ continuous $F : X \times [0,1] \to Y$ s.t. $F(x, 0) = f(x), F(x, 1) = g(x)$. This is a formal way of saying f can be continuously deformed into g; we write $f \simeq g$.

Definition: Topological spaces X, Y are homotopy equivalent (or htpy equivalent) if there exist continuous maps $f : X \to Y, g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y, g \circ f \simeq \mathrm{id}_X$; this is a weakening of the notion of homeomorphism, where we would require $f \circ g = \mathrm{id}_Y, g \circ f = \mathrm{id}_X$

Example: $S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$ (but these spaces are clearly not homeomorphic e.g. by compactness). Explicitly, take $\iota : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}, g : \mathbb{R}^{n-1} \setminus \{0\} \to S^{n-1} : v \mapsto \frac{v}{||v||}$. $g \circ \iota = \mathrm{id}_{S^{n-1}}, \iota \circ g \simeq \mathrm{id}$ by $(x, t) \mapsto tx + (1-t)\frac{x}{||x||}$.

Algebraic topology is the study of the correspondence from topological spaces up to homotopy equivalence to groups and homomorphisms. As mentioned before, we'll concentrate on "nice" spaces, including manifolds.

Definition: A (topological) manifold is a Hausdorff topological space locally homeomorphic to $\mathbb{R}^n$ for some fixed n, the dimension of the manifold. For example, the torus or two-holed torus are 2-manifolds, but the double cone or pinched torus are not manifolds (since they have special points no neighbourhood of which looks like $\mathbb{R}^n$). Manifolds have been something of a 20th century obsession that the lecturer hasn't quite grown out of, but they're also popular because algebraic topology works well on them - by contrast there is e.g. no good algebraic topology of fractal sets (most of the invariants from this course come out as 0 or large infinities). We will want to generalize a bit, e.g. to general algebraic varieties (manifolds are the same as smooth algebraic varieties over $\mathbb{C}$.

Finally, something this course is not about: the first attempt to algebraize spaces. Loops in a space with a fixed (distinguished) base point can be concatenated; this defines (with some work) a group structure on the set of homotopy classes of maps $S^1 \to X$ sending $0 \mapsto$ our distinguished point x. Analagously, there is a group structure on homotopy classes of maps $S^n \to X, 0 \mapsto x$. This is the nth homotopy group, $\pi_n(X)$; $\pi_1(X)$ is the fundamental group.

The problem with this that is even e.g. $\pi_n(S^2)$ is not known for all n; in fact the groups $\pi_n(X)$ are not all know n for any simply-connected manifold (and calculating them all for S^2 is probably a good candidate problem for a Fields medal).

2 (Co)chain Complexes

Recall: our aim is to "algebraise" topology, notions of connectivity, and in particular the idea of higher-dimensional "holes". We take a clue from electromagnetism: if you pass a current through a wire we get a magnetic field $\vec{v}$. Moving a pole through the field along a path, the work done is $\int_\gamma \mathbf{v} \cdot d\mathbf{s}$.

Fact: If γ is closed, this is quantised, and computes a linking number. Moreover, this is unchanged by deformations: given two paths γ_1, γ_2 with a surface Σ between them, $\int_{\gamma_1} \mathbf{v} \cdot d\mathbf{s} - \int_{\gamma_2} \mathbf{v} \cdot d\mathbf{s} = \int_\Sigma \nabla \times \mathbf{u} \cdot d\mathbf{S} = 0$, since $\nabla \times \mathbf{v} = 0$ by Maxwell. Note that this is better than being unchanged by small deformations; the surface Σ need not just be a rectangle, but may be some topologically interesting surface (e.g. γ_1, γ_2 may be the boundaries of small holes in a torus Σ). Thus this is a promising definition.

Analagously, an electric charge generates a field and we measure a flux across a bounding surface. The flux $\int_S \mathbf{e} \cdot d\mathbf{s}$ is quantised, and $\int_{S_1} \mathbf{e} \cdot d\mathbf{S} - \int S_2 \mathbf{e} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{e} dV = 0$.

With the benefit of hindsight, we formulate this as follows: let $X \subset \mathbb{R}^3$ be an open set, and define $\Omega^i(X)$ to be the space of (smooth) maps from X to $\mathbb{R}$ for $i = 0, 3$, and from X to $\mathbb{R}^3$ for $i = 1, 2$.

We have (linear) maps of these vector spaces: $\Omega^0(X) \to \Omega^1(X)$ by $f \mapsto \nabla f$, $\Omega^1(X) \to \Omega^2(X)$ by $\mathbf{u} \mapsto \nabla \times \mathbf{u}$, and $\Omega^2(X) \to \Omega^3(X)$ by $\mathbf{v} \mapsto \nabla \cdot \mathbf{v}$. The key observation here is that the composite of any two successive maps vanishes: $\nabla \times \nabla = 0 = \nabla \cdot \nabla \times$.

Now, algebraically, we can define: $H^0_{\mathrm{dR}}(X) = \ker(\nabla)$, $H^1_{\mathrm{dR}}(X) = \frac{\ker(\nabla\times)}{\mathrm{im}(\nabla)}$, $H^2_{\mathrm{dR}}(X) = \frac{\ker(\nabla\cdot)}{\mathrm{im}(\nabla\times)}$ (note that in this setting $\nabla\cdot : \Omega^2 \to \Omega^3$ is surjective, so there is nothing more we can sensibly write). These are called the de Rham (the dR distinguishing them from other varieties of cohomology groups) cohomology groups of $X \subset \mathbb{R}^3$. Think of $H^i_{\mathrm{dR}}(X)$ as "invariants of closed i-dimensional regions of X", e.g. for $i = 1$, an element of H^1_{dR} is (an equivalence class of) vector-valued functions which could be magnetic fields in X. Given a closed 1D subspace of X, i.e. just a curve, the invariant we get is the work done by moving a magnetic pole/charge along the curve in the given field. By Maxwell's equations, this is a quite robust invariant.

Example: $H^0_{\mathrm{dR}}(X)$ is the set of locally constant functions on X, so $= \mathbb{R}^k$ where k is the number of connected components of X. These H^i_{dR} are (for reasonable X) finite-dimensional, so we should not be scared by the superficial complexity of introducing the infinite-dimensional Ω^is.

We associate invariants to topological spaces in two steps: first we map the space X to a chain complex or cochain complex, then we take the (co) homology of this complex; it is important to distinguish these steps, because in cases of confusion or to make definitions we often need to return to the chain/cochain complex. In a third course in topology (where this is still intended as a first course), we would avoid the homology entirely and work directly with the chain complices.

Definition: A chain complex $(C_\star, d)$ is a sequence of abelian groups and homomorphisms $\cdots \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \ldots$ such that $d^2 = 0$, i.e. $d_{n-1} \circ d_n = 0 \forall n$. The indexing set $\star$ may be [a subset of] $\mathbb{Z}$, but we will always take it to be $\subset \mathbb{N}$. We will usually drop the indices on the d_i, and refer to all of them as d.

The homology $H(C_\star, d)$ has nth-graded piece $H_n(C_\star) = \frac{\ker(d:C_n \to C_{n-1})}{\operatorname{im}(d:C_{n-1} \to C_n}$; since $d^2 = 0$ the denominator is a subset of the numerator and so this is well defined. Elements of $\ker(d : C_n \to C_{n-1})$ are called (n-)cycles and elements of $\operatorname{im}(d : C_{n+1} \to C_n)$ are called boundaries (the lecturer will avoid the controversial topic of whether these are n- or $n+1$-boundaries and call them simply boundaries).

Definition: A cochain complex is a collection of abelian groups and homomorphisms $\cdots \to C^n \xrightarrow{d} C^{n+1} \xrightarrow{d} C^{n+2} \to \ldots$ such that $d^2 = 0$. Note that for chain complexes d lowers degree, while for cochain complexes d raises degree. The cohomology of the complex $D(C^\star, d)$ has graded pieces $H^n(C^\star) = \frac{\ker(d:C^n \to C^{n+1})}{\operatorname{im}(d:C^{n-1} \to C^n)}$; we have cocycles and coboundaries with the obvious definitions.

Our aim is to associate to X some theories: $H_\star(X)$ should have H_i buildt out of closed i-dimensional pieces of X, and $H^\star(X)$ should have H^i which associate numbers to closed i-dimensional regions in X. We want to do this for fairly general (in particular, not necessarily smooth, so not just subsets of $\mathbb{R}^3$) X; this means we have to use a rather abstract definition, the downside to which is that it's not immediately clear that it actually means this. We shall "build spaces out of lego".

Definition: an n-simplex is the convex hull of $(n+1)$ orderef points in $\mathbb{R}^n$, say $v_0, \ldots, v_n$, such that the vectors $v_i - v_0$ are all linearly independent $(1 \leq i \leq n)$. The standard n-simplex Δ_n is defined as $\{\mathbf{t} \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} t_i = 1, t_i \geq 0\}$.

Exercise: any n-simplex is canonically the image of Δ^n under a linear homeomorphism $t \mapsto \sum t_i v_i$ from Δ^n to $[v_0 \ldots v_n]$.

Suppose we have e.g. γ, a loop going once around a torus. We aim to see γ as a sum of 1D simplices.

Definition: note first that we can orient the edges of a simplex canonically, since the $\{v_i\}$ are ordered; say $v_i \to v_j$ if $i < j$. The ith face $\Delta^{n-1} \subset \Delta^n$ is the image of the subset of Δ^n where $t_i = 0$; we denote the ith face of $[v_0 \ldots v_n]$ by $[v_0 \ldots \widehat{v_i} \ldots v_n]$; this is an $(n-1)$ simplex and the orientations of its edges are consistent with those of the larger simplex.

Definition: the singular chain complex $C_\star(X)$ of a topological space X has $C_n(X) = \{\sum_{i=1}^N h_i \sigma_i : h_i \in \mathbb{Z}, \sigma_i : \Delta_n \xrightarrow{\text{continuous}} X$ are "n-simplices in X"$\}$ (i.e. the free abelian group on the set of continuous maps $\Delta_n \to X$). The boundary map d is defined on simplices in X and then extended linearly: on $\sigma : \Delta^n \to X$, $d\sigma = \sum_{i=0}^n (-1)^i \sigma |_{[v_0 \ldots \widehat{v_i} \ldots v_n]}$, for $\Delta^n = [v_0 \ldots v_n]$.

3 Singular (Co)homology

Recall the singular chain complex $C_\star(X,d)$: $C_n(X) = \{\sum_{i=1}^N h_i\sigma_i : h_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X$ an n-simplex in $X\}$. $d : C_n(X) \to C_{n-1}(X)$ is given by $d(\sigma) = \sum_{i=0}^n (-1)^i \sigma\mid_{[v_0 \ldots \widehat{v_i} \ldots v_n]}$ for $\sigma = [v_0 \ldots v_n]$, extended by linearity to $C_n(X)$.

Lemma: $d^2 = 0$ (i.e. $d \circ d : C_n \to C_{n-1} \to C_{n-2} = 0$ $[\forall n]$): $d \circ d(\sigma) = d(\sum_{i=0}^n (-1)^i \sigma\mid_{[v_0 \ldots \widehat{v_i} \ldots v_n]}) = \sum_{j<i} (-1)^i (-1)^j \sigma\mid_{[v_0 \ldots \widehat{v_j} \ldots \widehat{v_i} \ldots v_n]} + \sum_{i<j} (-1)^i (-1)^{j-1} \sigma\mid_{[v_0 \ldots \widehat{v_i} \ldots \widehat{v_j} \ldots v_n]}$, which is 0 by symmetry under exchange of i, j.

Remark: in the first lecture we mentioned that for $U \subset \mathbb{R}^3$ open and $H^\star_{\mathrm{dR}}$, "$d^2 = 0$" held since partial derivatives commute: $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$. This is a local property of space, rather than the purely formal, global algebraic result we have here. Almost all results we will cover (essentially, all results that do not depend on torsion properties) are also provable in the de Rahm setting, some being harder and some being easier to prove.

Informally, $d^2 = 0$ since d takes the boundary of some region made up of simplices, and boundaries have no boundary; this is "obvious" when using an informal notion of a boundary.

Similarly, $C^\star(X)$ the singular cochain complex has nth term $C^n(X) = \mathrm{Hom}(C_n(X), \mathbb{Z})$. $d^\star : C^n \to C^{n+1}$ (which we will call just d except for the next few lines) is defined by: $\psi \mapsto d^\star\psi$ where $d^\star\psi(\sigma) := \psi(d\sigma)$, for σ a $(n+1)$-simplex. Obviously $d^\star \circ d^\star = 0$, since $d^\star d^\star \psi(\sigma) = d^\star(\psi(d\sigma)) = \psi(d \circ d\sigma) = \psi(0) = 0$.

Definition: $H_\star(X, \mathbb{Z}) = H(C_\star, d)$ is the singular homology, $H^\star(X, \mathbb{Z}) = H(C^\star, d^\star)$ is the singular cohomology. $\mathbb{Z}$ here is called the coefficient ring, which we shall usually omit when it is $\mathbb{Z}$.

Trivial observation: $H_\star(X), H^\star(X)$ are homeomorphism-invariant. Subobservation: (Co) homology are functorial under continuous maps of spaces. If $f : X \to Y$ is continuous and $\sigma : \Delta^n \to X$ is continuous, $f \circ \sigma : \Delta^n \to Y$ is continuous so f induces a group homomorphism $f_\star : C_n(X) \to C_n(Y)$. Note $df_\star = f_\star d$; the reader should think through why this is manifest.

This note implies f induces a map $f_\star : H_\star(X) \to H_\star(Y)$ i.e. maps $H_n(X) \to H_n(Y) \forall n$. To see this, note: $df_\star = f_\star d$. Let $\psi = \sum_i h_i\sigma_i \in \ker(d : C_n(X) \to C_{n-1}(X))$, i.e. ψ is a representative of a class in $H_n(X)$. Then we want to write "$[f_\star(\psi)] = [\sum_i h_i(f \circ \sigma_i)] \in H_n(Y)$", but to make this definition we need that $\sum h_i(f \circ \sigma_i)$ is a cycle (i.e. $\in \ker(d : C_n(Y) \to C_{n-1}(Y))$. $d(\sum h_i(f \circ \sigma_i)) = d(\sum h_i f_\star(\sigma_i)) = \sum h_i(df_\star\sigma_i) = \sum h_i(f_\star d)\sigma = f_\star(d(\sum h_i\sigma_i)) = 0$ as $d\psi = 0$, i.e. $f_\star$ maps cycles to cycles; by the same argument it also maps boundaries to boundaries. So if we change the representative of a $H_\star$-class in X, $[\psi] \to [\psi + d\alpha]$, then $[f_\star\psi]$ is unchanged in $H_\star(Y)$.

Corollary: if $f : X \to Y$ is continuous then it induces a homomorphism $f_\star : H_\star(X) \to H_\star(Y)$. Analagously, f induces a pullback map $f^\star : H^\star(Y) \to H^\star(X)$ (note the direction of this).

Lemma: For $f : X \to Y, g : Y \to Z$, $(g \circ f)_\star = g_\star \circ f_\star$, $(g \circ f)^\star = f^\star \circ g^\star$, and moreover $\mathrm{id} : X \to X$ induces identity on the homotopic and cohomology; the proofs of these are straightforward. So we have functoriality.

This lemma implies our above "observation": if $x \cong Y$, say $f : X \to Y, g : Y \to X$ are such that $f \circ g = \mathrm{id}_Y, g \circ f = \mathrm{id}_X$, then consider $H_\star(X) \xrightarrow{f_\star} H_\star(Y) \xrightarrow{g_\star} H_\star(X) \xrightarrow{f_\star} H_\star(Y)$; the maps between the first and third and second and fourth terms are identities, so we have $H_\star(X) \cong H_\star(Y)$ via $f_\star, g_\star$.

Caveat: there is a natural map $H^n(X) \to \mathrm{Hom}(H_n(X), \mathbb{Z})$, as seen in the first example sheet for this course, but this is not in general an isomorphism (and in some sense "the world would be a lot less interesting" if it were).

First computations:

1. $H_\star(\mathrm{pt}) = \mathbb{Z}$ for $\star = 0$, 0 otherwise: consider the chain groups $C_k(X)$; these are free abelian on the spaces of continuous maps $\Delta^k \to X$, but there is only one such map, so $C_k(X) = \mathbb{Z} \forall k$. The chain complex is then $\mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \to 0$. The map $d : C_k \to C_{k-1}$ is multiplication by the signed number of faces of Δ^k, which we can see by looking alternates between being 1 and 0 (e.g. $d(\Delta^1) = 1 - 1 = 0, d(\Delta^2) = 1 + 1 - 1 = 1$ since the edge $[v_0 v_2]$ is taken in the negative direction. So the maps always have image either 0 or all space, and the $H_i(\mathrm{pt})$ [other than H_0] are $\frac{\ker}{\mathrm{im}}$ where each time either $\ker = \{0\}$ or im = all space, so we have the result Unfortunately, this is the only connected space for which we can compute all the $H_\star$s.

2. $H_0(X) = \mathbb{Z}^{\#\text{ path-components}}$: note first that if X is a union of different path-components, $X = \bigcup_\alpha X_\alpha$, the entire singular chain complex breaks up into pieces indexed by α, $C_\star(X, d) = \bigoplus_\alpha C_\star(X_\alpha, d) \Rightarrow H_\star(X) = \bigoplus_\alpha H_\star(X_\alpha)$. So we really only need that $H_0(X) \cong \mathbb{Z}$ if X is path-connected. Define $\phi : C_0(X) \to \mathbb{Z}$ by $\sum_{i=1}^N h_i \sigma_i \mapsto \sum h_i$, where $\sigma_i : \{\mathrm{pt}\} \to X$. Note (if $X \neq \emptyset$; we shall generally omit such caveats) this is surjective $C_0(X) \to \mathbb{Z}$. For any 1-simplex τ, $\tau : [v_0, v_1] \to X$, $\partial\tau = v_1 - v_0$ lies in $\ker\phi$. So $\mathrm{im}(d : C_1 \to C_0) \subset \ker\phi$. If $\sum n_i \sigma_i \in \ker\phi$, then we claim $\sum n_i \sigma_i \in \mathrm{im}(d)$; if we have this, then $\phi : H_0(X) \to \mathbb{Z}$ descends to homology and we are done.

For each i, choose τ_i linking σ_i to a base point $p \in X$. Now $d(\sum n_i \tau_i) = \sum n_i \sigma_i - \sum n_i p$, which $= \sum n_i \sigma_i$ since $\sum n_i = 0$. So we're done.

4 First Computations

Recall that we saw from the definition $H_\star(\mathrm{pt}) = \mathbb{Z}$ if $\star = 0$, 0 otherwise, and $H_0(X) = \mathbb{Z}^{\#\text{ path-components}}$. To make it possible to effectively compute homology in general, we need further tools, including predominently the following two theorems:

Recall we know that $H_\star$ is functorial: $f : X \to Y \Rightarrow f_\star : H_\star(X) \to H_\star(Y)$ preserving degree.

Theorem (Homotopy Invariance)

If $f, g : X \to Y$ with $f \simeq g$ homotopic, then $f_\star = g_\star : H_\star(X) \to H_\star(Y)$, and similarly $f^\star = g^\star : H^\star(Y) \to H^\star(X)$ (so if $X \simeq Y$ are homotopic spaces then $H_\star(X) \cong H_\star(Y), H^\star(X) \cong H^\star(Y)$). The proof of this is similar to that for our earlier result about isomorphic spaces.

Definition: An exact sequence is a chain complex with vanishing (co)homology groups, i.e. a sequence of abelian groups and homomorphisms $\cdots \to A_n \xrightarrow{d} A_{n-1} \xrightarrow{d_{n-1}} A_{n-2} \to \ldots$ such that $\ker(d_{n-1}) = \mathrm{im}(d_n)$ (for a chain complex, where we only require $d^2 = 0$, we have the same definition with a "$\subset$" in place of this last "="). Then Mayer-Vietoris lets us cut up spaces:

Theorem (Mayer-Vietoris)

Suppose $X = A \cup B$ is a union of two open sets. Then there is an exact sequence $\cdots \to H_i(X) \overset{d_{\mathrm{MV}}}{\to} H_{i-1}(A \cap B) \overset{(i_A, i_B)}{\to} H_{i-1}(A) \oplus H_{i-1}(B) \overset{j_A - j_B}{\to} H_{i-1}(X) \overset{d_{\mathrm{MV}}}{\to} H_{i-2}(A \cap B) \to \ldots$, where the maps are as follows: $i_A : A \cap B \hookrightarrow A, i_B : A \cap B \hookrightarrow B, j_A : A \hookrightarrow X, j_B : B \hookrightarrow X$ are the obvious inclusions, and the remaining Mayer-Vietoris boundary map $H_i(X) \to H_{i-1}(A \cap B)$ has the following interpretation: take an i-cycle in X and write it as the union of a chain (with boundary) in A and a chain in B, with cancelling boundaries in $A \cap B$. Then d_{MV} takes this common boundary. For example, suppose we have a double-holed torus with A and B each being a punctured torus, and γ a loop passing through both holes; we can write this as the sum of two half-loops with two points as the boundary of each, one in A and one in B; then $d_{\mathrm{MV}}(\gamma)$ is these two points, considered as a 0-cycle in $A \cap B$.

Moreover, the Mayer-Vietoris sequence is natural: if $f : X = A \cup B \to Y = A' \cup B'$ and $f(A) \subset A', f(B) \subset B'$ then there is an induced map of Mayer-Vietoris sequences $f_\star : H_i(X) \to H_i(Y), f_\star : H_{i-1}(A \cap B) \to H_{i-1}(A', \cap B'), f_\star : H_{i-1}(A) \oplus H_{i-1}(B) \to H_{i-1}(A') \oplus H_{i-1}(B'), \ldots$, where all the squares commute (E.g. $f_\star \circ d_{\mathrm{MV}} = d_{\mathrm{MV}} \circ f_\star$).

Remark: Note that in $H^\star$ the analagous boundary map is $\partial^\star_{\mathrm{MV}} : H^i(A \cap B) \to H^{i+1}(X)$.

We shall use these two theorems for a bit before proving them.

Example: The circle $S^1 = I \cup I$ can be written as a union of two intervals $A = I, B = I$ meeting in two [disjoint] intervals. By homotopy equivalence, may consider $A = \star$ [a point], $B = \star, A \cap B = \star \amalg \star$. The Mayer-Vietoris sequence reads $H_1(A \cap B) \to H_1(A) \oplus H_1(X) \to H_1(S^1) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(S^1)$; this is $0 \to 0 \oplus 0 \overset{\alpha}{\to} H_1(S^1) \overset{\phi}{\to} \mathbb{Z} \oplus \mathbb{Z} \overset{\beta}{\to} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$. The last map here is $(u, v) \mapsto u - v)$ and β is given by $(a, b) \mapsto (a + b, a + b)$ (by considering their "geometric" definitions). Now exactness tells us $\ker \phi = \mathrm{im}\alpha = 0$, i.e. ϕ is injective. $\mathrm{Im}\phi = \ker \beta = \mathbb{Z}$ spanned by $(1, -1)$, so $H_1(S^1) = \mathbb{Z}$.

Example: Spheres: $H_\star(S^n, \mathbb{Z}) = \mathbb{Z}$ for $\star = 0, n$, 0 otherwise (Considering $n > 0$, since S^0 is two points). We induct on n, and have just done $n = 1$: $\forall n \geq 2$, $S^n = A \cup B$ the union of two overlapping open hemispheres. A, B are contractible, i.e. homotopic to points $A \simeq \star, B \simeq \star$ and $A \cap B \simeq S^{n-1}$. Then in the Mayer-Vietoris sequence $H_i(S^{n-1}) \to H_i(\star) \oplus H_i(\star) \to H_i(S^n) \to H_{i-1}(S^{n-1}) \to H_{i-1}(\star) \oplus H_{i-1}(\star) \to \ldots$. If $i - 1 > 0$ we see $0 \to H_i(S^n) \to H_{i-1}(S^{n-1})$; the result that if we have $0 \to P \to Q \to 0$ exact then $P \cong Q$ is left as an exercise. This tells us that $H_i(S^n) \cong H_{i-1}(S^{n-1}) \forall i > 1$. If $i = 1$ we have $0 \to H_1(S^n) \to H_0(S^{n-1}) \overset{r}{\to} H_0(\star) \oplus H_0(\star)$ as a piece of the Mayer-Vietoris sequence, but as before the map [which I have labelled r] is $a \mapsto (a, a)$ so injective, so $H_1(S^n) = 0$.

Corollary: If $\mathbb{R}^m \cong \mathbb{R}^n$ then $m = n$: if $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism, then ϕ must map $\mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \phi(0)$ homeomorphicly. But these are S^{m-1} and S^{n-1} respectively, so we have $\phi_\star : H_\star(\mathbb{R}^m \setminus 0) = H_\star(S^{m-1}) \to H_\star(\mathbb{R}^n \setminus \phi(0)) = H_\star(S^{s-1})$, but the two sides are equal iff $m = n$.

Remark: There are continuous maps from $I = [0, 1]$ to $[0, 1] \times [0, 1]$ which are surjective (the "space-filling curves"), so this result was not quite as obvious as it might initially appear.

Remark: if $f : S^n \to S^n$ is any map, it induces $f_\star : H_n(S^n) \cong \mathbb{Z} \to H_n(S^n) \cong \mathbb{Z}$, which must be multiplication by some integer, which is the degree $\deg(f)$. Note that the degree of the identity is 1, while the degree of the constant map is 0 since we can factorize a constant map $S^n \to S^n$ as $S^n \to \mathrm{pt} \hookrightarrow S^n$, so our induced map is $H_n(S^n) \to H_n(\mathrm{pt}) \cong 0 \to H_n(S^n)$. This rigorises our account of the Brouwer fixed point theorem in the first lecture.

Example: The Klein bottle K has $H_\star(K,\mathbb{Z}) = \mathbb{Z}$ for $\star = 0$, $\mathbb{Z} + \frac{\mathbb{Z}}{2}$ if $\star = 1$, and 0 if $\star \geq 2$. Some of the essential information here is <u>torsion</u>, i.e. of shape $\frac{\mathbb{Z}}{p}$ for some $p \neq 0$. This would be invisible if we took our coefficients from $\mathbb{R}$ rather than $\mathbb{Z}$ (as is done in the de Rahm case).

Proof: K is the quotient space of the square with opposite pairs of edges identified, the sides in the same direction, the top and bottom edges in opposite direction. Cut this via letting A be a "strip" down the middle of the square, B the strips down the left and right sides (which is a connected space because of the identification). We see $K = A \cup B$ where A, B are Möbius strips, and $A \cap B \simeq S^1$. In the Mayer-Vietoris sequence we see $0 \to H_2(A \cup B) \to H_1(A \cap B) = \mathbb{Z} \xrightarrow{\psi} H_1(A) \oplus H_1(B) = \mathbb{Z} \oplus \mathbb{Z} \to H_1(A \cup B) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B)$ and this last map is injective. We almost (but not quite) know enough; once we know the map ψ we can determine everything else. We claim ψ is given by $1 \mapsto (2,2)$, because $A \cap B$ is the boundary of each Möbius band A, B, but A, B are generated by the <u>equatorial</u> circles of the bands [which the bands retract onto]. So (easily by algebra) ψ is injective so $H_2(K) = 0$, ϕ is surjective so $H_1(K) = \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathrm{im}\psi} = \mathbb{Z} \oplus \frac{\mathbb{Z}}{2}$.

5 Degree

Recall that $f : S^k \to S^k$ induces a map $f_\star : H_k(S^k) \cong \mathbb{Z} \to H_k(S^k) \cong \mathbb{Z}$; this must be multiplication by some integer $\deg(f)$ or $d(f) \in \mathbb{Z}$.

Properties: $d(f \circ g) = d(f)d(g), d(\mathrm{id}) = 1, d(\mathrm{constant}) = 0$. Indeed, if $f : S^k \to S^k$ is not surjective then it has degree 0, since in that case f factorizes as a map $S^k \to \mathbb{R}^k \to S^k$.

Lemma: Let $G \in O(n+1)$, the group of orthogonal $(n+1) \times (n+1)$ matrices. Then G acts on S^n and hence on $H_n(S^n, \mathbb{Z})$, and acts with degree $\det(G)$. (Note that if $G \notin O(n+1)$, i.e. G is a non-orthogonal matrix, it is not obvious that it acts on S^n). $O(n+1)$ has two connected components distinguished by $\det = \pm 1$. So by homotopy invariance of degree, it suffices to see that reflection in a hyperplane H has degree -1.

Divide S^n into two hemispheres each <u>preserved by the reflection</u> - i.e. by a hyperplane H' orthogonal to H. Pieces of the relevant Mayer-Vietoris sequences give a square: we have $0 \to H_n(S^n) \xrightarrow{d_{\mathrm{MV}}} H_{n-1}(S^{n-1}) \to 0$ and $0 \to H_n(S^n) \xrightarrow{d_{\mathrm{MV}}} H_{n-1}(S^{n-1}) \to 0$, with maps reflection in H from $H_n(S^n) \to H_n(S^n)$ and reflection in $H' \cap H$ from $H_{n-1}(S^{n-1})$ to $H_{n-1}(S^{n-1})$. We used when stating the Mayer-Vietoris theorem that the Mayer-Vietoris maps are natural, so this does commute. The d_{MV} maps are isomorphisms (indeed, the same isomorphism), so we reduce to the case $n = 1$ (the two reflections' induced maps must always be either both +1 or both -1).

For the circle, S^1, we computed $H_\star(S_1)$ using Mayer-Vietoris: split the circle into segments A, B with intersection two points p, q, then sequence is $0 \to H_1(S^1) \to H_0(p \amalg q) \to H_0(A) \oplus H_0(B) \to \dots$, with the map $H_0(p \amalg q) \xrightarrow{\phi} H_0(A) \oplus H_0(B)$ which are isomorphic to $\mathbb{Z} < p > \oplus \mathbb{Z} < q > \to \mathbb{Z} < \mathrm{pt} > \oplus \mathbb{Z} < \mathrm{pt} >$

being given by $(u, v) \mapsto (u + v, u + v)$. We saw $H_1(S^1) \cong \mathbb{Z}$ by exactness, and explicitly $H_1(S^1) \cong \ker(\phi) \cong \mathbb{Z} < (1, -1) > \subset \mathbb{S}^2 = H_0(p \amalg q)$. Reflection in the line H perpendicular to pq swaps p and q, so it acts on $\mathbb{Z}^2$ by $(u, v) \mapsto (v, u)$, so it maps on $\mathbb{Z} < 1, -1 >$ by multiplication by -1. Note that we have not been "clever" in our choice of the orientation of H relative to p and q; reflection in H must preserve the Mayer-Vietoris decomposition, so this was the only possible choice.

Corollary: The antipodal map on S^n has degree $(-1)^{n+2}$, as it's a composite of $n + 1$ reflections.

Corollary: The sphere S^n has a nowhere zero vector field if and only if n is odd (Definition: a <u>vector field</u> on a smooth manifold is an assignment $p \mapsto X_p$ of a tangent vector $X_p \in T_pM$ for each $p \in M$, i.e. it's an infinitesimal flow. Explicitly on $S^n \subset \mathbb{R}^{n+1}$, a vector field is an assignment $v : S^n \to \mathbb{R}^{n+1}$ such that $\langle x, v(x)\rangle = 0 \forall x$ (under the usual inner product $\langle , \rangle$)).

If $n = 2k-1$ odd, take $(x_1, y_1, \ldots, x_k, y_k) \in S^{2k-1} \subset \mathbb{R}^{2k} \mapsto (-y_1, x_1, -y_2, x_2, \ldots, -y_k, x_k)$ and this is a nowhere zero vector field.

Now suppose we have a nowhere zero vector field v. Take wlog $v(x)$ always of length 1, by $v \mapsto \frac{v}{\|v\|}$. Consider the family of maps $v_t : x \mapsto (\cos t)x + (\sin t)v(x)$; observe that $\forall t, x, |v_t(x)| = 1$, since $\langle x, v(x)\rangle = 0$ (orthogonality). At $t = 0$, $v_0 = \mathrm{id}$, and at $t = \pi$, $v_\pi = -\mathrm{id}$, i.e. $\mathrm{id} \sim -\mathrm{id}$, the antipodal map on S^n. So $\deg(\mathrm{id}) = 1 = \deg(\text{antipodal}) = (-1)^{n+1}$, and n is odd.

Special case: the "hairy ball theorem": "you can't comb a dog flat".

The antipodal map has no fixed points.

Lemma: if $f : S^n \to S^n$ has no fixed points, $f \simeq$ the antipodal map. The only possible way we can use the fact that f has no fixed points in a proof is to divide by something that we could not otherwise do because it might be 0, so we'll do that; we shall actually prove something slightly stronger: if $f(x) \neq g(x) \forall x$, for $f, g : S^n \to S^n$, then $f \simeq a \circ g$ where $a : S^n \to S^n$ is the antipodal map. Consider the maps $x \mapsto \frac{tf(x)-(1-t)g(x)}{\|tf(x)-(1-t)g(x)\|}$ for $x \in S^n, 0 \leq t \leq 1$. Note that if $tf(x) = (1 - t)g(x)$, taking the norm $t = 1 - t = \frac{1}{2}$ so then $f(x) = g(x)$, a contradiction; thus this is valid. At $t = 0$ this map is $a \circ g$ and at $t = 1$ it is f, so we have the result.

Corollary: if the group G acts freely on S^{2k}, then $G \leq \frac{\mathbb{Z}}{2}$.

Remark: $S^{2k-1} \subset \mathbb{C}^k$ is preserved by multiplication by $e^{i\theta}$, so S^{2k-1} has a free action of $\frac{\mathbb{Z}}{m}$ for every $m \geq 2$.

Proof: If $f : S^n \to S^n$ has no fixed point, then $f \simeq$ the antipodal map, so f has degree $(-1)^{n+1}$. But if G acts on S^n, $\deg : G \to \frac{\mathbb{Z}}{2}$ has all nontrivial elemnts going to -1, so $\ker(\deg) = \{e\}$, but deg is a homomorphism so $G \leq \frac{\mathbb{Z}}{2}$.

Note that Mayer-Vietoris can be used to compute a number of other things, which we will need to use but only have time to summarize here: $H_\star(\Sigma_g) = \mathbb{Z}$ if $\star = 0$, $\mathbb{Z}^{2g}$ for $\star = 1$, $\mathbb{Z}$ for $\star = 2$ and 0 otherwise (Σ_g is the surface of genus g).

Think about $X \vee Y = \frac{(X \amalg Y)}{\star_X = \star_Y}$ where each $\star$ is a single base-point, and $M_1 \# M_2$: for M_1, M_2 n-manifolds, we cut out balls from each and glue the boundaries. We have $T^2 \# T^2 = \Sigma_2$, etc.

Observe that every map of (closed) oriented surfaces has a degree, since the only thing we needed for the existence of degrees was that the "top" homology group was $\mathbb{Z}$, e.g. this is true for maps $T^2 \to S^2$.s

Example: Let $S^1 \amalg S^1 \subset \mathbb{R}^3$ be a link of two circles, i.e. an embedding of two disjoint circles into $\mathbb{R}^3$; they may be distinct, looped around each other like two links in a chain, or in some more complicated arrangement. Define $\phi : T^2 \to S^2$, the Gauss map of the link, by $(x, y) \in S^1 \times S^1 \mapsto \frac{x-y}{\|x-y\|}$ Definition: $\deg(\phi) = \mathrm{lk}(L_1, L-2)$ is the linking number of the two components.

Note: If $\mathrm{lk}(L_1, L_2) \neq 0$, you can't separate the two links spatially - to be more precise, no embedded S^2 in $\mathbb{R}^3$ separates the two components. Otherwise, deform this by moving one of the two loops away to infinity: ϕ is clearly not onto (since only a small portion of the sphere is available as directions between the two widely separated loops), so ϕ is of degree 0.

6 Local Degree

Intuitively, $H_n(S^n) \cong \mathbb{Z}$ and H_2(oriented surface) $= \mathbb{Z}$ because "there's a unique closed top-dimensional lump" to these spaces. E.g. for S^2 $H_2 = \mathbb{Z}$, but for $S^2 \vee S^2$, $H_2 = \mathbb{Z} \oplus \mathbb{Z}$, since the two spheres only meet in a point so there are two two-dimensional lumps. $H_2(\frac{S^2}{p\sim q})$, the sphere where we identify two points (so it looks like the pinched torus), $= \mathbb{Z}$. For an "open" surface e.g. a surface of genus 3 with a disc cut out, $H_2 = 0$ - there is no single lump.

The degree $\deg(f)$ measures how many times the domain "wraps onto" the range.

For example, consider the map $S^2 \to T^2$ obtained by projecting down and up onto a flat disc, then taking a homeomorphism from this to the square and identifying the edges appropriately to make it a torus. This has degree 0, since it factors through the disc and $H_2(\text{disc}) = 0$. Now consider the inverse image of a general point p on the torus; it will be a single point in the square (of course some points in the torus have preimage two points on the boundary of the square, but we are considering a general point), so a single point in the disc, so two points p^+, p^- on the sphere.

Locally near p^+, p^-, the maps $S^2 \to T^2$ is a homeomorphism, and these two homeomorphisms differ by reflection in the equator. So, informally, one of these has degree 1, the other degree -1, and these add together to give the overall degree 0.

Example: $S^1 \to S^1$ $z \mapsto z^k$ has degree k: $H_1(S^1) \cong \mathbb{Z}$, a generating 1-cycle is a sum of k simplicies each going $\frac{2\pi}{k}$ around the circle. Under the map, each of these 1-simplices maps to a 1-simplex which is now a cycle, and represents the generator of $H_1(S^1)$. So $f_\star : H_1(S^1) = \mathbb{Z} \to \mathbb{Z}$ is multiplication by k. Again consider the inverse image of a general point p, which is k points p_i. The map near each p_i is again a homeomorphism, but this time the different such local homeomorphisms differ by rotations of the domain - thus the k 1s add together to give the total degree k.s

Remark: If $f : X \to Y$ is a smooth map of closed smooth manifolds, Sard's Theorem says that for a dense set of regular values $y \in Y$, $f^{-1}(y) = \{x_1, \dots, x_k\}$ is finite, and there are neighbourhoods $U_i \ni x_i, V \ni y$ such that $f|_{U_i} : U_i \to V$ is a diffeomorphism.

Algebraic Digression

An exact sequence $0 \to A \to B \to C \to 0$ is called a short exact sequence. A SES of chain complexes is a diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \to & A_{n+1} & \to & A_n & \to & A_{n-1} & \to & \dots \\
 & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \\
\dots & \to & B_{n+1} & \to & B_n & \to & B_{n-1} & \to & \dots \\
 & & \downarrow \beta & & \downarrow \beta & & \downarrow \beta & & \\
\dots & \to & C_{n+1} & \to & C_n & \to & C_{n-1} & \to & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}.$$

The rows are chain complexes, with $d^2 = 0$ but not necessarily exact. The columns are all SES: $\mathrm{im}\alpha = \ker\beta$ always. Note that β is always surjective and α always injective. When we draw a diagram like this in this course, it is implicit that it commutes, unless otherwise stated.

Proposition: The SES of chain complexes defines or gives rise to an associated long exact sequence in homology: $0 \to A_\star \to B_\star \to C_\star \to 0 \Rightarrow H_k(A_\star) \to H_k(B_\star) \to H_k(C_\star) \xrightarrow{\partial} H_{k-1}(A_\star) \to H_{k-1}(B_\star) \to H_{k-1}(C_\star) \xrightarrow{\partial} H_{k-2}(A_\star) \to \dots$. The unlabelled maps are those naturally induced from the maps $A_\star \to B_\star \to C_\star$ (there are maps on homology since all the squares in the diagram commute). To define the connecting homomorphism $\partial : H_k(C_\star) \to H_{k-1}(A_\star)$: pick a cycle $\sigma \in C_k$ representing a homology class $[\sigma] \in H_k(C_\star)$. There is $b \in B_k$ such that $\beta(b) = \sigma$. Note $\beta(db) = d(\beta b) = d(\sigma) = 0$ since σ is a cycle, so $db \in \ker\beta = \mathrm{im}\alpha$. So there is $a \in A_{k-1}$ such that $\alpha(a) = db$. Note $da = 0$ since $\alpha(da) = d(\alpha(a)) = d(db) = 0$ since $d^2 = 0$, and α is injective. So a is itself a cycle and defines $[a] \in H_{k-1}(A_\star)$; define $\partial[\sigma] = [\alpha]$. Warning: we have made choices to get from one to another: we chose a representing cycle σ, and we chose b. So to prove this we need to check (as on the first example sheet for the course) i) ∂ is well defined ii) ∂ is linear and a homomorphism iii) the resulting sequence of groups really is exact.

Examples: 1. If $A \subset X$ is a subspace, $C_\star(A) \to C_\star(X)$ is the inclusion of singular chains from A to X. This is valid since if $\sigma : \Delta^n \to X$ lies in A, then all the faces of σ lie in A, so $d : C_\star(X) \to C_{\star-1}(X)$ preserves $C_\star(A) \to C_{\star-1}(A)$. We get "for free" an induced d on the quotient, so define $C_n(X, A) = \frac{C_n(X)}{C_n(A)}$; we get $d : C_n(X, A) \to C_{n-1}(X, A)$ from d on $C_\star(X)$. $0 \to C_\star(A) \to C_\star(X) \to C_\star(X, A) \to 0$ is a SES of chain complexes; the associated LES $H_k(A) \to H_k(X) \to H_k(X, A) \to H_{k-1}(A) \to H_{k-1}(X) \to \dots$ is called the "LES of the pair". A cycle for $C_\star(X, A)$ is a chain in X all of whose boundary lies in A; the boundary map $\partial : H_k(X, A) \to H_{k-1}(A)$ just takes the boundary of such a "relative cycle".

2. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of a space X. Let $C_\star(X, \mathcal{U})$ be the subcomplex of $C_\star(X)$ comprising sums of simplices each of which lies in some set in $\mathcal{U}$, i.e. $\{\sum_{i=1}^N n_i \sigma_i\}$, where each $\sigma_i : \Delta^n \to X$ has $\mathrm{im}\sigma_i \subset U_{\alpha_i}$, and $n_i \in \mathbb{Z}$. Again, $d : C_\star(X) \to C_\star(X)$ preserves $C_\star(X, \mathcal{U})$.

Theorem: $C_\star(X, \mathcal{U}) \hookrightarrow C_\star(X)$ induces an isomorphism on homology, i.e. $H(C_\star(X, \mathcal{U})) \cong H_\star(X)$.

Corollary (Mayer-Vietoris): If $X = A \cup B, \mathcal{U} = \{A, B\}$, then we have a SES $0 \to C_\star(A \cap B) \to C_\star(A \cap B) \to C_\star(A) \oplus C_\star(B) \xrightarrow{\beta} C_\star(X, \mathcal{U}) \to 0$, where the two nontrivial maps are respectively $\sigma \mapsto (\sigma, \sigma)$ and $(u, v) \mapsto u - v$. $C_\star(X, \mathcal{U})$

is defined to make β surjective. So we have a LES in homology, $H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(X, \mathcal{U}) \cong H_k(X) \to H_{k-1}(A \cap B) \to \dots$.

Corollary (Excision): If $Z, A \subset X, \mathrm{cl}(Z) \subset \mathrm{int}(A)$, then $H_\star(X, A) \cong H_\star(X \setminus Z, A \setminus Z)$ - "you can excise Z". For the intuition, note that at the level of spaces $\frac{X}{A}$ and $\frac{X \setminus Z}{A \setminus Z}$ are the same. For the proof, if $B = X \setminus Z$ then $X = A \cup B$ and the theorem says $\frac{C_n(B)}{C_n(A \cap B)} \cong \frac{C_n(X, \mathcal{U})}{C_n(A)} \hookrightarrow \frac{C_n(X)}{C_n(A)}$ induces isomorphism on homology; we do need the method of proof of the above theorem rather than just the theorem itself to know that passing to the quotient is ok. Both these quotients are free groups in the set of simplices in B not wholly lying in A. $H_\star(X, A) \cong H_\star(B, A \cap B)$.

Go back to a map $f : S^n \to S^n$. Suppose $f^{-1}(y) = \{x_1, \dots, x_k\}$ and $f : U_i \ni x_i \to V \ni y$. Excision lets us relate $H_\star(S^n, S^n \setminus x_i) \cong H_\star(U_i, U_i \setminus x_i)$, excising $S^n \setminus U_i$ whose closure does lie in $S^n \setminus x_i$. (TBC)

7 Axioms

Recall we are discussing local degree. Given $f : S^n \to S^n$ (or more generally a map of manifolds) and given $f^{-1}(y) = \{x_1, \dots, x_k\}$, we aim to find $\deg(f)$ from the geometry near the x_is.

We've added to our arsenal of results, the <u>Excision Theorem</u>: if $A \subset X, \mathrm{Cl}(z) \subset \mathrm{int}(A)$ then inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism $H_\star(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_\star(X, A)$ (recall $H_\star(X, A) = H(\frac{C_\star(X)}{C_\star(A)})$ and cycles measure "lumps of X with boundary in A").

Example: If M is a manifold and $x \in M$ has a neighbourhood $U \cong \mathbb{R}^n$, $U \ni x$, we can excise all of $M \setminus U$ to give $H_\star(M, M \setminus x) \xleftarrow{\cong} H_\star(M \setminus (M \setminus U), (M \setminus x) \setminus (M \setminus U)) = H_\star(U, U \setminus x)$.

The LES for relative homology (X, A) is $H_i(A) \to H_i(X) \to H_i(X, A) \xrightarrow{\partial} H_{i-1}(A) \to \dots$; in particular, $H_i(U \setminus x) \to H_i(U) = 0 \to H_i(U, U \setminus x) \xrightarrow{\partial} H_{i-1}(U \setminus x) \to H_{i-1}(U) = 0$ where the groups labelled $= 0$ are usually $=0$ because $U \cong \mathbb{R}^n \simeq \star$. So ∂ becomes an isomorphism and $H_\star(U, U \setminus x) = \mathbb{Z}$ for $\star = n$, 0 otherwise.

Given $f : S^n \to S^n$, $f^{-1}(y) = \{x_1, \dots, x_k\}$, suppose $U_i \ni x_i, V \ni y$ such that $f : U_i \to V$ has $x_i \mapsto y$. Choose the U_i disjoint, so f induces [maps] $(U_i, U_i \setminus x_i) \to (V, V \setminus y)$ and hence $H_n(U_i, U_i \setminus x_i) \cong \mathbb{Z} \to H_n(V, V \setminus y) \cong \mathbb{Z}$, which will be multiplication by the local degree of f, $\deg_{x_i}(f)$.

Note we can also use excision to identify $H_n(S^n, S^n \setminus \bigcup x_i) \cong \bigoplus H_n(U_i, U_i \setminus x_i)$ by excising $S^n \setminus \bigcup_i U_i$.

Consider the diagram
$$\begin{array}{ccccc}
 & \swarrow & H_n(U_i, U_i \setminus x_i) & \to & H_n(V, V \setminus y) \\
 & & \downarrow & & \downarrow \\
H_n(S^n, S^n \setminus x_i) & \leftarrow & H_n(S^n, S^n \setminus \bigcup x_i) & \to & H_n(S^n, S^n \setminus y) \\
 & & \uparrow & & \uparrow \\
 & \nwarrow & H_n(S^n) & \to & H_n(S^n)
\end{array}$$

The lower arrow in the middle column is the obvious inclusion; the upper one is inclusion as a summand, since the group in the middle is isomorphic to a direct sum of groups. The bottom horizontal map is induced from f i.e. is $\deg f$; the top horizontal map is $\deg_{x_i} f$, and we'll call the middle rightwards-pointing map $\tilde{f}$. The top left diagonal map and the downward map in the top right are isomorphisms by excision; the bottom left diagonal map and the upward map in the bottom right are isomorphisms by the LES $H_n(S^n \setminus y) \to H_n(S^n) \overset{\text{iso}}{\to} H_n(S^n, S^n \setminus y) \to H_{n-1}(S^n \setminus y) = H_{n-1}(\mathbb{R}^n) = 0 \to H_{n-1}(S^n)$. The triangle is the bottom left commutes, so the upwards injection at the bottom of the middle column is actually the diagonal inclusion $\mathbb{Z} \to \mathbb{Z}^{\# U_i}$ $1 \mapsto (1, 1, \dots, 1)$. $\deg f$ is defined as $f_\star(1)$ for $1 \in H_n(S^n)$; by commutativity this is $\tilde{f}_\star(1, 1, \dots, 1)$, which $= \sum_i \tilde{f}_\star(0, 0, \dots, 0, 1, 0, \dots, 0)$ where the 1 is in the ith place. We can lift this up to the top row as $\sum_i \deg_{x_i}(f)$ by the definition of local degree and commutativity of the top right square.

Example: Note that if $f\mid_{U_i}$ is a homeomorphism it has local degree ± 1 (with the sign depending on orientation, a subject to which we shall return). Consider $\mathbb{C} \to \mathbb{C}$: let p be a complex polynomial (in one variable) which extends to a map $p : S^2 = \mathbb{C} \cup \{\infty\} \to S^2$. The degree of p really is the same as the degree of p: if $p(z) = z^k + a_1 z^{k-1} + \cdots + az$, this is homotopic as a map $S^2 \to S^2$ to $\phi : z \mapsto z^k$. $\phi^{-1}(1)$ is a set of kth roots of unity; near a root of unity ϕ is locally a homeomorphism (by the open mapping theorem) so say it is of degree +1 at 1. Then it is of the same degree locally at other roots, since the local maps differ by global rotations of S^2 (which are of degree 1).

Warning: The map $z \mapsto az + 1$ as a varies from 1 to 0 is continuous as a family of polynomials, but <u>not</u> as a family of maps $S^2 \to S^2$: $S^2 \times [0, 1] \to S^2$ $(z, a) \mapsto az + 1$ is not continuous at $(\infty, 0)$.

The theory as developed so far has the following features: Given $(X, A), A \subset X$, we associate to it a family of groups $\{H_\star(X, A)\}_{\star \in \mathbb{Z}}$ and these satisfy: there is a LES $H_i(A) \to H_i(X) \to H_i(X, A) \to H_{i-1}(A) \to \dots$, we have functoriality $f : (X, A) \to (Y, B)$ induces $f_\star : H_\star(X, A) \to H_\star(Y, B)$, homotopy invariance: if $f \simeq g$ through maps of pairs then $f_\star = g_\star : H_\star(X, A) \to H_\star(Y, B)$. Excision: if $\mathrm{cl}(Z) \subset \mathrm{int}(A)$, $H_\star(X, A) \overset{\cong}{\leftarrow} H_\star(X \setminus Z, A \setminus Z)$, and unions: $H_\star(\amalg_\alpha X_\alpha) \simeq \bigoplus_\alpha H_\star(X_\alpha)$.

Remark: In fact, these imply that the Mayer-Vietoris sequence holds.

An assignment $(X, A) \to h_\star(X, A)$ of pairs of spaces to graded abelian groups ($h_\star(X)$ is shorthand for $h_\star(X, \emptyset)$) which satisfies these axioms is called a <u>generalized homology theory</u>

Roughly, any generalized homology theory that arises as the homology groups of a chain complex is our theory (tensored with something).

Example: in the first example sheet we saw the notion of the <u>suspension</u> of a space ΣX. In the first lecture, we briefly mentioned the existence of $\pi_i(X)$, defined as homotopy classes of maps $(S^i, \star) \to (X, x)$. Fact: there are natural maps $\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \pi_{i+2}(\Sigma^2 X) \to \dots$ which eventually become isomorphisms. The limit $\lim_{n\to\infty} \pi_{i+n}(\Sigma^n X) = \pi_i^{\mathrm{st}}(X)$, the <u>stable homotopy group</u>, is a generalized homology theory. There is a million dollar prize available for computing the stable homotopy groups of a point.

Definition: A <u>cell complex</u> is a space X built by: X_0 is a finite set, $X_1 = U_0 \cup D_1^1 \cup D_2^1 \cup \cdots \cup D_n^1$ where D_i^1 is a 1-disk (i.e. $[0, 1]$) attached to X_0 by a map $\partial D_i^1 \to X_0$, and more generally $X_k = X_{k-1} \cup D_1^k \cup \cdots \cup D_{n_k}^k$ where D_i^k is the closed k-disk and attached by a map $\partial D_i^k \to X_{k-1}$. Then $X = \bigcup_k X_k$ with the weak topology. X_k is called the <u>k-skeleton</u>; the D^i are i-cells.

Remarks: 1. X_k is a quotient space, $X_k = \frac{X_{k-1} \amalg D_1^k \amalg \cdots \amalg D_{n_k}^k}{\sim}$ where $\sim$ identifies each $p \in \partial D_i^k$ with its image in X_{k-1} 2. The weak topology is that $U \subset X$ is open if $U \cap X_k$ is open in $X_k \forall k$.

Examples: S^n is a point union an n-cell: X_0 is a point, X_k is a point $\forall k < n$, and $X_n = S^n = \text{pt} \cup D^n_1$. T^2 has a cell structure with one 0-cell, a corner of the square it is a quotient of, 2 1-cells, the two edges of the square, and one 2-cell, the face of the square.

Theorem (seen on the second example sheet): if $h_\star, k_\star$ are GHTs defined on pairs (X, A) with X a cell complex and $A \subset X$ a subcomplex, then if $h_\star(\text{pt}) \cong k_\star(\text{pt})$, any natural transformation $h_\star \to k_\star$ induces an isomorphism $h_\star(X, A) \cong k_\star(X, A)$ for all (X, A). Informally, this means the theory is completely defined by the groups of a single point. The proof of this result requires all our axioms but nothing more, hence why we choose that set of axioms.

8 Homotopy Invariance

Homology and cohomology are useful since they are insensitive to "inessential" deformations of maps and spaces. Recall spaces X, Y are homotopy equivalent $X \simeq Y$ if $\exists f : X \to Y, g : Y \to X$ such that $f \circ g \simeq \text{id}_Y, g \circ f \simeq \text{id}_X$; recall further that $p, q : X \to Y$ are homotopic $p \simeq q$ if $\exists F : X \times [0, 1] \to Y$ with $F(x, 0) = p(x), F(x, 1) = q(x) \forall x \in X$.

Theorem: If $f \simeq g : X \to Y$ then $f_\star = g_\star : H_\star(X) \to H_\star(Y)$ (and by a minor variation of the proof, $f^\star = g^\star : H^\star(Y) \to H^\star(X)$.

The algebraic way to show two maps of chain complexes coincide is to introduce an algebraic version of homotopy.

Definition: Given chain complexes $A_\star, B_\star$ and maps ϕ, ψ of these chain complexes (these are chain maps, $d\phi = \phi d, \psi d = d\psi$), then ϕ, ψ are <u>chain homotopic</u> if $\exists P : A_{n-1} \to B_n$ $(\forall n)$ such that $\phi - \psi = \partial P \pm P\partial$.

Lemma: chain homotopic chain maps induce the same maps on homology: if σ is a cycle, say $\sigma \in \ker(d : A_n \to A_{n-1})$, $\phi\sigma - \psi\sigma = (\partial P \pm P\partial)\sigma = \partial(P\sigma)$ is a boundary, so $[\phi(\sigma)] = [\psi(\sigma)]$ as elements of $H(B_\star)$.

Proof of the homotopy invariance theorem: given $f, g : X \to Y$, $f \overset{F}{\simeq} g$, let $i_0 : X \hookrightarrow X \times I$ be the map $x \mapsto (x, 0)$, $i_1 : X \hookrightarrow X \times I$ $x \mapsto (x, 1)$. Then $f = F \circ i_0 \Rightarrow f_\star = F_\star \circ (i_0)_\star, g = F \circ i_i \Rightarrow g_\star = F_\star \circ (i_1)_\star$, so it suffices to prove $(i_0)_\star = (i_1)_\star$; thus it is sufficient to show $(i_0)_\star$ and $(i_1)_\star$ are chain homotopic.

Our key ingredient for seeing this is a certain universal decomposition of $\Delta^n \times [0, 1]$ into $(n + 1)$-simplices Δ^{n+1}. (This is an important technique - for this theorem "anything will work", we just need some way to decompose, because the result is one which is very true, but when proving excision we will need to carefully choose a decomposition with certain properties).

We cut $\Delta^n \times [0, 1]$ into the $(n + 1)$-simplices $[v_0 \dots v_i w_i \dots w_n]$ for $0 \leq i \leq n$. Claim: these are $(n + 1)$-simplices and they exactly fill $\Delta^n \times [0, 1]$: let $\phi_i : \Delta^n \to [0, 1]$ be given by $(t_0, \dots, t_n) \mapsto t_{i+1} + \dots + t_n$. Observe $0 = \phi_n \leq \phi_{n-1} \leq \dots \leq \phi_0 \leq \phi_{-1} = 1$. The vertices $v_0, \dots, v_i, w_{i+1}, \dots, w_n$ all lie on the graph (i.e. plot) of ϕ_i, and w_i does not lie on this graph. The graphs of the ϕ_i are copies of Δ^n in $\Delta^n \times [0, 1]$ which project homeomorphicly to $[v_0 \dots v_n]$. The region between two successive such graphs is exactly one of the $[v_0 \dots v_i w_i \dots w_n]$, and since w_i is not in the graph of ϕ_i, the linear independence condition holds and we have our claim.

Define a prism operator $P : C_n(X) \to C_{n+1}(X \times [0,1])$ by $\sigma \mapsto \sum_{i=0}^{n}(-1)^i(\sigma \times 1) |_{[v_0 \ldots v_i w_i \ldots w_n]}$, for $\sigma : \Delta^n \to X \therefore \sigma \times 1 : \Delta^n \times I \to X \times I$.

Claim: $\partial P\sigma + P\partial\sigma = i_{0\star}\sigma - i_{1\star}\sigma$¡ i.e. P defines our chain homotopy. This is just the fact that the boundary of the prism minus the prism of the boundary is the bottom minus the top; explicitly, $\partial P\sigma = \sum_{j\leq i}(-1)^i(-1)^j\sigma \times 1 |_{[v_0 \ldots \widehat{v_j} \ldots v_i w_i \ldots w_n]}$ $+ \sum_{j\geq i}(-1)^i(-1)^{j+1}\sigma \times 1 |_{[v_0 \ldots v_i w_i \ldots \widehat{w_j} \ldots w_n]}$. This is $\sum_{i>j}(-1)^i(-1)^j\sigma \times 1 |_{[v_0 \ldots \widehat{v_j} \ldots v_i w_i \ldots w_n]}$ $+ \sum_{i<j}(-1)^i(-1)^{j+1}\sigma \times 1 |_{[v_0 \ldots v_i w_i \ldots \widehat{w_j} \ldots w_n]}$ $+\sigma \times 1 |_{[\widehat{v_0} w_0 \ldots w_n]}$ $-\sigma \times 1 |_{[v_0 \ldots v_n \widehat{w_n}]}$; the reader should check and convince himselves that the first two terms are $-P(\partial\sigma)$, while the third is the top and the fourth the bottom, so we are done; P is a chain homotopy between $(i_0)_\star$ and $(i_1)_\star$. (In many ways this proof is simply practice for the far more painful matter of proving excision).

The following result is proved using homotopy theory, but the proof will not be given in this course:

Whitehead Theorem: Let X, Y be simply connected cell complexes (recall a simply connected space X has $\pi_1(X, x) = 0$, or equivalently every continuous map $S^1 \to X$ extends to a map $\overline{D^2} \to X$). Suppose $f : X \to Y$ induces $f_\star : H_\star(X, \mathbb{Z}) \xrightarrow{\cong} H_\star(Y, \mathbb{Z})$ which is an isomorphism. Then f is a homotopy equivalence $X \xrightarrow{\simeq} Y$.

Warning: This does not mean that if we have $H_\star(X) \simeq H_\star(Y)$ for simply connected cell complexes X, Y then $X \simeq Y$, because the isomorphism may not be induced by any map f.

Remark: Let $X = \{\frac{(xz+1)^2-(yz+1)^2}{z}\} \subset \mathbb{C}^3$ be the given affine surface; then $X \simeq \mathrm{pt}$ but $X \not\cong \mathbb{C}^2$; indeed $X \cap S^5$ for S^5 a large sphere in $\mathbb{C}^3$ gives a 3-manifold which is not S^3; it is not even simply connected. So detecting homotopy equivalence is "not the be all and end all".

In the next few lectures, we shall cover the unpleasant matter of prooving the excision theorem. This is not something the reader will ever be asked to reproduce in an examination, but the lecturer feels compelled to provide a proof as a matter of honesty.

9 Excision

Theorem (Excision): If $Z \subset A \subset X, \mathrm{cl}(Z) \subset \mathrm{int}(A)$, then $H_\star(X, A) \overset{\cong}{\leftarrow}_{\text{inclusion}} H_\star(X \setminus Z, A \setminus Z)$. The key, as with homotopy invariance, will be a method for dividing simplices. We saw that excision follows from:

Theorem: If X is a space and $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ is a cover of X by sets whose interiors cover X, then $C_\star(X, \mathcal{U}) \hookrightarrow C_\star(X)$ is an isomorphism on homology. Recall $C_n(X, \mathcal{U}) = \{\sum_{i=1}^{n} a_i\sigma_i : a_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X, \mathrm{im}(\sigma_i) \subset U_{\alpha(i)} \text{ some } \alpha(i) \in I\}$.

Strategy of proof: we have $\iota : C_\star(X, \mathcal{U}) \hookrightarrow C_\star(X)$. We will construct $\rho : C_\star(X) \to C_\star(X, \mathcal{U})$ (these are both chain maps $d\iota = \iota d, d\rho = \rho d$) and construct $D : C_n(X) \to C_{n+1}(X)$ such that (A) $\partial D + D\partial = 1 - \iota\rho$ and (B) $\rho\iota = \mathrm{id}$. Then (A) implies $\iota_\star\rho_\star$ is the identity on homology, (B) implies $\rho_\star\iota_\star$ is the identity on homology, and together these imply $\iota_\star : H(C_\star(X, \mathcal{U})) \xrightarrow{\cong} H(C_\star(X)) = H(X)$ is an isomorphism as required.

Construction of D, step i: Divide Δ^n into smaller n-simplices by a chain map $\sigma \mapsto S(\sigma)$, $\partial S = S\partial$. Our S is barycentric subdivision: the barycentre of the simplex Δ^n is the point $\frac{1}{n+1}(1, 1, \ldots, 1)$, the "centre of mass". Let $S(\Delta^n)$ be the union of $[bw_0 \dots w_{n-1}]$ where $[w_0 \dots w_{n-1}]$ lies in some subdivided face $S(\Delta_i^{n-1})$ where Δ_i^{n-1} is the ith face (so the definition is inductive over n). Explicitly, the vertices of simplices in $S(\Delta^n)$ are all barycentres of all k-faces ($0 \le k \le n$) of Δ^n. If $[v_{i_0} \dots v_{i_k}] \subset [v_0 \dots v_n]$ is a k-face, its barycentre is $(t_0 \dots t_n)$ where $t_i = \frac{1}{k+1}$ if $i = i_j$ for some j, 0 if $i \neq$ any i_j.

Technical lemma: if σ is one of the n-simplices in $S(\Delta^n)$, $\operatorname{diam}(\sigma) \le \frac{n}{n+1}\operatorname{diam}(\Delta^n)$, independently of the shape of Δ^n (i.e. $\forall [v_0 \dots v_n] \in \mathbb{R}^n$); we'll prove this next lecture. Note that we do need this result for a general simplex, not just for Δ^n, as we want to iterate, and the canonical homeomorphism from Δ^n to a general n-simplex does not preserve length.

Construction of D, step ii: divide $\delta^n \times [0,1]$ into $(n+1)$-simplices such that on the lower boundary, we just get Δ^n, but on the upper boundary we see $S(\Delta^n)$. Our aim for building (this division), T, is to join simplices in $\Delta^n \times \{0\} \cup \partial\Delta^n \times [0,1]$ to the barycentre of $\Delta^n \times \{1\}$. Explicitly, T is constructed inductively in n so as to satisfy $\partial T + T\partial = 1 - S$.

For an n-simplex λ, $T\lambda = b_\lambda(\lambda - T\partial\lambda)$, where $b_\lambda : [v_0 \dots v_n] \mapsto [bv_0 \dots v_n]$ where b in the marycentre of the simplex λ. Assuming $\partial T + T\partial = 1 - S$ for simplices of smaller dimension, $\partial T\lambda = \partial(b_\lambda(\lambda - T\partial\lambda))$. Note $\partial b_\lambda = 1 - b_\lambda\partial$ (by considering the definition of b_λ: $\partial[bv_0 \dots v_n] = [v_0 \dots v_n] - [bv_1 \dots v_n] + \dots$; this is valid since b_λ uses its own fixed value λ, it doesn't just take the barycentre of the simplex it's applied to) so this is $= \lambda - T\partial\lambda - b_\lambda\partial(\lambda - T\partial\lambda)$ and we see $[\partial T(\partial\lambda)$ and - don't understand this$]$ $\partial T + T\partial = 1 - S$ inductively, so this is $\lambda - T\partial\lambda - b_\lambda(S\partial\lambda + T\partial\partial\lambda)$; $\partial\partial = 0$ so this is $\lambda - T\partial\lambda - S\lambda$. Noting $b_\lambda(S\partial\lambda) = S\lambda$, this implies $\partial T + T\partial = 1 - S$ on n-simplices λ, so inductively this always holds.

Construction of D, step iii): X may not be a nice space, but the open cover $\{\sigma^{-1}(\operatorname{int}U_\alpha)\}_{\alpha\in I}$ of $\Delta^n \xrightarrow{\sigma} X$ has a Lebesque number $\delta > 0$. So $\exists m(\sigma)$ such that every simplex in $S^{m(\sigma)}(\sigma)$ lies in U_α for some $\alpha \in I$ (this is where we use the technical "shrinking" lemma) (Recall a Lebesque number δ for a cover U_α of a metric space (Z, d) is some $\delta > 0$ such that every set of diameter $< \delta$ (i.e. every $B_\delta(x)$ for $x \in Z$) lies completely inside some set of the cover). Define $D : C_n(X) \to C_{n+1}(X)$ by $\sigma \mapsto D_{m(\sigma)}(\sigma)$ where $D_m = \sum i = 0^{m-1} TS^i$ (and note $D_0 \equiv 0$). Here we take $m(\sigma)$ to be the least possible value such that all subsimplices of $S^{m(\sigma)}(\sigma)$ lie in elements of $\mathcal{U}$ (recall it is important that $\rho\iota$ is just the identity). Note that for each m, $\partial D_m + D_m\partial = \sum_0^{m-1} \partial TS^i + TS^i\partial = \sum \partial TS^i + T\partial S^i$ (since $\partial S = S\partial$) $= \sum(\partial T + T\partial)S^i = \sum(1 - S)S^i) = 1 - S^m$. So $\partial D\sigma + D\partial\sigma = \sigma - (S^{m(\sigma)}(\sigma) + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma))$ (noting $\partial D\sigma$ is defined as $\partial(D_{m(\sigma)}(\sigma))^m$ (in $C_\star(X, \mathcal{U})$); we cannot do with with the other term because $m(\partial\sigma) \neq m(\sigma)$. We define $\rho(\sigma)$ to be this large bracket, so that this $= \sigma - \rho(\sigma)$.

Key fact: $\rho(\sigma)$ lies in $C_n(X, \mathcal{U})$. Why? Let $\sigma_j = \sigma \mid_{j\text{th face of } \sigma}$. Then certainly $m(\sigma_j) \le m(\sigma)$. The terms $TS^i(\sigma_j)$ which occur in $D(\partial\sigma)$ are all terms in $D_{m(\sigma)}(\partial\sigma)$. So the difference $D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$ is a sum of terms $TS^i\sigma_j$, for $i > m(\sigma_j)$. So $S^i\sigma_j$ is small and T preserves $C_\star(X, \mathcal{U})$. Clearly $S^{m(\sigma)}(\sigma) \in C_n(X, \mathcal{U})$.

10 Excision, Continued

Recall we have $S : C_n(X) \to C_n(X)$ the subdivision operator, $T : C_n(X) \to C_{n+1}(X)$ subdivision of $\Delta^n \times [0,1]$; $\partial T + T\partial = 1 - S$. $D : C_n(X) \to C_{n+1}(X)$ $\sigma \mapsto D_{m(\sigma)}(\sigma) = \sum_{i=1}^{m(\sigma)-1} TS^i\sigma$, taking $m(\sigma)$ minimal such that $S^{m(\sigma)}(\sigma)$ all lie in elements of the cover $\{U_\alpha\}_{\alpha\in I}$. We saw last time $\partial D\sigma + D\partial\sigma = \sigma - (S^{m(\sigma)}(\sigma) + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma))$; difining this large bracket to be $\rho(\sigma)$ we see $\rho(\sigma) \in C_n(X, \mathcal{U})$ (Note $D_{m(\sigma)}(\sigma)$ does not lie there - it is a sum whose early terms are not much smaller than σ).

We have $\partial D\sigma + D\partial\sigma = \sigma - \iota\rho(\sigma)$, $\rho : C_n(X) \to C_n(X, \mathcal{U})$, $\iota : C_n(X, \mathcal{U}) \to C_n(X)$. We need these to be chain maps; ι clearly is, but what about ρ? $\partial(\rho(\sigma)) = \partial\sigma - \partial(D\partial\sigma)$. $\partial D(\partial\sigma) + D(\partial\partial\sigma) = \partial\sigma - \rho(\partial\sigma)$ and $D(\partial\partial\sigma) = 0$, so this is $\rho(\partial\sigma)$ and ρ is a chain map as required.

Finally, if $\sigma \in C_n(X, \mathcal{U})$ then $m(\sigma) = 0$, but $D_0 \cong 0$ so $\rho \circ \iota : C_n(X, \mathcal{U}) \to C_n(X, \mathcal{U})$ is the identity, and ρ is a chain inverse for ι, via a chain homotopy D. So $\rho_\star, \iota_\star$ are isomorphisms on homology $H(C_\star(X, \mathcal{U})) \xrightarrow{\cong} H(C_\star(X))$.

It remains to prove the technical lemma from lecture 9: Let $[v_0 \dots v_n] \in \mathbb{R}^N$ be any simplex. Then each $\sigma \in S[v_0 \dots v_n]$ has diameter $\leq \frac{n}{n+1}$ taht of $[v_0 \dots v_n]$ (where by the diameter of a simplex we mean the diameter of the underlying geometric object in $\mathbb{R}^N$). Recall that points of the simplex have canonical coordinates $(t_0 \dots t_n), \sum t_i = 1$. Observe $\|v - \sum t_i v_i\| = \|(\sum t_i)v - \sum t_i v_i\| \leq \sum_i t_i \max\|v - v_i\| = \max\|v - v_i\|$; the distance from a fixed point to any point in the simplex is maximised by the distance to some vertex. So the diameter of a simplex is the maximal separation of two vertices. We'll proove the lemma inductively: suppose it is valid for all k-simplices for $k < n$. Let $\sigma \in S[v_0 \dots v_n]$. The case we have to consider is $\sigma = [bv_{i_1} \dots v_{i_n}]$ for some $\{i_1, \dots, i_n\} \subset$ the set of indices of new vertices; otherwise all the vertices lie in some face of $[v_0 \dots v_n]$, meaning this is actually a smaller-dimensional simplex. The diameter $\mathrm{diam}(\sigma) = \|b - v_i\|$ for some i; otherwise use the inductive hypothesis to bound $\|v_{i_j} - v_{i_k}\|$. Consider the face $[v_0 \dots \widehat{v_i} \dots v_n]$ of the original simylex. The barycentre b_i of this face has coordinates $(t_0 \dots t_n)$ where $t_j = \frac{1}{n}$ for $j \neq 0$, $t_i = 0$. By contrast b has coordinates $(t_0 \dots t_n)$ where $t_j = \frac{1}{n+1} \forall j$. Observe $b = \frac{n}{n+1} b_i + \frac{1}{n+1} v_i$. So b lies on the line between v_i and b_i, and $\mathrm{diameter}([b, v_{i_1}, \dots, v_{i_k}]) = \|b - v_i\| = \frac{n}{n+1}\|b_i - v_i\| \leq \frac{n}{n+1}\mathrm{diam}[v_0 \dots v_n]$.

Remarks: When we deduced the Mayer-Vietoris Theorem (and indeed excision) from the statement that $C_\star(X, \mathcal{U}) \hookrightarrow C_\star(X)$ is a homology isomorphism, the lecturer stated that the homology isomorphism induced a homology isomorphism $\frac{C_\star(X,\mathcal{U})}{C_\star(A)} \hookrightarrow \frac{C_\star(X)}{C_\star(A)}$ for $\mathcal{U} = A \cup B$. This is true since the equation $\partial D + D\partial = 1 - \iota\rho$ descends to the quotient group by $C_\star(A)$, since all of our operators $S, T, D, \dots$ preserve the property of lying in A. Alternatively, one could use the 5-lemma.

Remark: The theorems we've proved all hold in cohomology, e.g. $H^\star(X, A) \overset{\cong}{\hookrightarrow} H^\star(X \setminus Z, A \setminus Z)$ if $\mathrm{cl}(Z) \subset \mathrm{int}(A)$. In the axioms for a generalized cohomology theory, we have functoriality, homotopy invariance, LES for pairs and excision just as for homology, but the "unions" axiom changes: $h^\star(\amalg X_\alpha) \cong \prod_\alpha H^\star(X_\alpha)$ (so $H^\star(\amalg X_\alpha) = \prod_\alpha H^\star(X_\alpha)$) - direct product, not direct sum.

Recall we are studying cell complexes: $X = \bigcup_{k\geq 0} X_k$, X_k is the k-skeleton, $X_k = X_{k-1} \cup$ finitely many k-cells D^k. Define: reduced homology $\tilde{H}_\star(X) := H_\star(X, \mathrm{pt})$. So $\tilde{H}_\star(X) = H_\star(X)$ for $\star > 0$, $\tilde{H}_0(X) \oplus \mathbb{Z} = H_0(X)$.

Lemma: If X is a cell complex and $A \subset X$ a subcomplex, $H_\star(X, A) \cong \tilde{H}_\star(\frac{X}{A})$. Think: a "SES of spaces" $A \hookrightarrow X \hookrightarrow \frac{X}{A}$ gives rise to a LES in homology $\cdots \to \tilde{H}_i(A) \to \tilde{H}_i(X) \to \tilde{H}_i(\frac{X}{A}) \to \tilde{H}_{i-1}(A) \to \dots$. (Exercise: this is the LES of the pair). To prove this we use the technical fact that if $A \subset X$ is a subcomplex of a cell complex, there are open neighbourhoods $V \supset A$ in X which are homotopy equivalent to A and such that $A \hookrightarrow V$ is a deformation retract.

Then (by the 5-lemma; see example sheet 2) $H_\star(X,A) \cong H_\star(X,V)$. So consider:
$\begin{array}{ccccc} H_k(X,A) & \to & H_k(X,V) & \leftarrow & H_k(X \setminus A, V \setminus A) \\ \downarrow & & & & \downarrow \\ H_k(\frac{X}{A}, \frac{A}{A} & \to & H_k(\frac{X}{A}, \frac{V}{A}) & \leftarrow & H_k(\frac{X}{A} \setminus \frac{A}{A}, \frac{V}{A} \setminus \frac{A}{A}) \end{array}$. The top left and bottom left maps are isomorphisms by the 5-lemma, the top right and bottom right are isomorphisms by excision, and the right hand side is an isomorphism since the quotient map $X \to \frac{X}{A}$ is a homeomorphism on the complement of A. So the left hand side must also be an isomorphism.

Observe, if X is a cell complex, $X \supset X_k \supset X_{k-1} \supset \cdots \supset X_1 \supset X_0$ and $\frac{X_k}{X_{k-1}} = \bigvee_{i=1}^{i_k} S_i^k$ is a wedge of k-spheres, since $X^k = X^{k-1} \cup \overline{D^k} \cup \cdots \cup \overline{D^k}$ glued by maps $\partial D^k \to X_{k-1}$.

11 Cellular Homology

Let $X = \bigcup_{k \geq 0} X_k$ be a cell complex.

Lemma: i) $H_k(X_k, X_{k-1}) \cong \mathbb{Z}^{n_k}$, the free abelian group on the set of k-cells, $H_i(X_k, X_{k-1}) = 0$ for $i \neq k$ ii) $H_i(X_k) = 0$ if $i > k$; inclusion $X_k \hookrightarrow X$ induces an isomorphism $H_i(X_k) \xrightarrow{\cong} H_i(X) \forall i < k$: Recall if $A \subset X$ is a subcomplex, $H_\star(X,A) \cong \tilde{H}_\star(\frac{X}{A})$ (where $\frac{X}{A}$ is the quotient collapsing A to a point). $\frac{X_k}{X_{k-1}} = \bigvee_{i=1}^{n_k} S^k$ and Mayer-Vietoris implies $H^\star(\bigvee_{i=1}^{n_k} S^k) = \mathbb{Z}^{n_k}$ for $\star = k$, $\mathbb{Z}$ for $\star = 0$ and 0 otherwise. This proves i). For ii) we consider the LES of the pair (X_k, X_{k-1}) (we will use X_k and X^k at random to denote the same thing; note that for homology groups $H^\star, H_\star$ are different) $\cdots \to H_{i+1}(X_k, X_{k-1}) \to H_i(X_{k-1}) \to H_i(X_k) \to H_i(X_k, X_{k-1}) \to H_{i-1}(X_{k-1}) \to \ldots$. If $i > k$, we get $0 \to H_i(X_{k-1}) \to H_i(X_k) \to 0$, so this is an isomorphism and inductively $H_i(X_k) \cong H_i(X_{k-1}) \cong \cdots \cong H_i(X_0) = 0$ (for $i > 0$). If $i < k$, $H_i(X^k) \cong H_i(X^{k+1}) \cong \cdots \cong H_i(X^{k+n})$ for any finite $n > 0$. If X is compact then $X = X^N$ for N sufficiently large and this proves ii); for a general, perhaps infinite cell complex X, we complete the argument by: any simplex $\sigma : \Delta^n \to X$ has compact image, so a simplex lies in some X^n. Similarly therefore finite sums of simplexes also lie in some X^n, so if $\alpha \in H_i(X)$, $\exists n$ such that $\alpha \in H_i(X^n) \hookrightarrow H_i(X)$. With a little more thought, this says that for large enough n, $H_i(X^n) \to H_i(X)$ is onto. Similarly, a chain between two cycles representing a relation or identity at the level of homology is a finite sum of simplices, so actually the finite X^n also see relations between cycles. So $H_i(X^n) \to H_i(X)$ is an isomorphism for n large enough, and then the finite induction proves ii).

Example: 1. For a sequence $(n_1, n_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$, there is a connected cell complex such that $H_i(X) \cong \mathbb{Z}^{n_i} \forall i > 0$, namely $X = \bigvee_{i=1}^{n_1} S_i^1 \vee \bigvee_{j=1}^{n_2} S_j^2 \vee \ldots$. This was not immediately obvious, and tells us that we will not be able to obtain any arithmetic restrictions on homology groups for general spaces - though finding such restrictions for particular types of spaces is still a very important area of topology. 2. D^2 has one 0-cell, one 1-cell and one 2-cell. $H_\star = \mathbb{Z}$ for $\star = 0$, 0 otherwise. (3.) T^2 has one 0-cell, 2 1-cells and one 2-cell; $H_\star = \mathbb{Z}$ for $\star = 0$, $\mathbb{Z}^2$ for $\star = 1$, $\mathbb{Z}$ for $\star = 2$ and 0 otherwise. 4. The Klein bottle has one 0-cell, two 1-cells and a 2-cell; $H_\star = \mathbb{Z}$ for $\star = 0$, $\mathbb{Z} \oplus \frac{\mathbb{Z}}{2}$ for $\star = 1$ and 0 otherwise. 5. Σ_2 has one 0-cell, four 1-cells and one 2-cell; $H_\star = \mathbb{Z}$ for $\star = 0$, $\mathbb{Z}^4$ for $\star = 1$, $\mathbb{Z}$ for $\star = 2$ and 0 otherwise. 6. $S^2 \vee S^2$ has one 0-cell and two 2-cells; $H_\star = \mathbb{Z}$ for $\star = 0, 2$.

Observe that the rank of $H_k(X)$ is $\leq$ the number of k-cells. This suggests that the homology could be induced from some chain complex where this would hold - but our current chain groups are infinite-dimensional.

Definition: The cellular chain group $C_k^{\text{cell}}(X) = \mathbb{Z}^{n_k} = H_k(X^k, X^{k-1})$ is free abelian on k-cells of X. The cellular boundary operator $d^{\text{cell}} : H_k(X^k, X^{k-1}) \to H_{k-1}(X^{k-1}, X^{k-2})$ is defined as the map $H_k(X^k, X^{k-1}) \to H_k(X^{k-1}) \to H_{k-1}(X^{k-1}, X^{k-2}) = \text{incl}_{\text{LES}} \circ \partial_{\text{LES}}$, where incl_{LES} is the inclusion from the LES of the pair (X^{k-1}, X^{k-2}) and ∂_{LES} is the boundary map from the LES of the pair (X^k, X^{k-1}).

Lemma: $d^{\text{cell}} \circ d^{\text{cell}} = 0$ (so $(C_\star^{\text{cell}}, d^{\text{cell}})$ is a valid chain complex): $d^{\text{cell}} \circ d^{\text{cell}} = \iota \circ \partial \circ \iota \circ \partial$, but the central ∂ and ι are successive maps in the LES of the pair (X^{k-1}, X^{k-2}) so compose to 0.

Theorem: $H_\star^{\text{cell}}(X) \cong H_\star(X)$; in particular $H_\star^{\text{cell}}(X) = H(C_\star^{\text{cell}}, d^{\text{cell}})$ is a topological invariant of X. Note that $C_\star^{\text{cell}}(X)$ is certainly not a topological invariant - it depends on the skeleton i.e. the choice of cell decomposition.

We have pieces of LES of pairs: $0 \to H_k(X^k) \xrightarrow{\iota} H_k(X^k, X^{k-1})$ so ι is an injection, and $H_{k+1}(X^{k+1}, X^k) \xrightarrow{\partial} H_k(X^k) \to H_k(X^{k+1}) \cong H_k(X) \to 0 \cong H_k(X^{k+1}, X^k)$, the two $\cong$s being by the earlier lemma. So $H(X) = \frac{H_k(X^k)}{\text{im}(\partial: H_{k+1}(X^{k+1}, X^k) \to H_k(X^k))}$ which composing both with the injection $\iota = \frac{\iota(H_k(X^k)}{\text{im}(d^{\text{cell}})}$. This $= \frac{\ker(\partial: H_k(X^k, X^{k-1}) \to H_{k-1}(X^{k-1}))}{\text{im}(d^{\text{cell}}: C_{k+1}^{\text{cell}} \to C_k^{\text{cell}})} = \frac{\ker(d^{\text{cell}}: C_k^{\text{cell}} \to C_{k-1}^{\text{cell}})}{\text{im}(d^{\text{cell}}: C_{k+1}^{\text{cell}} \to C_k^{\text{cell}})}$ since $\iota : H_{k-1}(X^{k-1}) \to H_{k-1}(X^{k-1}, X^{k-2})$ is injective.

Corollary: If X is a cell complex 1. $H_q(X)$ is a finitely generated abelian group of range $\leq n_q$ the number of q-cells 2. If $H_q(X) \neq 0$, every cell structure on X has q-cells 3. If X is compact, $H_\star(X, \mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-vector space (and similarly for any other coefficient ring) 4. If X has only even-dimensional cells, $H_\star(X) \cong C_\star^{\text{cell}}(X)$ for this cell decomposition.

Remark: 4. is worth noting, since various spaces of importance in algebraic geometry have such cell structures, e.g. $\mathbb{CP}^n$ or $\text{Gr}_k(\mathbb{C}^n)$ (the Graussmanion of k-planes in $\mathbb{C}^n$).

12 Time for Change

(Ohio: nice touch)

Recall $d^{\text{cell}} : C_k^{\text{cell}} \to C_{k-1}^{\text{Cell}}$ where the groups are free abelian on k-cells and $(k-1)$-cells respectively. Computational recipe: $d_k^{\text{cell}}([D_\alpha^k]) = \sum_\beta d_{\alpha\beta}[D_\beta^{k-1}]$ where $d_{\alpha\beta}$ is the degree of the map of spheres $S^{k-1} = \partial D_\alpha^k \xrightarrow{\phi} X^{k-1} \to \frac{X^{k-1}}{X^{k-2}} = \bigvee_\beta S_\beta^{k-1} \to S_\beta^{k-1}$, where ϕ is the attaching map gluing the boundary of the α k-cell to the $(k-1)$-skeleton and the final map is a quotient. Here α indexes k-cells and β indexes $(k-1)$-cells; proving this result is a good exercise in the definitions.

Example: Real projective space $\mathbb{RP}^n$ is the space of unoriented lines in $\mathbb{R}^{n+1}$, $= \frac{S^n}{\pm 1}$. $\mathbb{RP}^2 = \frac{S^2}{\pm 1} = D^2 \cup \frac{\text{equator}}{\pm 1} = D^2 \cup \mathbb{RP}^1$; more generally $\mathbb{RP}^n = D^n \cup \mathbb{RP}^{n-1} = \cdots = D^n \cup \cdots \cup D^1 \cup \text{pt}$, so $\mathbb{RP}^n$ has a natural cell structure with one cell of each dimension $0 \leq \star \leq n$. $C_\star^{\text{cell}} = 0 \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \to 0$, where the first $\mathbb{Z}$ has degree $\star = n$ and the last $\star = 0$. What are the maps? In forming $\mathbb{RP}^k$, $D^k \to \mathbb{RP}^{k-1}$ (= X^{k-1} the $(k-1)$-skeleton) is attached via the map $S^{k-1} \to \mathbb{RP}^{k-1}$ which is the natural quotient map. This gives a map

$S^{k-1} \to \mathbb{RP}^{k-1} \to \frac{\mathbb{RP}^{k-1}}{\mathbb{RP}^{k-2}} = S^{k-1}$, which is 2:1, and if $x \in S^{k-1}$, it has two preimages where the map is locally a homeomorphism. The two local maps differ by the antipodal map, so the degree is $1 + \text{degree(antipodal)} = 1 + (-1)^k$, i.e. $d_k^{\text{cell}} : \mathbb{Z} \to \mathbb{Z}$ is multiplication by $1 + (-1)^k$.

So $C_\star^{\text{cell}}$ is, for n even, $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$, and for n odd, $0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$. So $H_\star(\mathbb{RP}^n, \mathbb{Z}) = \mathbb{Z}$ for $\star = 0$ or $\star = n$ and n odd, $\frac{\mathbb{Z}}{2}$ for $0 < \star < n$ and $\star$ odd, and 0 otherwise.

Remark: One can define homology using chain complexes with other coefficient groups than $\mathbb{Z}$; any abelian group works. I.e. we look at $C_\star(X, G) = \{\sum_i \alpha_i \sigma_i : \alpha_i \in G, \sigma_i : \Delta^\star \to X\}$. For example, it can be very useful to take $G = \frac{\mathbb{Z}}{p}$. For $G = \frac{\mathbb{Z}}{2}$, there is still the cellular homology, with $C_\star^{\text{cell}}$ the $\frac{\mathbb{Z}}{2}$-vector space generated by $\star$-cells. For $\mathbb{RP}^n$ the complex decomes $0 \to \frac{\mathbb{Z}}{2} \xrightarrow{0} \frac{\mathbb{Z}}{2} \xrightarrow{0} \frac{\mathbb{Z}}{2} \to \cdots \to \frac{\mathbb{Z}}{2} \to 0$, since $d^{\text{cell}}(D_\alpha) = \sum d_{\alpha\beta}(\text{mod } 2)[D_\beta]$, so $H_\star(\mathbb{RP}^n, \frac{\mathbb{Z}}{2}) = \frac{\mathbb{Z}}{2}$ for $0 \leq \star \leq n$, 0 otherwise. (This kind of thing will be very important later, when we are doing cohomology, since we will want to add a ring structure to our groups. For now, just note that this can be done, it can change the groups, and this change is sometimes to simplify them).

Definition: A covering space $p : X \to B$ is a fibre bundle with discrete fibres, i.e. its a map such that $\forall b \in B \exists$ open $U \ni b$ such that $p^{-1}(U) = \amalg_{\alpha \in A} V_\alpha$ and $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism (and p is <u>onto</u>).

Examples: 1. $\mathbb{R} \to S^1$ $t \mapsto e^{2\pi i t}$ 2. $S^1 \to S^1$ $z \mapsto z^2$ for S^1 viewed as $\subset \mathbb{C}$ 3. cutting the surface of genus 3 "down the middle", so its left and right halves look like tori with two open tubes protuding from each, then mapping 2:1 by rotation about a central axis onto a single such surface, and finally identifying the two cut ends of tubes, resulting in a surface of genus 2. 4. $S^n \to \mathbb{RP}^n$.

Fact: If $\sigma : Z \to B$ is a map from a contractible nice (locally path-connected) space, then σ <u>lifts</u> to X, and the lift is uniquely determined by its value at a point: $\begin{array}{ccc} & & X \\ & \overset{\exists\tilde{\sigma}}{\nearrow} & \downarrow p \\ Z & \xrightarrow{\sigma} & B \end{array}$. We are interested in the case where σ is a simplex (hence the notation).

Lemma: If $p : X \to B$ is a double cover, i.e. every point has two preimages, there is a LES $\cdots \to H_r(B, \frac{\mathbb{Z}}{2}) \to H_r(X, \frac{\mathbb{Z}}{2} \xrightarrow{p_\star} H_r(B, \frac{\mathbb{Z}}{2}) \to H_{r-1}(B, \frac{\mathbb{Z}}{2}) \to H_{r-1}(X, \frac{\mathbb{Z}}{2}) \xrightarrow{p_\star} \ldots$: this is the LES associated to a SES of chain complexes given by $0 \to C_\star(B, \frac{\mathbb{Z}}{2}) \to C_\star(X, \frac{\mathbb{Z}}{2}) \to C_\star(B, \frac{\mathbb{Z}}{2}) \to 0$, where the first nontrivial map is $\sigma \mapsto \tilde{\sigma_1} + \tilde{\sigma_2}$ (where the $\tilde{\sigma_i}$ are the two lifts of σ, by the lemma) and the other is $\tau \mapsto p(\tau)$. Note the composite $\sigma \mapsto p(\tilde{\sigma_1}) + p(\tilde{\sigma_2}) = 2\sigma \equiv 0$ since we are considering coefficients in $\frac{\mathbb{Z}}{2}$. Surjectivity on the right follows by the fact again and exactness, then the rest is straightforward.

Remark: We will build this LES another way later, cf Gysin sequences, at least for $S^n \to \mathbb{RP}^n$.

Theorem (Borsuk-Ulam): If $f : S^n \to S^n$ is odd, $f(-x) = -f(x) \forall x$, then f has odd degree: f induces a map $\mathbb{RP}^n \xrightarrow{\overline{f}} \mathbb{RP}^n$, since f is odd. We have:

$$\begin{array}{ccccccccc} 0 & \to & C_i(\mathbb{RP}^n, \frac{\mathbb{Z}}{2}) & \to & C_i(S^n, \frac{\mathbb{Z}}{2}) & \xrightarrow{p} & C_i(\mathbb{RP}^n, \frac{\mathbb{Z}}{2}) & \to & 0 \\ & & \downarrow \overline{f} & & \downarrow f & & \downarrow \overline{f} & & \\ 0 & \to & C_i(\mathbb{RP}^n, \frac{\mathbb{Z}}{2}) & \to & C_i(S^n, \frac{\mathbb{Z}}{2}) & \xrightarrow{p} & C_i(\mathbb{RP}^n, \frac{\mathbb{Z}}{2}) & \to & 0 \end{array}$$

. The squares in this diagram commute, since $pf = \overline{f}p$. In the LES (with all coefficient rings being $\frac{\mathbb{Z}}{2}$), $0 \to H_n(\mathbb{RP}^n) \to H_n(S^n) \to H_n(\mathbb{RP}^n) \to H_{n-1}(\mathbb{RP}^n) \to 0 \to \cdots \to 0 \to H_i(\mathbb{RP}^n) \to H_{i-1}(\mathbb{RP}^n) \to 0 \to \cdots \to H_1(\mathbb{RP}^n) \to H_0(\mathbb{RP}^n) \to H_0(S^n) \to H_0(\mathbb{RP}^n) \to 0$, where the 0s in the middle come from $H_i(S^n) = 0$ if $0 < i < n$; the initial zero is $H_{n+1}(\mathbb{RP}^n)$. $f, \overline{f}$ obviously induce isomorphisms on H_0; using pieces like

$$\begin{array}{ccccccc} 0 & \to & H_i(\mathbb{RP}^n) & \to & H_{i-1}(\mathbb{RP}^n) & \to & 0 \\ & & \downarrow \overline{f} & & \downarrow \overline{f} & & \\ 0 & \to & H_i(\mathbb{RP}^n) & \to & H_{i-1}(\mathbb{RP}^n) & \to & 0 \end{array}$$

we induct; for the top case, the second nontrivial map in the LES is multiplication by 2 (i.e. 0), so the first is an isomorphism. So f is an isomorphism on $H_n(S^n, \frac{\mathbb{Z}}{2})$ and so the degree of f is odd.

Corollary: 1. If $g : S^n \to \mathbb{R}^n$ is continuous, $\exists x \in S^n$ such that $g(x) = g(-x)$ (e.g. if $n = 2$, there are always antipodal points on the Earth's surface at the same temperature and pressure). 2. Cheese-and-pickle sandwich theorem: if $A_1 \dots A_n$ are bounded measurable sets in $\mathbb{R}^n$, there exists a hyperplane cutting each of them into equal volumes (thus we can share the sandwich each getting even amounts of bread, cheese and pickle).

13 Euler Characteristic

Definition: Let X be a cell complex. 1. The jth Betti number $b_j(X) = \mathrm{rk}_{\mathbb{Q}} H_j(X, \mathbb{Q})$ 2. The Euler characteristic $\chi(X)$ (or $e(X)$ is $\sum_{j \geq 0} (-1)^j b_j(X)$, which makes sense for X compact. (Similarly for more general spaces; these are the definitions when they make sense).

Proposition: Let X be a compact cell complex. Then $\chi(X) = \sum_{j\geq 0}(-1)^j(\#j\text{-cells})$; in particular the RHS is independent of choice of cell decomposition: in fact, if $(C_\star, d)$ is any chain complex for which each C_i is finitely generated and only finitely many are nonzero, then $\sum_{j\geq 0}(-1)^j \mathrm{rk}(H_j(C_\star, d)) = \sum_{j \geq 0}(-1)^j \mathrm{rk}(C_j)$. To see this, we split up the chain complex into a collection of short exact sequences $0 \to \ker(d_k) =: Z_k \to C_k \xrightarrow{d_k} \mathrm{im}(d_k) =: B_{k-1} \to 0$, where $C_\star = \cdots \to C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} C_{k-2} \to \dots$ and $H_k(C_\star) = \frac{Z_k}{B_{k+1}}$. So if $Z_k = \mathrm{rk}(Z_k)$ and $b_k = \mathrm{rk}(B_k)$, then $r_k = \mathrm{rk}(C_k) = z_k + b_{k-1}$, for instance since all the terms in the SES are free (so non-canonically $C_k \cong Z_k \oplus B_{k-1}$, by choosing a splitting $\rho : B_{k-1} \to C_k$ such that $d_k \circ \rho = \mathrm{id}$. So $r_0 - r_1 + r_2 - r_3 + \cdots = z_0 - (z_1 + b_0) + (z_2 + b_1) - (z_3 + b_2) + \cdots = (z_0 - b_0) - (z_1 - b_1) + \cdots = \mathrm{rk}H_0 - \mathrm{rk}H_1 + \cdots = \sum_{j\geq 0}(-1)^j \mathrm{rk}(H_j)$ (being sloppy about what happens at the top, but the lecturer assures us that this comes out in the wash; the reader may check if he likes).

So $\chi(X)$ is often very easy to compute, and sometimes sufficies to distinguish spaces.

Remark: Later, we'll see that for M a (closed) manifold, $\chi(X)$ is equivalent data to a distinguished element of $H^\star(M)$. But in general, and for now, we know nothing about how e.g. maps act on χ.

Examples and properties: 1. $\chi(S^n) = 2$ for n even, 0 for n odd. 2. $\chi(\Sigma_g) = 2 - 2g$ 3. $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ if $X = A \cup B$ is a union of subcomplexes (or more generally of open sets).

Aside: χ is the unique homotopy-invariant $\mathbb{Z}$-valued function on cell complexes such that 3. holds and $\chi(\text{pt}) = 1, \chi(\emptyset) = 0$. Compare with volume (in fact, there are many possible volume functions, and in some sense it is an accident that a particular one has come to be known as volume. However, χ is the only integer-valued one).

4. $\chi(X \times Y) = \chi(X) \times \chi(Y)$ since $X \times Y$ has a cell complex structure such that k-cells of $X \times Y$ are products (i-cell of X) $\times$ (($k-i$)-cell of Y). 5. A <u>fibre bundle</u> $\pi : X \to B$ is a map which is locally trivial, i.e. $\forall b \in B \exists U \ni b$ open such that $\pi^{-1}U \xrightarrow{\cong} U \times F$ by a map taking $\pi^{-1}(q) \overset{\phi_q}{\mapsto} \{q\} \times F \forall q \in U$; F is the <u>fibre</u> of the bundle. Examples: $S^3 \xrightarrow{h} S^2$ the Hopf map, taking $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$ to $[z_1 : Z_2] \in \mathbb{CP}^1$ is a fibre bundle with fibre S^1. A covering space $p : X \to B$ is a fibre bundle with discrete fibres. A vector bundle is a fibre bundle with $F \cong \mathbb{R}^n$ and the local trivialisations ϕ being linear on the fibres, ϕ_q linear.

3. and 4. imply $\chi(X) = \chi()\chi(F)$. Exercise: if $\tilde{X} \to X$ is a d-sheeted cover, $\chi(\tilde{X}) = d\chi(X)$ (cf the "lifting" results for cells).

Theorem: Let X be a finite cell complex. Then $H^0(X, \mathbb{Z}) \cong \frac{H_j(X,\mathbb{Z})}{\text{Torsion}} \oplus \text{Torsion}(H_{j-1}(X, \mathbb{Z}))$; to get $H^\star$ from $H_\star$ we "keep the same $\mathbb{Z}$-summands and shift the torsion $\frac{\mathbb{Z}}{k}$ summands up one degree". E.g. $H_\star(\text{Klein}) = \mathbb{Z}$ for $\star = 0$, $\mathbb{Z} \oplus \frac{\mathbb{Z}}{2}$ for $\star = 1$, and 0 for $\star \geq 2$, so $H^\star(\text{Klein}) = \mathbb{Z}$ for $\star = 0, 1$, $\frac{\mathbb{Z}}{2}$ for $\star = 2$, and 0 otherwise.

Remark: The <u>Universal Coefficient Theorem</u> says that this is true for any space such that $H_j(X)$ is fintely generated for each j. We use the stronger assumption that $\exists$ finitely generated <u>chain groups</u> e.g. $C_\star^{\text{cell}}$. (This is in some sense not the right proof for this result; however, we don't have time for the homological algebra required for the "good" proof. Therefore, the reader should not worry about this proof (however, it is of course a perfectly valid, rigorous proof)). As before, we take the (finitely generated) chain complex and split the associated SES $0 \to Z_n \to C_n \to B_{n-1} \to 0$ (Z_n is the group of cycles, B_{n-1} that of boundaries), writing (non-canonically) $C_n \cong Z_n \oplus K_n$ where $K_n \cong B_{n-1}$ (so the subscripts record the degree in which we live in the chain complex). Homology is the quotient: $0 \to B_n \to Z_n \to H_n \to 0$. So the homology groups are actually the homology groups of a collection of many very short chain complexes $0 \to K_{n+1} \to Z_n \to 0$ (note: this map has cokernel, so contributes homology in only one degree, namely n). So it suffices to proove the theorem for a chain complex of the shap $0 \to \mathbb{Z}^a \to \mathbb{Z}^b \to 0$ ("that's what we have after taking bases").

The Smith normal form theorem says that we can choose a $\mathbb{Z}$-basis such that the matrix has shap $\begin{pmatrix} d_1 & & & & & & \\ & d_2 & & & & & \\ & & \cdots & & & & \\ & & & d_k & & & \\ & & & & 0 & & \\ & & & & & \cdots & \\ & & & & & & 0 \end{pmatrix}$ with $d_1 \mid d_2, d_2 \mid d_3, \ldots, d_{k-1} \mid d_k$. So the map $0 \to \mathbb{Z}^a \xrightarrow{\text{matrix}} \mathbb{Z}^b \to 0$ breaks into subcomplexes of shape $0 \to \mathbb{Z}_{(i)} \xrightarrow{\times d} \mathbb{Z}_{(i-1)} \to 0$ and $0 \to \mathbb{Z}_{(i)} \xrightarrow{0} \mathbb{Z}_{(i-1)} \to 0$ for (i), $(i-1)$ some degrees. The cohomology groups are given by dualising these complexes,

$0 \leftarrow \mathbb{Z}_{(i)} \xleftarrow{d} \mathbb{Z}_{(i-1)} \leftarrow 0, 0 \leftarrow \mathbb{Z}_{(i)} \xleftarrow{0} 0$. These have cohomology and homology related as in the theorem.

Digression (Morse theory)

Let M be a shooth closed manifold. Let $f : M \to \mathbb{R}$ be a function with non-degenerate critical points, i.e. if in local coordinates (x_i) $(\frac{\partial f}{\partial x_i}) \equiv 0$ (x is critical) then $\det(\frac{\partial^2 f}{\partial x_i \partial x_j}) \neq 0$ (the reader should check this is independent of the choice of coordinates). Such a function is called a Morse function, and they exist in abundence (they are dense in $C^\infty(M, \mathbb{R})$). Pick a Riemannian metric g on M. Then every point of M (other than critical points) lies on a unique flow-line of f, i.e. a solution of $\dot{x}(t) = \nabla f(x(t))$ where ∇ is the gradient wrt metric g. Let index(p) for p critical be the number of negative eigenvalues of $\mathrm{Hess}_p(f) = (\frac{\partial^2 f}{\partial x_i \partial x_j}) |_p$.

Fact (Morse) (Gass' Law): the subset of M swept out by the gradient flow lines descending from a critical point of index k form a k-cell (if the metric g is generic). So M admits the structure of a cell complex (and things will be finite since M is closed, unless f or g is silly).

Remark: In fact, Morse homology has $C_k^{\mathrm{Morse}}(M)$ = free abelian on index k critical points of a Morse function and d_k^{Morse} counting isolated gradient flow lines. (This is a useful formulation since it can give valid answers for infinite spaces where other homology theories would give $\frac{\infty}{\infty}$ or similar useless nonsense). One key use of homology in geometry is to estimate numbers of critical points of smooth functions.

14 Cup-product

From now on we focus our attention on $H^\star(X)$, which turns out to be a ring.

Definition: Let $\phi \in C^k(X), \psi \in C^l(X)$ be cochains. Then $\phi \cup \psi \in C^{k+l}(X)$, the cup-product (of ϕ and ψ), is defined via, for $\sigma : \Delta^{k+l} \to X$, say $\sigma = [v_0 \dots v_{k+l}]$, $(\phi \cup \psi)(\sigma) := \phi([v_0 \dots v_k]) \cdot \psi([v_k \dots v_{k+l}])$, where the product on the right hand side is the ordinary product in $\mathbb{Z}$ (or more generally, in the coefficient ring).

Lemma: $\partial(\phi \cdot \psi) = \partial\phi \cdot \psi + (-1)^k \phi \cdot \partial\psi$ (the $\phi \cdot \psi$ being the cup-product; we shall generally omit the $\cup$ and may even simply write it as $\phi\psi$), where $k = \deg(\phi)$: $(\partial\phi \cdot \psi)[v_0 \dots vk+l+1] = (\partial\phi)([v_0 \dots v_{k+1}])\psi([v_{k+1} \dots v_{k+l+1}]) = \sum_{i=0}^{k+1}(-1)^i \phi([v_0 \dots \widehat{v_i} \dots v_{k+1}])\psi([v_{k+1} \dots v_{k+l+1}])$. $(-1)^k(\phi \cdot \partial\psi)[v_0 \dots v_{k+l+1}] = \phi([v_0 \dots v_k]) \sum_k^{k+l+1}(-1)^i \psi[v_k \dots \widehat{v_i} \dots$ Modulo errors introduced by the lecturer, the last term of the first of these results cancels with the first term of the second, and the rest of the sum gives the LHS of our result: $\partial(\phi\psi)(\sigma) = (\phi \cup \psi)(\sum_{i=0}^{k+l+1}(-1)^i[v_0 \dots \widehat{v_i} \dots v_{k+l+1}])$.

Corollary: The cup-product descends to cohomology, inducing $H^k(X) \cup H^l(X) \to H^{k+l}(X)$: the product of cycles is a cycle, and the product with a boundary is a boundary.

Remarks: 1. $f : X \to Y$ induces a map of cohomology rings $f^\star : H^\star(Y) \to H^\star(X)$. 2. If X is path-connected, $H^\star(X)$ has a unit $1 \in H^0(X) \cong \mathbb{Z}$, the generator $\phi \in C^0(X)$ takes the value $+1$ on a point $\sigma : \Delta^0 \to X$ (in general, $H^\star(X, R)$ is a unital ring if R has a unit).

Proposition: The product is graded commutative, $\phi \cdot \psi = (-1)^{kl}\psi \cdot \phi$ in cohomology (note we are now working in the cohomology; ϕ, ψ now refer to the equivalence classes in $H^\star$ they represent, rather than directly to the maps in $C^\star$), i.e. $H^\star(X)$ is a graded commutative ring.

Corollary: Therefore, if $\deg(\phi)$ is odd, $\phi \cdot \phi$ is a 2-torsion element. E.g. for the torus $(S^1)^n$, we computed that $H^j(T^n)$ has rank $\binom{n}{j}$ and $H^\star$ is torsion-free. So for every $\phi \in H^1(T^n) \cong \mathbb{Z}^n$, $\phi \cdot \phi = 0$.

The proof of the above proposition "imitates" the proof of homotopy invariance; we use a prism-type operator but this time with "order-reversal of vertices", so that the top face is equal to the bottom face but with order of vertices reversed. Define $\rho : C_n(X) \to C_n(X)$ by $[v_0 \dots v_n] \mapsto [v_n \dots v_0]\epsilon_n$, where $\epsilon_n = (-1)^{\frac{n(n+1)}{2}}$ (this is the sign of this permutation of vertices, or the number of transpositions made). We claim ρ is a chain map, chain homotopic to the identity (via a prism operator). If we take the claim on trust (ha!) (sic) for a moment, then we will have $\rho^\star\phi \cdot \rho^\star\psi(\sigma) = \phi(\epsilon_k \cdot [v_k \dots v_0])\psi(\epsilon_l[v_{k+l} \dots v_k]), \rho^\star(\psi\phi)(\sigma) = \epsilon_{k+l}\psi([v_{k+l} \dots v_k])\phi([v_k \dots v_0])$ $(\star)$. The reader may check $\epsilon_{k+l} = (-1)^{kl}\epsilon_k\epsilon_l$, so $(\star)$ says $\epsilon_k\epsilon_l\rho^\star\phi\rho^\star\psi = \epsilon_{k+l}\rho^\star(\psi\phi)$, which implies $\rho^\star\phi \cdot \rho^\star\psi = (-1)^{kl}\rho^\star(\psi\phi)$, on chains, so $\phi \cdot \psi = (-1)^{kl}\psi \cdot \phi$ on cohomology (since $\rho^\star \equiv \mathrm{id}$ on cohomology).

Claim 1: ρ is a chain map. $(\partial\rho)(\sigma) = \epsilon_n \sum_i (-1)^i \sigma \mid_{[v_n \dots \widehat{v}_{n-i} \dots v_0]}$, $\rho(\partial\sigma) = \rho(\sum_i (-1)^i \sigma \mid_{[v_0 \dots \widehat{v}_i \dots v_n]}$ $) = \epsilon_{n-1} \sum_i (-1)^{n-i} \sigma \mid_{[v_n \dots \widehat{v}_{n-i} \dots v_0]}$ and (the reader should check) $\epsilon_n = (-1)^n \epsilon_{n-1}$. Thus $\partial\rho = \rho\partial$ and ρ is a chain map.

Claim 2: ρ is chain homotopic to the identity (and hence induces the identity on $H^\star$, as we used above): i.e. we will construct $P : C_n(X) \to C_{n+1}(X)$ such that $\partial P + P\partial = \rho - 1$. Let $\pi : \Delta^n \times [0,1] \to \Delta^n$ be projection, and let $P\sigma = \sum_i (-1)^i \epsilon_{n-i}(\sigma\pi) \mid_{[v_0 \dots v_i w_n \dots w_i]}$ (these are the same $(n+1)$-simplicies in $\Delta^n \times [0,1]$ as those used for P in the homotopy invariance proof, except that we have re-ordered the top. Again, $\partial P + P\partial = \rho - 1$ will be the statement that "the boundary of the prism = the prism on the boundary + the top - the base"). $\partial P\sigma = \sum_{j \le i} (-1)^i (-1)^j \epsilon_{n-i}[v_0 \dots \widehat{v}_j \dots v_i w_n \dots w_i] + \sum_{j \ge i} (-1)^i (-1)^{i+1+n-j} \epsilon_{n-i}[v_0 \dots w_i w_n \dots \widehat{w}_j \dots w_i]$. $P\partial\sigma = \sum_{i<j} (-1)^i (-1)^j \epsilon_{n-i-1}[v_0 \dots v_i w_n \dots \widehat{w}_j \dots w_i] + \sum_{i>j} (-1)^{i-1} (-1)^j \epsilon_{n-i}[v_0 \dots \widehat{v}_j \dots v_i w_n \dots w_i]$.

Explicitly evaluating these terms would only serve to confuse the lecturer, so instead the reader should check the following: a) the $j = i$ terms in the first sums give $\epsilon_n[w_n \dots w_0] - [v_0 \dots v_n] + \sum_{i>0}(\epsilon_{n-i})[v_0 \dots v_{i-1} w_n \dots w_i] + \sum_{i<n} (-1)^{n+i+1} \epsilon_{n-i}[v_0 \dots v_i w_n \dots w_{i+1}]$ and here the last two terms cancel via $-\epsilon_{n-i} = (-1)^{n-i}\epsilon_{n+i+1}$, and b) the terms with $j \ne i$ in the expression for $\partial P\sigma$ give $P\partial\sigma$ using $\epsilon_{n-i} = (-1)^{n-i}\epsilon_{n-i-1}$.

Remarks: Given spaces X, Y we have natural maps $\pi_X : X \times Y \to X, \pi_Y$. We can define a product $H^i(X) \times H^j(Y) \to H^{i+j}(X \times Y)$ by $(c_1, c_2) \mapsto \pi_X^\star c_1 \cup \pi_Y^\star c_2$. There is a relative cup-product $H^i(X) \times H^j(X,A) \to H^{i+j}(X,A)$ since $\psi \in C^j(X,A)$ by definition vanishes on chains lying in A, so $(\phi \cdot \psi)(\sigma) = 0$ if all of σ lies in A, since its "back face" - which we "feed into" ψ - does. See the second example sheet for a more general relative cup-product.

15 Unknown

The first set of examples for which we can compute cohomology rings are product spaces:

Theorem (Kunneth): If X is a cell complex and Y a cell complex such that $H^\star(Y)$ is finitely generated free (as an abelian group), then $H^\star(X \times Y) \simeq H^\star(X) \otimes H^\star(Y)$ so $H^\star(X \times Y) \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y)$.

Remarks: So $b_n(X \times Y) = \sum b_i(X) b_{n-i}(Y)$, which refines $\chi(X \times Y) = \chi(X)\chi(Y)$. Contrast: $\chi(Z) = \chi(Y)\chi(B)$ if Z is a fibre bundle over B with fibre F, but <u>not</u> $H^\star(Z) = H^\star(F)H^\star(B)$, e.g. $S^1 \to S^3 \overset{\text{Hopf}}{\to} S^2$.

Proof of Kunneth: Consider the two associations $(X, A) \mapsto h^\star(X, A) = H^\star(X, A) \otimes H^\star(Y), k^\star(X, A) (= H^\star(X \times Y, A \times Y)$. Cross-product $H^\star(X) \otimes H^\star(Y) \to H^\star(X \times Y)$ $(\alpha, \beta) \mapsto \pi_1^\star(\alpha) \cup \pi_2^\star(\beta)$ induces a map $\Phi : H^\star(X, A) \otimes H^\star(Y) \to H^\star(X \times Y, A \times Y)$ since if $\phi \in C^k(X)$ vanishes on chains in A, and $\psi \in C^i(Y)$, then $\phi \cdot \psi \in C^{k+i}(X, Y)$ will vanish on chains in $A \times Y$. Note also $h^\star(\text{pt}, \phi) \simeq k^\star(\text{pt}, \phi)$ induced by $\Phi : h^\star \to k^\star$. So general theory implies that on all complexes, $h^\star(*X, A) \simeq k^\star(X, A)$, provided these <u>are</u> cohomology theory. For $k^\star$ the proofs of the axioms carry over; for $h^\star$, most things are easy. The existence of LES (and tensoring) is not trivial, but holds because tensoring with a free, finitely generated thing <u>does</u> preserve exactness (cf: $\otimes \frac{\mathbb{Z}}{2}$ does <u>not</u> preserve exact sequences, e.g. $0 \to \mathbb{Z} \overset{\times 2}{\to} \mathbb{Z}$ and $0 \to \frac{\mathbb{Z}}{2} \overset{\times 2}{\to} \frac{\mathbb{Z}}{2}$).

Corollary: $H^\star(S^2 \times S^2) = \frac{\mathbb{Z}[x,y]}{x^2 = y^2 = 0}$, $\deg x = \deg y = 2$ (the degree-4 element is xy). $H^\star(T^n) = H^\star(S^1 \times \cdots \times S^1) = \Lambda(x_1, \ldots, x_n), |x_i| = 1$.

Definition: The <u>cuplength</u> $\mathrm{cl}(x) = \max\{N : \exists \alpha_1 \ldots \alpha_n \in H^\star(X)$ such that $\alpha_1 \alpha_2 \ldots \alpha_N \neq 0 \in H^\star(X)\}$.

Theorem: Let $\mathcal{M}$ be a closed smooth manifold. Any smooth $f : \mathcal{M} \to \mathbb{R}$ has $> \mathrm{cl}(\mathcal{M})$ critical points.

Remark: Morse theory implies that if critical points are nondegenerate then there are $\geq \sum_j b_j(\mathcal{M})$ critical points.

Remark: If $f : \mathcal{M} \to \mathbb{R}$ has only two critical points then $\mathcal{M} \simeq$ the sphere.

Proof of theorem: we'll show the number of critical points is bounded below by a "category".

Definition: We say X has <u>category</u> k (LS-category or Lyusternik-Schnirelman) if k is minimal such that $X = \bigcup_{j=1}^k U_j$ with U_i contractible.

Observe: $\mathcal{M}$ has finite category.

In example sheet 2 we'll see $\nu(X) > \mathrm{cl}(X)$ for ν a category. $\nu(X)$ satisfies the following: to each $U \in X$ assign an integer (where U are the minimal number of contractibles to cover) such that i) $A \subset X \Rightarrow \exists$ open $U \supset A$ such that $\nu(A) = \nu(B)$ ii) $\nu(A \cup B) \leq \nu(A) + \nu(B)$ iii) $A \subset B \Rightarrow \nu(A) \leq \nu(B)$ iv) ν is homotopy invariant v) $\nu(\emptyset) = 0, \nu(\text{pt}) = 1$. Any such function from subsets$(X) \to \mathbb{Z}$ is called a category.

Take $f : \mathcal{M} \to \mathbb{R}$; for $c \in \mathbb{R}$ let $\mathcal{M}_c = f^{-1}(-\infty, c]$. Pick a metric on $\mathcal{M}$, then f defines a gradient flow (of $-\nabla f$) by homeomorphisms $\phi^t : \mathcal{M} \to \mathcal{M}$. If c is not critical, i.e. if $f^{-1}(c)$ contains no critical points, then the flow is nontrivial on the level set, so $\exists t, \delta > 0$ such that $\phi^t(\mathcal{M}^{c+\delta}) \subset \mathcal{M}^{c-\delta}$. Let $c_j = \sup\{c : \nu(\mathcal{M}^c) < j\}$, so $c_1 = \min(f), c_{\nu(\mathcal{M})} = \max(f)$. Observe c_j is a critical value of f $\forall j$.

Claim: $c_j < c_{j+1}$ or $f^{-1}(c_j)$ contains infinitely many critical points. Note a finite set in a manifold always lies in an open set U of category 1. Suppose $f(c_j)$ has $< \infty$ critical points. Considre $\mathcal{M}^{c_j+\delta} \leq \nu(\mathcal{M}^{c_j+\delta} - U) + 1 \leq \nu(\mathcal{M}^{c_j-\delta}) + 1$ since $\phi^t(\mathcal{M}^{c_j+\delta} - U) \subset \mathcal{M}^{c_j-\delta}$. But $\nu(\mathcal{M}^{c_j-\delta}) < j$ by the definition of c_j, so $c_{j+1} \geq c_j + \delta > c_j$; this clearly suffices.

Example: $\mathrm{cl}(T^n) = n$ (if $x_1, \ldots, x_n$ are generators of degree 1 cohomology coming from factors, then $x_1 \wedge \cdots \wedge x_n$ generates $H^n(T^n) \simeq \mathbb{Z}$). So any $f : T^n \to \mathbb{R}$ has $\geq (nH)$ critical points (since the no. of critical points is $\geq$ the category $>$ the cup length).

16 Vector Bundles

Recall a vector bundle $E \to X$ is a fibre bundle with linear fibres and with fibrewise-linear local trivialisations, i.e. a family $\{E_x\}_{x\in X}$ of real vector spaces paramaterised by X and a topology on $\bigcup_{x\in X} E_x = E$ such that $\forall x \in X \exists$ open $U \ni x$ such that $E\mid_U \cong U \times \mathbb{R}^n, E_y \mapsto \{y\} \times \mathbb{R}^n$ is linear isomorphism. A section $S : X \to E$ is a map such that $\pi \circ S = \mathrm{id}_X$. The zero-section is the map $x \mapsto 0 \in E_x \forall x$. So if $X \equiv$ the 0-section is the real line and E the plane, with each E_x being a vertical line, then the plot of a smooth function is a section, but a line which "curves back on itself" crossing one of the vertical lines more than once is not.

Examples: 1. Let $X = \mathrm{Gr}_k\mathbb{R}^n$ the set of k-dimension real subspaces in $\mathbb{R}^n$ ($\mathrm{Gr}_1(\mathbb{R}^n) = \mathbb{RP}^{n-1}$). There's a tautological vector bundle $E \to X$ where fibre at x is the subspaces $\langle x \rangle \subset \mathbb{R}^n$, so $\overline{E = \{(x,v)} : v \in \langle x \rangle\} \subset X \times \mathbb{R}^n$. If $\langle , \rangle$ is an inner product on $\mathbb{R}^n$, for $x \in X$, $U := \{y \in X : E_y \cap E_x^\perp = \{0\}\}$ (where $\perp$ is orthogonal subspace wrt this inner product), then $E\mid_U \to U \times E_x$ by $(y, \xi) \mapsto (y, \mathrm{pr}_{\langle x \rangle}\xi)$, where $\mathrm{pr} : E_y = \langle y \rangle \to E_x = \langle x \rangle$ is orthogonal projection. This map is a linear isomorphism on $y \in U$.

2. If M is a smooth manifold, the tangent bundle TM is a naturally associated vector bundle over M. If $M \subset \mathbb{R}^N$, $\overline{TM \subset \mathbb{R}^N \times \mathbb{R}^N}$ can be defined as $\{(m, v) : m \in M, v \in T_mM\}$ where T_mM is the vector subspace of $\mathbb{R}^N$ generated by tangent vectors to curves in M at m: for $\gamma : (-\epsilon, \epsilon) \to M \subset \mathbb{R}^N$ with $0 \mapsto m$, the vector $\gamma'(0) \in \mathbb{R}^N$. So T_mM is $\langle \gamma'(0) : \gamma \text{ varies} \rangle$. This is a vector space of dimension $\dim_{\mathbb{R}} M$; it is not obvious that this depends only on M and not on its embedding in $\mathbb{R}^N$, but this is in fact true. Example: $M = S^n \subset \mathbb{R}^{n+1}$, $TM = \{(x, v) : \|x\| = 1, \langle v, x \rangle = 0\} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

3. If M is a manifold and $Y \subset M$ a smooth submanifold, say closed, the normal bundle $\nu_{\frac{Y}{M}}$ has fibre at $y \in Y$ $\frac{T_yM}{T_yY}$. If M has a metric we could (non-canonically) identify $(\nu_{\frac{M}{Y}})_y = (T_yY)^\perp \subset T_yM$. It is a basic fact of differential topology that $TM, \nu_{\frac{Y}{M}}$ are vector bundles.

Our aim is to associate cohomology classes to submanifolds, at least in nice cases.

Definition: the vector bundle E is oriented if for each $x \in X$ we have a generator $\epsilon_x \in H^n(E_x, E_x \backslash 0) \cong \mathbb{Z}$ (where $n = \mathrm{rk}(E) = \dim_{\mathbb{R}}(E_x)$) and these choices are coherent, in the sense that if U is a trivialising open set, for $E\mid_X \xrightarrow{\cong} U \times E_x$ by $E_y \mapsto \{y\} \times E_x$ (which is linear so has $E_y \setminus 0 \to E_x \setminus 0$), the induced map $H^n(E_x, E_x \setminus 0) \to H^n(E_y, E_y \setminus 0)$ takes $\epsilon_x \to \epsilon_y$.

Definition: A smooth manifold M is oriented if TM is oriented. A submanifold $Y \subset M$ is co-oriented if $\nu_{\frac{Y}{M}}$ is oriented.

Excercise: If M and Y are oriented then Y is co-oriented; we will in fact prove everything one needs to show this, but will not come back and point out that we have done so.

Further fact from differential topology: Tubular neighbourhood theorem:

if M is a smooth manifold, $Y \subset M$ a smooth closed submanifold, then $\exists$ a diffeomorphism of an open neighbourhood U_Y of Y in M $U_Y \to \nu_{\frac{Y}{M}}$ $Y \overset{\text{id}}{\mapsto}$ 0 − section. Moreover, if $Y, Z \subset M$ are smooth closed submanifolds meeting transversely ($\forall x \in Y \cap Z, T_xY + T_xZ = T_xM$, which by dimension count implies $(\nu_{\frac{Y\cap Z}{M}})_x = (\nu_{\frac{Y}{M}})_x \oplus (\nu_{\frac{Z}{M}})_x$) then $\exists$ tubular neighbourhoods such that $U_{Y\cap Z} \cong U_Y \cap U_Z$ such that $(U_{Y\cap Z})_x \cong (U_Y)_x \times (U_Z)_x \forall x \in Y \cap Z$.

Theorem (Thom isomorphism theorem): Let X be any space and $E \xrightarrow{\pi} X$ an oriented vector bundle of rank n ($\dim E_x = n$). Let $E^\sharp = E \setminus 0$ − section. i) $H^k(E, E^\sharp) = 0$ for $k < n$ ii) $\exists$ a unique class $U_E \in H^n(E, E^\sharp)$ such that $U_E \mid_{E_x} = \epsilon_x \forall x \in X$, where $U_e \mid_{E_x}$ means the pullback under inclusion, i.e. restriction, iii) $\alpha \mapsto \pi^\star \alpha \cdot U_E$ is an isomorphism $H^k(X) \to H^{k+n}(E, E^\sharp) \forall k$. This U_E is the Thom class.

Under the map $H^n(E, E^\sharp) \to H^n(E) \xrightarrow{\cong} H^n(X)$ (this last an isomorphism since $E \simeq X$ via linear retraction in fibres - the 0-section $\subset E$ is a deformation retract), $U_E \to e_E \in H^n(X)$, the Euler class of E.

Now let $Y \subset M$ be a closed oriented submanifold of a closed oriented manifold (so Y is co-oriented; in fact this is all we shall use). Then under $H^k(\nu, \nu^\sharp) \cong H^k(U_Y, U_Y \setminus Y)$ by the tubular neighbourhood theorem, $\cong H^k(M, M \setminus Y)$ by excising $M \setminus U_Y$ from M, $\to H^k(M)$ by the LES of the pair, $U_{\nu_{\frac{Y}{M}}} \to \epsilon_Y$. ϵ_Y is the cohomology class associated to the submanifold Y.

Theorem: If Y, Z are closed oriented submanifolds of an oriented manifold M and if Y, Z intersect transversely, then $\epsilon_{Y\cap Z} = \epsilon_Y \cdot \epsilon_Z$ where this is the cup-product, i.e. "cup-product reflects transverse intersection": the Thom isomorphism theorem says U_E is unique, so it's represented by any cocycle c (taking $E = \nu_Y$) such that i) c vanishes on $M \setminus U_Y$ and $\epsilon_Y \in H^k(M)$ and ii) $c \mid_{(U_Y)_y}$ is the distinguished generator $\epsilon_Y \in H^{\text{codim}}(\nu_y, \nu_y^\sharp)$. Now $U_{Y\cap Z} \xrightarrow{\cong} U_Y \cap U_Z$ and $\epsilon_Y \cdot \epsilon_Z$ does vanish on $M \setminus U_{Y\cap Z}$, and it does restrict properly since $(\nu_{Y\cap Z})_x = (\nu_Y)_x \oplus (\nu_Z)_x$ and $H^{k+l}(\mathbb{R}^{k+l}, \mathbb{R}^{k+l} \setminus 0) \xleftarrow{\text{cross-product}} H^k(\mathbb{R}^k, \mathbb{R}^k \setminus 0) \otimes H^l(\mathbb{R}^l, \mathbb{R}^l \setminus 0)$ (where $k = \text{rk}(\nu_Y)$, $l = \text{rk}(\nu_Z)$ and so $k + l = \text{rk}(\nu_{Y\cap Z})$); this cross-product is an isomorphism, so indeed $\epsilon_Y \cdot \epsilon_Z$ is a fibrewise generator for $H^{k+l}(\nu_{Y\cap Z}, \nu^\sharp_{Y\cap Z})$.

17 Orientations

Recall: A vector bundle $E \to X$ is oriented if we can coherently choose generators $\epsilon_x \in H^n(E_x, E_x \setminus 0) \forall x \in X$ where $n = \text{rk}E = \dim_{\mathbb{R}} E_x$.

Examples and remarks: 1. In linear algebra, an orientation of a vector space V is a choice of ordered basis $\{v_1, \ldots, v_n\}$ for V; two choices define the same orientation if the change of basis matrix from one to the other has determinant > 0; up to equivalence there are two possible orientations.

A choice of basis for V defines a linear map $V \to \mathbb{R}^n$ (which we "keep at home in a box"; we fix one orientation on it for all time). For a fixed generator ϵ_n of $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ the isomorphism $V \to \mathbb{R}^n$ yields an isomorphism $H^n(V, V \setminus 0) \cong H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ and we take ϵ_V to be the element corresponding to ϵ_n.

Corollary: A complex vector space is canonically oriented: we choose a (complex) basis for $V_{\mathbb{C}}$ and take real and imaginary parts to give a basis of $V_{\mathbb{R}}$.

Different choices are related by elements of $GL_n\mathbb{C} \le GL_{2n}\mathbb{R}$, and the determinant of all of these is ¿0. So a complex vector bundle (e.g. the tangent bundle of a smooth complex variety) carries a canonical orientation.

2. Example: If M is a simply connected manifold, M is orientable i.e. TM admits an orientation (There are then two such, and in general neither is distinguished). Sketch of the proof: consider $\{(m, \epsilon_m) : m \in M, \epsilon_m$ a generator $\in H^n(T_mM, T_mM \setminus 0)\} =: M'$. There are two such generators corresponding to each point, as we always have $H^n(\dots) \cong \mathbb{Z}$ (non-canonically). The natural topology on M' is such that the $2:1$ map $M' \to M$ is a covering map (i.e. open sets are defined to be each of the two preimages of each open set in M). An orientation of M is a <u>section</u> of the cover, i.e. a map $M \to M'$ such that the composite $M \to M' \xrightarrow{\pi} M' = \mathrm{id}$. If $\pi_1(M) = 0$ there are <u>no</u> nontrivial covers (i.e. all covers are $M \times F$ for fibre F) so $M' = M \amalg M$ and there are two sections. (This result tells us slightly more; M is oriented if there is no [surjective?] homomorphism $\pi_1(M) \to \mathbb{Z}$).

3. If $Y \subset M$ is a closed smooth submanifold, orientations of Y and M define a co-orientation of Y: at each $y \in Y$, $T_yY \oplus (\nu_{\frac{Y}{M}})_Y \cong T_yM$. Such a splitting is given by choosing a metric on M, which is a contractible choice (i.e. the space of such metrics is contractible - so we have still made "essentially no" choices). We declare that an ordered basis of $(\nu_{\frac{Y}{M}})_y$ to define the positive orientation <u>if</u> [the ordered basis formed by some fixed ordered basis of T_yY followed by this ordered basis] is in the equivalence class of a positive ordered basis of T_yM (So in fact, orientations of any two of these three bundles define an orientation of the third).

Remark: $H^k(T_yY, T_yY \setminus 0) \otimes H^l((\nu_{\frac{Y}{M}})_y, (\nu_{\frac{Y}{M}})_y \setminus 0) \xrightarrow{\text{cross product}} H^{k+l}(T_yM, T_yM \setminus 0)$ For $\epsilon_y \in$ the last of these [and some fixed $\overline{\epsilon}$ in the first], there is a unique generator ϵ_ν of the second such that $\overline{\epsilon} \otimes \epsilon_\nu \mapsto \epsilon_y$ (and not $-\epsilon_y$).

4. If $Y, Z \subset M$ are closed submanifolds and all three are oriented and Y and Z intersect transversely (notation: $Y \pitchfork Z$) then an ordering of (Y, Z) defines a co-orientation on $Y \cap Z$: $\nu_{Y\cap Z} \cong \nu_Y \oplus \nu_Z$, where we use an ordering of (Y, Z) on the RHS. We know the "positive" equivalence class of bases in both of the right hand operands, and this defines the "positive" equivalence class of ordered bases on $\nu_{Y\cap Z}$. Observe: if we change the order of Y, Z and view $\nu_{Y\cap Z} \cong \nu_Z \oplus \nu_Y$, then an ordered basis on the LHS has changed from the previous situation by a factor of $(-1)^{\mathrm{rk}\nu_Y \cdot \mathrm{rk}\nu_Z} = (-1)^{\mathrm{codim}(Y)\mathrm{codim}(Z)}$, since we have done that many transpositions in $(v_1^y, \dots, v_n^y, w_1^z, \dots, w_k^z) \mapsto (w_1^z, \dots, w_k^z, v_1^y, \dots, v_n^y)$. Cf $\epsilon_{Y\cap Z} = \epsilon_Y \cdot \epsilon_Z = (-1)^{|\epsilon_Y|\cdot|\epsilon_Z|}\epsilon_Z \cdot \epsilon_Y$, so our formula for $\epsilon_{Y\cap Z}$ in the previous lecture <u>is</u> consistent.

5. Observe that the tubular neighbourhood theorem applied to a point $p \in M$ says $T_pM \xrightarrow{\cong} U \ni p$ an open neighbourhood of p (an explicit map is given by the exponential map of a metric - ignore this if you haven't done differential geometry). Then $H^n(T_pM, T_pM \setminus 0) \cong H^n(U, U \setminus 0)$, which by excision is $\cong H^n(M, M \setminus p)$. So an orientation on M is a coherent choice of generators for $H^n(M, M \setminus p)$ as p varies M, coherent meaning constant in local open sets relative to homeomorphisms to $\mathbb{R}^n$ and preserved by transition maps of an atlas. This makes good sense for a general topological manifold, <u>not</u> necessarily smooth (we are used to thinking of e.g. the cube which although not smooth can be smoothly deformed into the sphere. But as may be seen in the 4-manifolds

course next term, there are 4-manifolds which cannot be made smooth by any such deformations).

Example: $H^\star(\mathbb{CP}^n, \mathbb{Z}) = \frac{\mathbb{Z}[x]}{(x^{n+1})}$ for $|x| = \deg x = 2$ (recall $H^i(\mathbb{CP}^n) = \mathbb{Z}$ for $i = 0, 2, \ldots, 2n$, 0 otherwise. Pf: $\mathbb{CP}^n$ is oriented and the submanifolds $\mathbb{CP}^i \subset \mathbb{CP}^n$ are oriented (since they are all complex manifolds). Moreover, generically, the submanifolds $\mathbb{CP}^i$ and $\mathbb{CP}^{n-i}$ meet transversely in a single point (i.e. transverse linear subspaces of $\mathbb{C}^{n+1}$ meet only at the origin, and $\mathbb{C}^{i+1}, \mathbb{C}^{n-i+1}$ generically meet in a complex line). But [then] $\epsilon_{\mathbb{CP}^i} \cdot \epsilon_{\mathbb{CP}^{n-i}} = \epsilon_{\text{pt}} \in H^n(\mathbb{CP}^n) \cong \mathbb{Z}$. Recall ϵ_{pt} came from the map $H^{2n}(\mathbb{CP}^n, \mathbb{CP}^n \setminus \text{pt}) = \mathbb{Z} \to H^{2n}(\mathbb{CP}^n) = \mathbb{Z}$ in the LES; the next part of this LES is $\to H^{2n}(\mathbb{P}^n \setminus \text{pt})$, which $= 0$ as $\mathbb{CP}^n \setminus \text{pt} \cong \mathbb{CP}^{n-1}$. So the map $\mathbb{Z} \to \mathbb{Z}$ was a surjection so an isomorphism, so ϵ_{pt} really is the generator of $H^{2n}(\mathbb{CP}^n) \cong \mathbb{Z}$; in particular it is certainly $\neq 0$. So $\epsilon_{\mathbb{CP}^i} \neq 0 \forall i$, since $\epsilon_{\mathbb{CP}^i} \cdot \epsilon_{\mathbb{CP}^{n-i}} \neq 0$. Now using the fact that $\mathbb{CP}^{n-i} \cap \mathbb{CP}^{n-1} = \mathbb{CP}^{n-2}$ (for two general copies of $\mathbb{P}^{n-1}$ in $\mathbb{P}^n$) iteratively, we see that if $x = \epsilon_{\mathbb{CP}^{n-1}} \in H^2(\mathbb{CP}^n)$, then $x^i = \epsilon_{\mathbb{CP}^{n-i}}$ and have result.

Remark: The power of the argument here is that $\epsilon_{\text{pt}} \neq 0 \Rightarrow \epsilon_X \neq 0$ for many $X \subset M$; specifically if $\exists Y$ such that $X \pitchfork Y = \text{pt}$. In fact, for all our examples (spheres, surfaces, $\mathbb{RP}^i$, products), there is a cell decomposition with a unique top dimensional cell, so $M = M^{n-1} \cup D^n$ where M^{n-1} is the $(n-1)$-skeleton, glued via $\partial D^n \to M^{n-1}$. In that case, $M \setminus \text{pt} \simeq M^{n-1}$, so then $\epsilon_{\text{pt}} \neq 0$; in fact it generates a $\mathbb{Z}$-summand (assuming M orientable). In fact, $M \setminus \text{pt} \simeq (n-1)$-dimensional cell complex $\forall$ closed smooth M, but the proof of this is beyond this course, requiring e.g. Morse theory.

Example: $H^\star(S^2 \times S^4; \mathbb{Z}) = \frac{\mathbb{Z}[x,y]}{x^2=y^2=0}, |x| = 2, |y| = 4$. $\mathbb{CP}^3 \not\simeq S^2 \times S^4$ even though both are closed, simply connected, orientable and have the same $H^i(X, \mathbb{Z}) \forall i$. Why? We just saw the natural cup-product map $H^2(\mathbb{CP}^3) \otimes H^2(\mathbb{CP}^3) \to H^4(\mathbb{CP}^3)$ is $x \otimes x \mapsto x^2$, <u>non-zero</u>, but $H^2(S^2 \times S^4) \otimes H^2(S^2 \times S^4) \to H^4(S^2 \times S^4)$ will have $\epsilon_{S^4} \otimes \epsilon_{S^4} \mapsto \epsilon_{S^4 \cap S^4}$. But, by viewing $S^2 \times S^4$ as a "square" with S^2 along the side and S^4 along the bottom, we see that two general copies of S^4 meeting transversely do not intersect at all, so this is $\epsilon_\emptyset = 0$. So $H^\star(\mathbb{CP}^3) \not\cong H^\star(S^2 \times S^4)$ as <u>rings</u>.

18 Thom Isomorphism

Theorem: Let $E \xrightarrow{\pi} X$ be an oriented vector bundle of (real) rank n. Recall $E^\sharp :=$ $E \setminus$ the 0-section. i) $H^k(E, E^\sharp) = 0 \forall k < n$ ii) $\exists!$ class $U_E \in H^n(E, E^\sharp)$ such that $UE\ |_{E_x} = \epsilon_x$ the orientation class of the fibre $\forall x \in X$ (then U_E is the <u>Thom class</u> of E) iii) $\alpha \mapsto \pi^\star \alpha \cdot U_E$ $H^k(X) \xrightarrow{\cdot U_E} H^{k+n}(E, E^\sharp)$ is an isomorphism $\forall k$: we'll only prove the special case where X has a finite trivialising cover for E (this is always true if X is compact, or if X is a manifold "of finite type", i.e. one with a finite trivialising cover). Induct on the number of sets in such a cover: the base case is where $E \to X$ is a product $E = X \times \mathbb{R}^n$. The Künneth theorem gives $H^\star(X) \otimes H^\star(Y) \xrightarrow{\cong} H^\star(X \times Y, X \times Z)$. Take $(Y, Z) = (\mathbb{R}^n, \mathbb{R}^n \setminus 0)$, then all three results are straightforward.

Inductive step: look at the LES for the pair $(E, E^\sharp)$: $0 \to H^n(E, E^\sharp) \to H^n(E\ |_A, E^\sharp\ |_A) \oplus H^n(E\ |_B, E^\sharp\ |_B) \to H^n(E\ |_{A\cap B}, E^\sharp\ |_{A\cap B}) \to \ldots$, where A, B are sets in the base covered by $< k$ trivialising sets and $A \cup B$ is covered by k such. $E\mid_U$ is the restriction, i.e. for $U \overset{i}{\hookrightarrow} X$, $i^\star E \to U$ is the bundle over U with fibre $(i^\star E)_x := E_{i(x)} \forall x \in U$. (Remark: this "pullback bundle" makes sense for any map $f : Y \to X$, not just inclusion of an open set). The first 0 is the group $H^{n-1}(E\mid_{A\cap B}, E^\sharp\mid_{A\cap B}$, which =0 by the inductive hypothesis.

By induction we know $U_{E|_A}, U_{E|_B}, U_{E|_{A\cap B}}$ all exist. By uniqueness of $U_{E|_{A\cap B}}$, each of $U_{E|_A}$ and $U_{E|_B}$ do restrict to $U_{E|_{A\cap B}}$ under the inclusions $A\cap B \hookrightarrow A, A\cap B \hookrightarrow B$. So $(U_{E|_A}, U_{E|_B}) \mapsto 0 \in H^n(E\mid_{A\cap B}, E^\sharp\mid_{A\cap B})$ in the LES. So by exactness, $\exists U_E \in H^n(E, E^\sharp)$ such that $U_E \mapsto (U_{E|_A}, U_{E|_B})$, unique by injectivity of the map $H^n(E, E^\sharp) \to H^n(E\mid_A, E^\sharp\mid_A) \oplus H^n(E\mid_B, E^\sharp\mid_B)$.

At this stave we have ii) from the statement; for i), use the LES for $k < n$. We want iii): look at

$$\begin{array}{ccccccc}
H^{k+n}(E\mid_A, E^\sharp\mid_A) \oplus H^{k+n}(E\mid_B, E^\sharp\mid_B) & \to & H^{k+n}(E\mid_{A\cap B}, E^\sharp\mid_{A\cap B}) & \to & H^{k+n+1}(E, E^\sharp) & \to & H^{k+n+1}(E\mid_A, E^\sharp\mid_A) \oplus H \\
\uparrow & \star & \uparrow \times U_{E|_{A\cap B}} & \dagger & \uparrow \times U_E & & \uparrow \times U_{E|_A} \oplus \\
H^k(A) \oplus H^k(B) & \to & H^k(A\cap B) & \to & H^{k+1}(A\cup B) & \to & H^{k+1}(A) \oplus H
\end{array}$$

. The vertical maps are as in the statement of the theorem (in part iii)); the first two and fourth (and fifth, not present in the diagram due to space reasons) are isomorphisms by induction, so if the diagram commutes we are done by the 5-lemma.

Commutativity of the square $\star$, or any of the equivalent unlabelled squares, is just that cup-product is respected by maps of spaces. But we need to check commutativity of $\dagger$ (note the Mayer-Vietoris (boundary) map cannot be compatible with cup-product in a naive way, because cup product followed by boundary acting on two simplicies would have degree the sum of their degrees - 1, while boundary followed by cup product would have degree the sum of their degrees - 2).

Let ϕ be a cocycle representing U_E (on $X = A \cup B$). The restrictions of ϕ to $E\mid_A, E\mid_B, E\mid_{A\cap B}$ are cocycles representing the relevant Thom classes. Let $[c] \in H^k(A\cap B)$. Then $d_{\mathrm{MV}}(c)$ is obtained as follows: write $c = c_1 - c_2$ for $c_1 \in C^k(A), c_2 \in C^k(B)$; these are cochains rather than cocycles. Then $[a] := d_{\mathrm{MV}}(c) \in H^{k+1}(A\cup B)$ is given by patching $\partial c_1, \partial c_2$ where ∂ is the differential in the singular cochain complex $C^\star$. So $d_{\mathrm{MV}}(\pi^\star c \cdot U_{E|_{A\cap B}})$ is obtained by patching $\partial(\pi^\star c_1 \cdot \phi\mid_{E|_A})$ and $\partial(\pi^\star c_2 \cdot \phi\mid_{E|_B})$. Since $\phi\mid_\bullet$ is a cocycle, use the Leibnitz rule: these are $\partial(\pi^\star c_1)\cdot\phi\mid_{E|_A}$ and $\partial(\pi^\star c_2)\cdot\phi\mid_{E|_B}$ and $\partial\pi^\star = \pi^\star\partial$: i.e. patching $\pi^\star(\partial c_1)\cdot\phi\mid_{E|_A}$ and $\pi^\star(\partial c_2)\phi\mid_{E|_B}$ gives exactly such a patching of chains.

Remark: Recall the Euler class $e_E = u_E\mid_X \in H^n(X, \mathbb{Z})$, where $n = \mathrm{rk}_{\mathbb{R}}(E)$. This satisfies the following naturality property: for $f : Y \to X$ a map of spaces and $E \overset{\pi}{\to} X$ a vector bundle, $f^\star E \to Y$ the pullback bundle has $e_{f^\star E} = f^\star e_E \in H^{\mathrm{rk}E}(Y)$. A rule $E \mapsto c(E) \in H^\star(X)$ for bundles $E \to X$ with this property is called a characteristic class; we shall see another one in the 4-manifolds course.

Example: Gysin sequence:

$$\begin{array}{ccccccccc}
H^{k+n}(E, E^\sharp) & \to & H^{k+n}(E) & \to & H^{k+n}(E^\sharp) & \to & H^{k+n+1}(E, E^\sharp) & \to & \dots \\
\uparrow \text{Thom} & & \uparrow\cong \text{homotopy} & & \uparrow\cong & & \uparrow \text{Thom} & & \\
H^k(x) & \to & H^{k+n}(X) & \to & H^{k+n}(S(E)) & \to & H^{k+1}(X) & \to & \dots
\end{array}$$

, where $S(E)$ is the fibre bundle over X with fibre S^{n+1}, made of the unit spheres in fibres of E (wrt some metric, or alternatively the [spaces of] lines in fibres of E).

The lower line $\cdots \to H^k(X) \to H^{k+n}(X) \to H^{k+n}(S(E)) \to H^{k+1}(X) \to \dots$ (where the first map here is cup-product with the Euler class e_E) is the Gysin sequence. This gives us another way to find out information on $H^\star(X)$ as a ring, since the cup-product enters the exact sequence - this is the first time we've seen this (Of course this is only useful if we can understand $H^\star(S(E))$, but we often can).

For instance, if $E \to \mathbb{CP}^n$ is the tautological line bundle (a complex rank 1 bundle, so a real rank 2 bundle) then $S(E) = S^{2n+1} \subset \mathbb{C}^{n+1}$ the sphere. Similarly, the tautological real line bundle over $\mathbb{RP}^n$ has $S(E) = S^n$. On example sheet 2 we will see that $H^\star(\mathbb{CP}^n; \mathbb{Z}), H^\star(\mathbb{RP}^n; \frac{\mathbb{Z}}{2})$ can be obtained from the Gysin sequence (This is "strictly simpler" than our earlier computation, in that here we are using only classical algebraic topology, wheras before we needed some differential topology).

Remark: If E is a real line bundle, the Gysin sequence becomes (with $\frac{\mathbb{Z}}{2}$ coefficients) the sequence we saw before for double covers of spaces.

19 Compact Supports

In studying cohomology of manifolds, we observe empirically:

Theorem: Let M be a closed connected manifold of dimension n (over $\mathbb{R}$), then i) $H^n(M, \mathbb{Z}) \cong \mathbb{Z}$ iff M is orientable, ii) $H^n(M, \mathbb{Z} \cong \frac{\mathbb{Z}}{2}$ iff M not orientable (cf S^n, $\mathbb{RP}^n$, σ_g, Klein bottle, products etc.)

Remark: This is not true for [general] n-dimensional cell complexes, e.g. $H^n(S^n \vee S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$.

As with the Thom isomorphism, we might hope to prove this inductively over sets in an open cover over M. But $H^n(\mathbb{R}^n) = 0$, so the base case fails. Our solution is to introduce a theory where this is not so: cohomology with compact supports (but note that this will not satisfy the axioms of a generalized homology theory; nevertheless it is useful).

Definition: The cochain complex of singular cochains with compat supports $C^\star_{\mathrm{ct}}(X)$ has groups $C^k_{\mathrm{ct}}(X) = \{\phi \in C^k(X) : \exists \text{ compact } K = K_\phi \text{ such that } \phi\,|_{X\setminus K} \equiv 0\}$ (i.e. $\phi(\sigma) = 0$ for $\sigma : \Delta^k \to X$ with image$(\sigma) \subset X \setminus K$). The usual ∂-operator preserves $C^\star_{\mathrm{ct}}(X)$ ($\partial\phi\,|_{X\setminus K} = 0$ if $\phi\,|_{X\setminus K} = 0$).

The cohomology groups of this complex, $H^\star_{\mathrm{ct}}(X)$, are the "cohomology with compact supports".

Remarks: 1. If X is compact, $C^\star_{\mathrm{ct}} = C^\star$ (by taking $K = X$) 2. Usually, we'd say a function has compact supports if $\mathrm{supp}(f) = \{x : f(x) \neq 0\}$ is compact. The naive analogue of this would be: ϕ has compact support if $\phi(\sigma) \neq 0$ only for σ landing in a compact set K_ϕ. But e.g. if $\sigma \in C^0(\mathbb{R})$ vanishes on simplices not lying in $\{0\} \in \mathbb{R}$, then $\partial\phi\,|_{[0,N]} = \phi(0) - \phi(N) \neq 0 \forall N$, so this definition would not be respected by the boundary operator. (By contrast, in de Rham cohomology theory $\Omega^k_{\mathrm{ct}}(M) = \{\omega \in \Lambda^k T^\star M : \mathrm{supp}(\omega) \text{ is compact}\}$ is preserved by d_{dR}).

Alternative definition for $C^\star_{\mathrm{ct}}, H^\star_{\mathrm{ct}}$: $H^\star_{\mathrm{ct}}(X) = \varinjlim_{K \text{ compact, ordered by inclusion}} H^\star(X, X \setminus K)$. If $K_1 \subset K_2$ are compact sets in X, $X \setminus K_1 \supset X \setminus K_2$ and there is a natural map of pairs $(X, X\setminus K_2) \to (X, X\setminus K_1)$, so there is a pullback $H^\star(X, X\setminus K_1) \to H^\star(X, X\setminus K_2)$. Now if I have many compact sets $K_1 \subset K_2 \subset \dots$ or $\{K\}_{a\in A}$, A has a partial order corresponding to inclusions of K_a, $\{H^\star(X, X \setminus K_a) \to H^\star(X, X \setminus K_b) \text{ if } a < b\}$, $H^\star(X, X \setminus K_1) \to H^\star(X, X \setminus K_2) \to \dots$.

Definition: Let $\{G_a\}_{a\in A}$ be a collection of abelian groups indexed by a partially ordered set A¿ Suppose $\forall a, b \in A \exists c \in A$ sucht that $a < c, b < c$. Suppase I have group homomorphisms $\rho_{ab} : G_a \to G_b$ whenever $a < b$, satisfying the composition law that if $a < b < c$ then $\rho_{bc} \circ \rho_{ab} = \rho_{ac}$. Then define $\varinjlim G_a = \frac{\amalg_a G_a}{\sim}$ where $x \in G_a \sim \rho_{ab}(x)$ for $a < b$, $= \frac{\bigoplus G_a}{\langle x - \rho_{ab}(x)\rangle_{a<b}}$. So two elements in a G_a are identified if they become "eventually equal".

For $x \in G_a, y \in G_b$, $\exists c$ such that $a < c, b < c$, and then $x \sim \rho_{ac}(x) \in G_c, y \sim \rho_{bc}(y) \in G_c$. Define $[x] + [y] = [\rho_{ac}(x) + \rho_{bc}(y)]$.

Exercise: The definition of $C^\star_{\text{ct}}(X)$ given originally, at the level of groups, exactly had $C^k_{\text{ct}} = \bigcup_{K\subset X \text{ compact}} C^k(X, X \setminus K)$. So the original definition of $H^\star_{\text{ct}}$ agrees with this direct definition (In practice, this is the more useful definition).

Note: If in the set A there's a subset A' such that every $a \in A$ is $<$ some $a' \in A'$, then $\varinjlim_A G_a = \varinjlim_{A'} G_a$, since everything on the LHS gets identified with something on the RHS.

Example: $H^\star_{\text{ct}}(\mathbb{R}^n) = \mathbb{Z}$ for $\star = n$, 0 otherwise: compact sets in $\mathbb{R}^n$ can be ugly, but any one lies in some $B_R(0)$ the closed ball of radius R. So we want: $\varinjlim_R H^\star(\mathbb{R}^n, \mathbb{R}^n \setminus B_R(0))$, but by homotopy invariance $H^\star(|mathbbR^n, \mathbb{R}^n \setminus B_R) = H^\star(\mathbb{R}^n, \mathbb{R}^n \setminus 0) = H^\star(S^{n-1})$, so we get $\varinjlim \mathbb{Z}$, with all the maps in the sequence of groups being the identity (if $\star = n$), 0 otherwise.

Remark: $H^\star_{\text{ct}}(\text{pt}) = H^\star(\text{pt}) = \mathbb{Z}$ for $\star = 0$, 0 otherwise, so $H^\star_{\text{ct}}$ is not a homotopy invariant. It's not even functorial under general maps of spaces: for $f : X \overset{\text{cts}}{\to} Y$ there is no guarantee of anything $H^\star_{\text{ct}} \leftrightarrows H^\star_{\text{ct}}(Y)$. But, obviously, $H^\star_{\text{ct}}$ is a homeomorphism invariant. Suppose $i : U \hookrightarrow X$ is inclusion of an open set (or a homeomorphism onto an open set). Then there is a map, extension by zero, from $H^\star_{\text{ct}}(U) \to H^\star_{\text{ct}}(X)$. If $K \subset U$ is compact, then we can always write $K = K' \cap U$ for $K' \subset X$ compact. If $K_1 \subset K_2 \subset U$ and$K_i = K'i \cap U$ for $K_i \in X$, excision of $X \setminus U$ shows $H^\star(X, X \setminus K'_i) \cong H^\star(U, U \setminus K_i)$. Now vary over all $K \subset U$ and all $K' \subset X$, and there are more compact sets in X than in U, so we have the natural map $\varinjlim_K H^\star(U, U \setminus K) \to \varinjlim_{K'} H^\star(X, X \setminus K')$. This is extension by zero.

Example: If $U \hookrightarrow \mathbb{R}^n$ is the inclusion of an open disk, then $i_\star : H^\star_{\text{ct}}(U) \to H^\star_{\text{ct}}(\mathbb{R}^n)$ is an isomorphism.

Remark: If M is a manifold, we observed that M is orientable iff we can coherently choose generators ϵ_U (for small open sets U in M). Again equivalently, we could ask for generators of $H^n_{\text{ct}}(U)$ for open disks $U \subset M$; $H^n_{\text{ct}} \cong H^n(U, U \setminus \text{pt})$ by our computation, $\cong H^n(M, M \setminus p)$ by excision.

We should now prove an M-V theorem for this theorem, but doing so is fiddly; we may return to it later.

20 Cohomology of Manifolds

Recgall $H^\star_{\text{ct}}(X) = \varinjlim_{K\subset X \text{ compact}} H^\star(X, X \setminus K)$. This is functiorial (covariantly, which is unusual for cohomology) under open embeddings: $U \hookrightarrow V \Rightarrow H^\star_{\text{ct}}(U) \to H^\star_{\text{ct}}(V)$.

Proposition: If $X = U \cup V$ is a union of two open sets, there is a M-V sequence $\cdots \to H^{i-1}_{\text{ct}}(X) \to H^i_{\text{ct}}(U \cap V) \to H^i_{\text{ct}}(U) \oplus H^i_{\text{ct}}(V) \to H^i_{\text{ct}}(X) \to H^{i+1}_{\text{ct}}(U \cap V) \to \dots$; we'll prove this later.

Remark: Note that the maps here are covariant maps of spaces as in homology, but the boundary map raises the degree, as in cohomology. So this sequence does not look the same as either of those.

Recall also that in this language, M (not necessarily smooth) is <u>oriented</u> if M admits a coherent system of generators $\epsilon_U \in H^n_{\mathrm{ct}}(U)$ for small open disks $U \subset M$. Exercise: Reformulate this in terms of the existence of an "orientation-preserving" atlas.

Theorem: Let M be a connected manifold. a) If M is oriented, there is a unique map $\int_M : H^n_{\mathrm{ct}}(M) \xrightarrow{\cong} \mathbb{Z}$ such that for $U \overset{i}{\hookrightarrow} M$ an open disk, $\int_M i_\star \epsilon_U = 1$. b) If M is not orientable, $H^n_{\mathrm{ct}}(M) \cong \frac{\mathbb{Z}}{2}$. Although the theorem is true in its stated generality, we'll only prove the case where M has <u>finite type</u> i.e. admits a cover by finitely many sets U_i such that all U_i are disks and the iterated intersections of U_i are disks or empty (e.g. this is true for M closed). We'll induct on "type k", i.e. the minimal number of such sets needed to cover. The base step $M = \mathbb{R}^n$: $H^n_{\mathrm{ct}}(\mathbb{R}^n) \cong \mathbb{Z}$, and a choice of generator $\epsilon_{\mathbb{R}^n}$ proves the theorem in this case.

Choose a nice cover $M = U_1 \cup \cdots \cup U_N$ such that $WSi = U_1 \cup \cdots \cup U_i$ is connected $\forall i$. Suppose W_i is orientable and $W_i \cap U_{i+1}$ is connected, then we have MV: $H^n_{\mathrm{ct}}(W_i \cap U_{i+1}) \to H^n_{\mathrm{ct}}(W_i) \oplus H^n_{\mathrm{ct}}(U_{i+1}) \to H^n_{\mathrm{ct}}(W_{i+1}) \to 0$ (the 0 since the M-V sequence immediately implies: if M is a manifold of finite type, then $H^i_{\mathrm{ct}}(M)$ is non-zero for at most $0 \le i \le n$). By induction, and assuming W_i is orientable and of lower type than M, W_{i+1} and $U_{i+1} \cong \mathbb{R}^n$, so the start of this sequence is $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$.

Let $V \hookrightarrow W_i \cap V_{i+1}$ be an embedding of an open disk (which exists by connectedness). Then $H^n_{\mathrm{ct}}(V) \to H^n_{\mathrm{ct}}(U_{i+1})$ is an isomorphism, so $H^n_{\mathrm{ct}}(W_i \cap U_{i+1}) \to H^n_{\mathrm{ct}}(U_{i+1})$ is an isomorphism (the above map factors through the LHS), so $H^n_{\mathrm{ct}}(W_i) \to H^n_{\mathrm{ct}}(W_{i+1})$ is an isomorphism by exactness of the M-V sequence, so $H^n_{\mathrm{ct}}(W_{i+1}) \cong \mathbb{Z}$.

We define $\int_{W_{i+1}} i_\star \epsilon_V = 1$, where $V \subset U_{i+1}$ is oriented via U_{i+1} (This does define $\int_{W_{i+1}} : H^n_{\mathrm{ct}}(W_{i+1}) \to \mathbb{Z}$ coherently for all open disks in W_{i+1}.

More generally: For W_i orientable and $W_i \cap U_{i+1} = V_1 \amalg \cdots \amalg V_p$ a union of disks (so each V_i is connected orientable), we have MV: $H^n_{\mathrm{ct}}(V_1) \oplus \cdots \oplus H^n_{\mathrm{ct}}(V_p) \to H^n_{\mathrm{ct}}(W_i) \oplus H^n_{\mathrm{ct}}(U_{i+1}) \to H^n_{\mathrm{ct}}(W_{i+1}) \to 0$, and all the groups in the first and second term are $\mathbb{Z}$. Again we embed open disks $U \hookrightarrow V_j \hookrightarrow U_{i+1}$. Then under the extension-by-zero maps $(0, \dots, 0, 1, 0, \dots, 0) \mapsto (\pm 1, 1)$, where we orient V_j (the 1 being in the jth place) via its inclusion in U_{i+1}, there are two cases:

If all the images of the $(0, \dots, 0, 1, 0, \dots, 0)$ are equal, wlog $(1, 1)$, then $H^n_{\mathrm{ct}}(W_{i+1}) \cong \mathbb{Z}$ by exactess of the sequence, and again we can orient W_{i+1} by declaring that $\int_{W_{i+1}} i_\star \epsilon_U = 1$ for the small disks as before. If this always happens (i.e. is the case $\forall i$), then we get case a) of the theorem.

Otherwise, there is some i such that $W_i \cap U_{i+1}$ is disconnected, and images of the map include both $(\pm 1, 1)$. Then by exactness $H^n_{\mathrm{ct}}(W_{i+1} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (1,1), (-1,1) \rangle} \cong \frac{\mathbb{Z}}{2}$. Now if $j > i+1$ and $W_{j+1} = W_j \cup U_{j+1}$, where inductively $H^n_{\mathrm{ct}}(W_j) \cong \frac{\mathbb{Z}}{2}$, $W_j \cap U_{j+1} = \amalg V_\alpha$, the sequence $\bigoplus_\alpha H^n_{\mathrm{ct}}(V_\alpha) \to H^n_{\mathrm{ct}} W_j) \oplus H^n_{\mathrm{ct}}(U_{j+1}) \to H^n_{\mathrm{ct}}(W_j + 1) \to 0$ is $\bigoplus_\alpha \mathbb{Z} \xrightarrow{\psi} \frac{\mathbb{Z}}{2} \oplus \mathbb{Z} \to ? \to 0$. Inductively, the map $V_\alpha \hookrightarrow W_j$ induces a <u>surjection $H^n_{\mathrm{ct}}(V_\alpha) \to H^n_{\mathrm{ct}}(W_j)$</u>, since we also got this from exactness at the previous stage. So orienting the $V_\alpha \hookrightarrow U_{j+1}$ as usual, all generators on the LHS map by ψ to $(1, 1)$, which by exactness implies $H^n_{\mathrm{ct}}(W_{j+1}) \cong \frac{\mathbb{Z}}{2}$, proving b).

Remark: For either manifolds or vector bundles, this means everything is orientable if we take $\frac{\mathbb{Z}}{2}$ coefficients.

Corollary: For oriented closed manifolds, maps have a well-defined degree, i.e. their action on $H^n(M, \mathbb{Z}) \cong \mathbb{Z}$.

Corollary: Let M^n, Y^{n-k} be closed oriented manifolds, $Y \subset M$ of codimension k. Let $\alpha \in H^{n-k}(Y)$. Then $\int_M \alpha \cdot \epsilon_Y = \int_Y \alpha\mid_Y$, i.e. we may think of ϵ_Y as a "delta-function along Y": the LHS is $\int_{U_Y} \phi^\star \alpha \cdot U_{\nu_{\frac{Y}{M}}}$ where $U_Y \cong \nu_{\frac{Y}{M}} \overset{\phi}{\hookrightarrow} M$, U_ν is the Thom class (the equality follows from the definition/characterization of ϵ_Y). If $\iota : Y \hookrightarrow M, \pi : \nu_Y \hookrightarrow Y$ then $\iota \circ \pi \simeq \phi$ (tubular neighbourhood theorem). So we must prove $\int_{U_Y} \pi^\star \beta \cdot U_\nu = \int_Y \beta$ for $\beta \in H^{n-k}(Y) \cong \mathbb{Z}$. If $V \overset{j}{\hookrightarrow} Y$ is the inclusion of a disk, $H^n_{\text{ct}}(Y)$ is generated by $j_\star \epsilon_V$, so it suffices to prove the result for this β. If V is small, $\nu_{\frac{Y}{M}}\mid_V \cong V \times \mathbb{R}^k$ will be trivial. Then $U_{\nu_{\frac{Y}{M}}}\mid_{V\times\mathbb{R}^k} \cong \text{pr}_2^\star \epsilon_k$ by the definition of the Thom class, where ϵ_k generates $H^k_{\text{ct}}(\mathbb{R}^k)$. So we have reduced to $H^k_{\text{ct}}(\mathbb{R}^k)(\otimes H^{n-k}_{\text{ct}}(\mathbb{R}^{n-k}) \overset{\cong}{\to} H^n_{\text{ct}}(\mathbb{R}^n)$, which we know.

21 Cohomology of Manifolds II

Recall we used a M-V sequence for $H^\star_{\text{ct}}$: if $X = U \cup V$ is a union of open sets, we have $\cdots \to H^i_{\text{ct}}(U \cap V) \to H^i_{\text{ct}}(U) \oplus H^i_{\text{ct}}(V) \to H^i_{\text{ct}}(X) \to H^{i+1}_{\text{ct}}(U \cap V) \to \ldots$. The first step in proving this is to establish a relative M-V sequence:

Lemma: Let $(X, Y) = (A \cup B, C \cup D)$ with $C \subset A, D \subset B$. Then $\exists$ a LES $\cdots \to H^n(X, Y) \to H^n(A, C) \oplus H^n(B, D) \to H^n(A \cap B, C \cap D) \to H^{n+1}(X, Y) \to \ldots$ (this reduces to the M-V sequence if $Y = \emptyset$). Recall first that $C^\star(A+B)$, the group of cochains defined on simplicies lying in A or B, has a natural restriction map $C^\star(A + B) \leftarrow C^\star(X = A \cup B)$, and the proof of excision says this is an isomorphism in cohomology (dual statement: the inclusion $C_\star(A + B) \to C_\star(X)$ was an isomorphism in homology). We now consider the following large diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^n(A+B, C+D) & \overset{\phi}{\to} & C^n(A, C) \oplus C^n(B, D) & \overset{\psi}{\to} & C^n(A \cap B, C^\cap D) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^n(A+B) & \overset{\phi}{\to} & C^n(A) \oplus C^n(B) & \overset{\psi}{\to} & C^n(A \cap B) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^n(C+D) & \overset{\phi}{\to} & C^n(C) \oplus C^n(D) & \overset{\psi}{\to} & C^n(C \cap D) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}.$$

The top left entry is <u>defined</u> to make the left hand column exact (explicitly, it is dual to: $C_n(A + B, C + D)$ = free on simplices lying in either A or B and not contained in C or D).

Theorem: i) All three columns are exact (in fact by definition of their first terms) ii) the second and third rows are also exact, in particular $\psi \circ \phi = 0$. $\psi \circ \phi = 0$ in row 2 $\Rightarrow \psi \circ \phi = 0$ in row 1, as the groups in row 1 are subgroups of those in row 2.

So we have a SES of cochain complexes where the SES is vertically and the chain complexes are the rows. So there is a LES in cohomology, where two of every three terms vanish by ii). This implies the third term also vanishes, so all three rows are exact.

Now we know the top row is exact, we can take the associated LES in cohomology. Claim: this is the LES of the lemma. We have a natural map to the first column from $0 \to C^\star(X, Y) \to C^\star(X) \to C^\star(Y) \to 0$, so we get a map of associated LES but in two out of every three places this is an isomorphism, by our initial comment on excision. So by the 5-lemma $C^\star(A + B, C + D)$ computes $H^\star(X, Y)$ and we have the result.

Corollary: The M-V sequence for compact supports $\cdots \to H^{i-1}_{\mathrm{ct}}(X) \to H^i_{\mathrm{ct}}(U \cap V) \to H^i_{\mathrm{ct}}(U) \oplus H^i_{\mathrm{ct}}(V) \to H^i_{\mathrm{ct}}(X) \to H^{i+1}_{\mathrm{ct}}(U \cap V) \to \dots$ is exact: let $K \subset U, L \subset V$ be compact. Then using the previous lemma we have a LES $H^i(X, X \setminus K \cap L) \to H^i(X, X \setminus K) \oplus H^i(X, X \setminus L) \to H^i(X, X \setminus K \cup L) \to H^{i+1}(X, X \setminus K \cap L) \to \dots$ (here $A = B = X, C = X \setminus K, D = X \setminus L$ so $Y = C \cup D = X \setminus (K \cup L)$). Excise $X \setminus U \cap V$, $X \setminus U$ and $X \setminus V$ as appropriate; we obtain: $H^i(U \cap V, U \cap V \setminus K \cap L) \to H^i(U, U \setminus K) \oplus H^i(V, V \setminus L) \to H^i(X, X \setminus K \cup L) \to H^{i+1}(U \cap V, U \cap V \setminus K \cap L) \to \dots$. Observe: every compact set in X is of the form $K \cup L$ for some compact $K \subset U, L \subset V$ and every compact set in $U \cap V$ is of the form $K \cap L$ for some compact $K \subset U, L \subset V$. Now take $\varinjlim_{K \subset U, L \subset V \text{ compact}}$: we have a sequence $H^i_{\mathrm{ct}}(U \cap V) \to H^i_{\mathrm{ct}}(U) \oplus H^i_{\mathrm{ct}}(V) \to H^i_{\mathrm{ct}}(X) \to H^{i+1}_{\mathrm{ct}}(U \cap V) \to \dots$. And the direct limit of exact sequences is exact (this is an exercise in homological algebra, or the reader may take it on trust).

Remark: the inverse limit of exact sequences need not be exact.

Now we return to thinking about cohomology of manifolds. $H^n_{\mathrm{ct}}(M) \cong \mathbb{Z}$ if M is orientable.

Some examples of Betti numbers $b_j(M), 0 \le j \le n$: for S^n these are $1, 0, \dots, 0, 1$, for T^n, $1, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n} = 1$, for σ_g, $1, 2g, 1$, for $\mathbb{CP}^n$, $1, 0, 1, 0, \dots, 1, 0, 1$, and for $\mathbb{RP}^{\mathrm{odd}}$, $1, 0, \dots, 0, 1$. One observes that for orientable n-manifolds, $b_k(M) = k_{n-k}(M)$.

Poincaré Duality Theorem (version 1): If M is an oriented manifold (always assuming it's closed and connected), the pairing $H^k(M, \mathbb{Q}) \times H^{n-k}(M, \mathbb{Q}) \to \mathbb{Q}$ $(\alpha, \beta) \mapsto \int_M \alpha \cdot \beta$ is non-degenerate, i.e. $\forall \alpha \neq 0 \exists \beta$ such that $\int_M \alpha\beta = 1$. Then $H^k(M, \mathbb{Q}) \cong H^{n-k}(M, \mathbb{Q})^\star$ are dual $\mathbb{Q}$-vector spaces (so in particular they have the same rank).

Remark: Over $\mathbb{Z}$, if α is <u>primitive</u> i.e. $\alpha \neq k \cdot \alpha'$ for $\alpha' \in H^k(M, \mathbb{Z}), |k| > 1, k \in \mathbb{Z}$, then $\exists \beta \in H^{n-k}(M, \mathbb{Z})$ such that $\int_M \alpha\beta = 1$ (Note that primitive elements are always non-torsion, but the converse does not hold).

Note: Poincaré duality gives another way of computing cohomology rings, e.g. $H^\star(\mathbb{CP}^n)$: $H^k = \mathbb{Z}$ for k even, $0 \le k \le 2n$. Then inductively $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ is an isomorphism on cohomology up to degree $2n - 2$ (by MV, or cellular cohomology). Assume $H^i(\mathbb{CP}^n) = \mathbb{Z}\alpha^i$, $\alpha \in H^2(\mathbb{CP}^n)$, for $i < 2n$. Well, $H^2 \times H^{n-2} \to \mathbb{Z}$ is non-degenerate, $\alpha \in H^2$ so $\exists \beta \in H^{n-2}$ with $\int \alpha \cdot \beta = 1$. And inductively $\beta = k\alpha^{n-1}$ for some $k \in \mathbb{Z} \Rightarrow k = \pm 1$. So $\alpha^n \neq 0$ and we can choose that it generates $H^{2n}(\mathbb{CP}^n)$. So $H^\star(\mathbb{CP}^n) = \frac{\mathbb{Z}[\alpha]}{\alpha^{n+1}}$. (This is basically the same as our first proof using $\epsilon_{\mathbb{CP}^i} \cdot \epsilon_{\mathbb{CP}^{n-i}} = \delta_{\mathrm{pt}} \neq 0$).

22 Poincaé Duality

Recall the key theorem on the cohomology of manifolds: Poincaré Duality (Version 1): The pairing $H^i(M,\mathbb{F}) \times H^{n-i}(M,\mathbb{F}) \to \mathbb{F}$ $(\alpha,\beta) \mapsto \int_M \alpha\beta$ is non-degenerate, where i) M is a closed connected $\mathbb{F}$-oriented manifold and ii) $\mathbb{F}$ is a field (e.g. $\mathbb{F} = \mathbb{Q}$, or $\mathbb{F} = \frac{\mathbb{Z}}{2}$ for cases where M is not $\mathbb{Q}$-orientable). In fact, there is a more refined statement valid over $\mathbb{Z}$.

Definition: On any space X, the <u>cap-product</u> $C_l(X) \cap C^k(X) \to C_{l-k}(X)$ is given by $\sigma = [v_0 \dots v_l] \cap \phi \mapsto \phi(v_0 \dots v_k)[v_k \dots v_l]$.

Lemma: $\partial(\sigma \cap \phi) = (-1)^k(\partial\sigma \cap \phi - \sigma \cap \partial\phi)$, by computation.

As usual, such a formula implies that the cap product descends to homology/cohomology, $H_l(X) \cap H^k(X) \to H_{l-k}(X)$. If M is oriented and closed, we know $H_n(M,\mathbb{Z}) \cong \mathbb{Z}$ with a distinguished generator $[M]$, the <u>fundamental class</u>. Therefore, we get a map $D : H^k(M;\mathbb{Z}) \overset{[M]\cap -}{\to} H_{n-k}(M;\mathbb{Z})$.

Poincaré Duality (version 2): D is an isomorphism (over $\mathbb{Z}$).

Relation to version 1: A computational exercise shows that: for $a \in C_i(X), \phi \in C^k(X), \psi \in C^{i-k}(X)$, $\psi(a \cap \phi) = (\phi \cup \psi)(a) \in \mathbb{Z}$. Recall also (on the first example sheet) that there's always a natural surjection $H^p(X,\mathbb{Z}) \overset{\Gamma_p}{\to} \mathrm{Hom}(H_p(X),\mathbb{Z})$. Suppose $\psi \in C^{i-k}$ defining $\psi \in H^{i-k}(M)$ satisfies $\int_M \phi \cup \psi = 0 \forall \phi \in H^{-}(M)$ (i.e. the group of relevant dual degree, in this case $H^{n-(i-k)}$), i.e. $0 = \langle \phi \cup \psi, [M]\rangle$ by definition of $[M]$, i.e. $0 = \psi([M] \cap \phi)$. But PD version 2 says $M \cap -$ is an isomorphism, i.e. if ψ is degenerate for pairing then $\psi \in \ker(\Gamma_{i-k})$. But, over $\mathbb{Q}$, Γ is an isomorphism (e.g. because we know the groups have the same dimension as rational vector spaces). So in fact ψ must have been a torsion class over $\mathbb{Z}$, and over $\mathbb{Q}$ the pairing is non-degenerate.

Remark: So over $\mathbb{Z}$, if ψ is primitive as a cohomology class, i.e. not equal to some integer times ψ' for some $\psi' \in H^\star(M,\mathbb{Z})$, with the absolute value of the integer > 1, then $\exists\phi$ such that $\int_M \phi \cdot \psi \neq 0$ (in fact it =1).

Classical idea for proving PD

Suppose our manifold M can be triangulated (i.e. is homeomorphic to some simplicial complex). Take the <u>dual</u> decomposition of M: we have a vertex for the center of each face (i.e. top-dimensional simplex), an edge across each codimension-1 facet of the original triangulation, and so on. E.g. consider the icosahedron as a triangulation of S^2, then the dual decomposition is the dodecahedron; this is <u>not</u> a triangulation, but it <u>is</u> a cell decomposition. By construction "if things work", for each simplex or cell of the initial decomposition of degree i, there is a unique $(n-i)$-cell in the dual decomposition such that it hits the chosen i-cell once and doesn't hit the other i-cells (counting intersections algebraicly with signs). So we get a natural identification $\begin{array}{ccc} C_i^{\text{original}} & \leftrightarrow & C^{n-i}_{\text{dual}} \\ \downarrow & & \downarrow \\ C_{i-1}^{\text{original}} & \leftrightarrow & C^{n-(i-1)}_{\text{dual}} \end{array}$ (note we are taking $C^\star$ on the right not $C_\star$, so that ∂ is compatible. So $C_\star^{\text{cell,original}}$ and $C^\star_{\text{cell,dual}}$ (the latter taken "in reverse", as "$C^{n-\star}_{\text{cell,dual}}$") are naturally identified, so $H_i(M) \cong H^{n-i}(M)$.

Problems: The dual of a triangulation is not always a cell decomposition. Also, not all topological manifolds admit triangulation. But if we only care about the case where M is smooth, we remarked that M admits Morse functions $f : M \to \mathbb{R}$. One can compute $H_\star(M)$ from a complex where $C_k(f) = \mathbb{Z}\langle$critical points of f of <u>index</u> k$\rangle$, where the index of x is the number of negative eigenvalues of the Hessian of f at x. Then $d : C_k(f) \to C_{k-1}(f)$ counts flow lines of f. In this setting, if we replace $f \to -f$, critical points of f of index k correspond precisely to critical points of $-f$ of index $n-k$ (and flow lines are the same, just in the other direction). So this gives a much nicer proof of Poincaré duality than any other known method.

Modern Proof of PD

Generalize the map $D : H^k(M,\mathbb{Z}) \overset{-\cap[M]}{\to} H_{n-k}(M,\mathbb{Z})$ to a map $D : H^k_{\text{ct}}(M) \to H_{n-k}(M)$, which exists even for M not closed. (Note that this map is <u>not</u> capping with some fixed chain; it's the limit over compact $K \subset M$ of capping with something depending on K). Look at the MV sequence, using induction over a covering of M.

$$\begin{array}{ccccccccc} H^k_{\text{ct}}(U\cap V) & \to & H^k_{\text{ct}}(U)\oplus H^k_{\text{ct}}(V) & \to & H^k_{\text{ct}}(M) & \to & H^{k+1}_{\text{ct}}(U\cap V) & \to & \dots \\ \downarrow\cong D & & \downarrow\cong D & & \downarrow D & ? & \downarrow\cong D & & \downarrow\cong D \\ H_{n-k}(U\cap V) & \to & H_{n-k}(U)\oplus H_{n-k}(V) & \to & H_{n-k}(M) & \to & H_{n-k-1}(U\cap V) & \to & \dots \end{array}.$$

In the square ?, one finds that $d_{\text{mv}} \circ D = (-1)^k D \circ d_{\text{mv}}$ (checking this is unilluminating; the reader may consult Hatcher if so desired), where k is the degree of the H^k_{ct} term at the top left of the square. The sign <u>doesn't</u> depend on which element in $H^k_{\text{ct}}(M)$ you start with, so (lemma) the 5-lemma still applies, and then we can deduce the theorem by induction.

This result is very easy in de Rham theory, and beautiful in Morse theory.

A key consequence of Poincaré Duality is (example sheet 3):

Corollary: If M, N are oriented n-manifolds and $f : M \to N$ has non-zero degree, then $f^\star : H^\star(N,\mathbb{Q}) \hookrightarrow H^\star(M,\mathbb{Q})$.

23 Fixed Points

Throughout this section we take $\mathbb{Q}$ or $\mathbb{R}$ coefficients for cohomology. Let M be a closed oriented manifold, of (real) dimension n. Let $\{a_i\}$ be a basis of $H^\star(M)$ (i.e. bases for each group viewed as a vector space, $a_i \in H^{d_i}(M)$. Poincaré duality implies there is a dual basis $\{a_j^\star\}$, defined by $\int_M a_i a_j^\star = \delta_{ij}$. Let $\Delta \subset M\times M$ be the diagonal submanifold, $\Delta = \{(m,m) \in M\times M\}$. A fact from differential topology: $\nu_{\frac{\Delta}{M\times M}} \cong TM$ as real vector bundles, indeed as oriented real vector bundles (considered as over M). Δ has a cohomology class $\epsilon_\Delta \in H^n(M\times M)$. $H^\star(M\times M) \cong H^\star(M)\otimes_{\mathbb{Q}} H^\star(M)$.

Proposition: $\epsilon_\Delta = \sum_i (-1)^{d_i} a_i \otimes a_i^\star$: Using PD on $M\times M$ it suffices to show $\int_{M\times M}(\xi\otimes\eta)\cdot\epsilon_\Delta = \int_{M\times M}(\xi\otimes\eta)\cdot(\sum_i(-1)^{d_i}a_i\otimes a_i^\star) \forall \xi \in H^{n-p}(M), \eta \in H^p(M) \forall p$. From a previous result, $\int_X \alpha\cdot\epsilon_Y = \int_Y \alpha$ for $Y \hookrightarrow X$ a closed oriented submanifold of a closed oriented manifold. So we need: $\int_M \xi\cdot\eta = \int_{M\times M}(\xi\otimes\eta)(\sum(-1)^{d_i}a_i\otimes a_i^\star)$. For the RHS, $\int_{M\times M}\xi\otimes\eta\cdot a_i\otimes a_i^\star = (-1)^{p^2}\int_M \xi\cdot a_i \int_M \eta\cdot a_i$ if $d_i = p$, 0 otherwise. So we need $\int_M \xi\cdot\eta = \sum_i \int_M \xi\cdot a_i \int_M \eta\cdot a_i^\star$ ($(-1)^{p+p^2} = (-1)^{p(p+1)} = 1$). We want this

$\forall \xi$ and $\forall \eta$, but both sides are linear in ξ and η. The $\{a_i\}$ form a basis for $H^\star(M)$, so take $\eta = a_j$. Then the statement becomes $\int_M \xi \cdot a_j = \sum_i \int_M \xi a_i \int_M a_j a_i^\star$, but this last integral is δ_{ij} by the definition of $a_i^\star$¡ so we have the result.

Note: we sometimes see this sum written as $\sum_i (-1)^{n+d_i} a_i \otimes a_i^\star$, which is the same by duality.

Corollary (Gauss-Bonet): If M is smooth, closed and oriented, $\chi(M) = \int_M e_{TM}$. In particular, if $\chi(M) \neq 0$ then $e_{TM} \neq 0$, which means TM has no nowhere-zero section, i.e. $\chi(M) \neq 0 \Rightarrow M$ has no nowhere-vanishing vector field: recall, if $E \to X$ is an oriented vector bundle, $e_E = u_E \mid_X = (\text{zero})^\star u_E$ where zero : $X \to E$ is the 0-section, $= s^\star u_E$ for any section $s : X \to E$ (since all sections are homotopic). But $u_E \in H^\star(E, E^\sharp)$ so if $s(X) \subset E^\sharp$ then $s^\star u_E = e_E = 0$. Returning to the proof, $\nu_{\frac{\Delta}{M\times M}} \cong TM$ and so $\epsilon_\Delta \mid_\Delta = e_{TM}$. Then $\int_M e_{TM} = \int_\Delta \epsilon_\Delta \mid_\Delta$, which by the formula $= \sum_i (-1)^{d_i} \int_M a_i a_i^\star$, but the $\{a_i\}$ form a basis so this $= \sum_k (-1)^k \text{rank}(H^k(M)) = \chi(M)$, as required.

A vector field on a smooth manifold generates a flow. The zeroes of the vector field correspond to the fixed points of the time-ϵ map of the flow. So Gauss-Bonet is an existence theorem for fixed points of maps. There are more general existence theorems for fixed points:

Suppose $f : M \to M$ is any map of a closed smooth manifold. It is a fact from differential topology that f is homotopic to a smooth map.

Definition: f has <u>non-degenerate</u> fixed points if Γ_f (the graph of f) and Δ meet transitively in $M \times M$, i.e. $\forall p \in \Gamma_f \cap \Delta, p = (a,a) = (a, f(a)), T_p\Gamma_f + T_p\Delta = T_p(M \times M)$ $(\star)$. If f has non-degenerate fixed points, each fixed point is necessarily isolated, and each comes with a <u>sign</u> depending on whether the orientations agree or disagree in $(\star)$ (Equivalently: for $F = \text{id} \times f : M \times M \to M \times M$, does $DF_x : T_x \to T_{F(x)}$ preserve or reverse orientation?)

Theorem (Lefschetz Fixed Point Theorem): If M is closed oriented and smooth and $f : M \to M$ is smooth, if f has non-degenerate fixed points, the number of these (counted with sign) is $L(f) := \sum_{k\geq 0} (-1)^k \text{tr}(f^\star : H^k(M) \to H^k(M))$, the "Lefschetz number".

Note: If $L(f) \neq 0$ then f has a fixed point (even if f not smooth)

Note: If $f \simeq \text{id}$, $L(f) = \chi(M)$.

Proof of LFPT: The signed count of fixed points, by definition/construction, is $\int_M \epsilon_{F^{-1}(\Delta)}$, $F^{-1}(\Delta)$ being a finite set, the fixed points of f. But $\epsilon_{F^{-1}(\Delta)} = F^\star \epsilon_\Delta$ by uniqueness of cohomology classes associated to submanifolds, $= (\text{id} \times f)^\star \epsilon_\Delta$, and $\epsilon_\Delta = \sum_i (-1)^{d_i} a_i a_i^\star$, so this is $\sum_i (-1)^{d_i} a_i \cdot f^\star(a_i^\star)$ where $a_i \cdot f^\star(a_i^\star)$ is the (i,i) matrix entry for $f : H^{d_i}(M) \to H^{d_i}(M)$ wrt our chosen basis $\{a_i\}$. So this <u>is</u> $L(f)$, as required.

Remark: LFPT in the form "$L(f) \neq 0 \Rightarrow \exists$ a fixed point" actually holds for fairly general spaces.

Corollary: Let $f : \mathbb{CP}^{2k} \to \mathbb{CP}^{2k}$ be any map. Then f has a fixed point; thus no (nontrivial) finite group acts freely on $\mathbb{CP}^2$: f induces a ring homomorphism $f^\star : H^\star(\mathbb{P}^{2k}) \circlearrowleft$, and this ring is $\frac{\mathbb{Q}[\alpha]}{\langle \alpha^{2k+1} = 0\rangle}, |\alpha| = 2$. $f^\star(\alpha) = \iota \cdot \alpha$ for $\iota \in \mathbb{Z}$, since f induces a map on $H^\star(\mathbb{CP}^{2k}, \mathbb{Z}) \Rightarrow f^\star(\alpha^j) = \iota^j \cdot \alpha^j \Rightarrow L(f) = 1 + \iota + \iota^2 + \cdots = \frac{1-\iota^{2k-1}}{1-\iota} \neq 0$ (for $\iota = 1$ we have $1 + \iota + \iota^2 + \cdots \neq 0$ at the previous stage). So $\text{Fix}(f) \neq \emptyset$.

A nonexaminable lecture was also given at the end of this course, but I did not take notes.