Waves

May 12, 2008

This course has been given in the tripos for at least 50 years; at some stages it was known as "waves in fluids and solid media", but the current title is more accurate as our focus lies exclusively upon the waves rather than the media.

1 Introduction

The dictionary defines a wave as a disturbance which passes through a medium leaving it unchanged. Of course this is inadequate in many respects, but it makes a useful starting point.

Examples: waves on a string (as covered in IB methods), sound waves (or acoustics) (the first variety we will cover in this course, electromagnetic waves (or optics) (as seen in IB electromagnetism; we will leave these to other courses), seismic waves (which we shall cover; this is now the only place in the tripos where the important applied mathematical area of solid mechanics still appears), water waves (as seen in IB fluids, and which will be covered in this course). Some more arguable examples are quantum mechanics (which we shall not cover) ard more abstract waves such as traffic waves (which we will touch on briefly).

Important Properties of Waves

1. Generation - "how to design an efficient loudspeaker"

2. Propagation - How fast to waves travel, do all waves in a medium travel at the same speed (spoilers: no)? Dispersion.

3. Transmission, reflection, scattering, diffraction (shadow formation) - these are all phenomena which occur when a wave hits a discontinuity in material properties; they are useful for e.g. body imaging, seismology, noninvasive materials testing, forces on oil rigs. We will not concentrate overmuch on these phenomena; this is primarily a course in waves rather than their interactions with other things.

4. Formation of Shocks - sonic booms, breaking of water waves, traffic jams; these are all avatars of nonlinearity. We will spend some time on these.

5. Instabilities - waves are $e^{i\omega t}$ phenomena, while (linear) instabilities are $e^{\sigma t}$ phenomena, and in the complex plane these are the same thing; therefore we can use what we know about waves to study instabilities as well.

Some books are listed in the schedules; the actual course book is that by Billingham and King. Lighthill gave an earlier version of this course, and his book is very clever, but consists of clever equations written in words; the lecturer found it almost unreadable. Some books not on the schedules are Hudson's

"Excitation and propagation of elastic waves" which is useful for the solid waves section; also Van Dyk's "An Album of Fluid Motion" is nice to look at, and helpful in understanding many phenomena. Finally, the Archimedeans have a good, comprehensive set of notes from a previous version of this course available online.

2 Sound Waves

Consider a compressible inviscid fluid; it has density $\rho(\vec{x},t)$, pressure $p(\vec{x},t)$ and velocity $\vec{u}(\vec{x},\vec{t})$.

2.1 Mass conservation

Consider a fixed (Eulerian) volume V, with boundary S and surface normal $\vec{n}$. The mass in V is $\int_V \rho dV$; the mass crossing S into V per unit time is $-\int_S \rho\vec{u}\cdot\vec{n}dS$ so $\frac{d}{dt}\int \rho dV = -\int \rho\vec{u}\cdot\vec{n}dS = -\int \vec{\nabla}\cdot(\rho\vec{u})dV$ by the divergence theorem, so $\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{u})=0$; $\rho\vec{u}$ is called the mass flux. We can also express this as $\frac{D\rho}{Dt}=-\rho\vec{\nabla}\cdot\vec{u}$.

At a boundary we are (usually) given $\vec{u}\cdot\vec{n}$; a rigid boundary means $\vec{u}\cdot\vec{n}=0$ (note there is no restriction on flow parallel to the boundary for an inviscid fluid).

2.2 Momentum conservation

The momentum in V is $\int_V \rho\vec{u}dV$; this changes in time because of the flux of momentum into V, $-\int \rho\vec{u}\vec{u}\cdot\vec{n}dS$ per unit time, the surface force on S, $-\int_S p\vec{n}dS$, and the body force on V $\int_V \rho\vec{F}dV$, so $\frac{d}{dt}\int \rho\vec{u}dV = -\int \rho\vec{u}\vec{u}\cdot\vec{n}dS - \int p\vec{n}dS + \int \rho\vec{F}dV \Rightarrow \frac{\partial}{\partial t}(\rho\vec{u})+\vec{\nabla}\cdot(\rho\vec{u}\vec{u}) = -\vec{\nabla}p+\rho\vec{F}$; here $\vec{u}\vec{u}$ is a second-rank tensor, and the term in it is the vector $\nabla_i\rho u_i u_j$. So $\rho\frac{D\vec{u}}{Dt}=-\nabla p+\rho\vec{F}$, just as in IB (but note that if we are considering a viscous fluid, we not olly have to add the term $\mu\nabla^2\vec{u}$, but also another term $\xi\nabla\nabla\cdot\vec{u}$.

At a free boundary the pressure p is constant. If $\vec{F}=0$, we can express the above as $\frac{\partial}{\partial t}+\vec{\nabla}\cdot(p\vec{I}+\rho\vec{u}\vec{u})=0$ where $\vec{I}$ is the identity tensor $[\delta_{ij}]$; $p\vec{I}+\rho\vec{u}\vec{u}$ is called the momentum flux. In general, the flux of an nth rank tensor is an $(n+1)$th rank tensor.

$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{u})=0$

2.3 Equation of state

We assume that $p=p(\rho)$, a prescribed function; this closes the equations. For a liquid this is approximately true, but for a gas we in fact have $p=p(\rho,T)$ where T is temperature $T(\vec{x},t)$, so we need another equation to find T. For a perfect gas, $p=\rho RT$ where R is a constant; thus if T is constant we have $p\propto\rho$. But if T varies in the motion so as to keep the entropy fixed (which turns out to be appropriate for a fast motion like a sound wave) (this is called adiabatic motion, among other names) then we find (and this is essentially the "big result" of the statistical physics course in part II) $p=p_0(\frac{\rho}{\rho_0})^\gamma$ where γ is a constant and p_0, ρ_0 are constants, the values measured in some reference state. For a diatomic gas $\gamma = \frac{7}{5} = 1.4$ (for a monatomic gas $\gamma = \frac{5}{3}$).

Energy of Compression

If a unit mass of gas is compressed, its volume is $V = \frac{1}{\rho}$. The work done per unit mass in compressing a gas to from a reference state ρ_0 is thus $\int_{\rho_0}^{\rho} p\frac{d\rho}{\rho^2}$, so the compressive energy per unit volume is $w := \rho\int_{\rho_0}^{\rho} p\frac{d\rho}{\rho^2}$; this is quite difficult to calculate if we do not think of calculating the energy per unit mass first. For a perfect gas at constant entropy we have $w = \rho\int_{\rho_0}^{\rho} \rho^\gamma \frac{d\rho}{\rho^2}\frac{p_0}{\rho_0^\gamma} = \cdots = \frac{1}{\gamma-1}(p - \frac{p_0}{\rho_0}\rho)$.

2.4 Sound

Consider a base state $\vec{u} = 0, \rho = \rho_0, p = p_0$ and a small perturbation: $p = p_0 + \tilde{p}(\vec{x}, t), |\tilde{p}| << p_0, \rho = \rho_0 + \tilde{\rho}(\vec{x}, t), |\tilde{\rho}| << p, \vec{u} = \vec{u}(\vec{x}, t)$. Substitute these into the above equations and linearise: we have $\frac{\partial\tilde{\rho}}{\partial t} + \rho_0\vec{\nabla}\cdot\vec{u} = 0, \rho_0\frac{\partial\vec{u}}{\partial t} = -\vec{\nabla}\tilde{p}, \tilde{p} = c_0^2\tilde{\rho}$ where $c_0^2 = \frac{dp}{d\rho}\,|_{\rho=\rho_p}$.

Example: For a perfect gas and fast motion, $c_0^2 = \gamma\frac{p_0}{\rho_0}$. Note that if T was constant we would instead have $c_0^2 = \frac{p_0}{\rho_0}$, but experiment shows this is wrong; doing what appears a sensible approach giving a sensible result is entirely wrong. This mistake was made by Newton. c_0 will turn out to be the speed of sound; it is around 340 m/s in air and 1480 m/s in water.

Eliminating $\tilde{p}, \vec{u}$ we get $\frac{\partial^2\tilde{\rho}}{\partial t^2} = c_0^2\nabla^2\tilde{\rho}$, the 3D wave equation with (sound) speed c_0; some books write this as $\Box^2\tilde{\rho} = 0$); clearly $\tilde{p}$ also satisfies this equation (as it is $\propto \tilde{p}$).

Note that $\frac{\partial}{\partial t}\vec{\nabla}\times\vec{u} = 0 \therefore$ if $\vec{\nabla}\times\vec{u} = 0$ initially, then $\vec{\nabla}\times\vec{u} = 0 \forall t$, so we can write $\vec{u} = \vec{\nabla}\phi$ for some acoustic velocity potential ϕ. Then $\frac{\partial\phi}{\partial t} = -\frac{\tilde{p}}{\rho_0} = -c_0^2\frac{\tilde{\rho}}{\rho_0}$ so ϕ also satisfies this wave equation, and therefore so does $\vec{u}$.

Our wave equation is <u>linear</u>, so it satisfies a <u>superposition principle</u>: if ϕ_1, ϕ_2 are solutions then so is $\lambda\phi_1 + \mu\phi_2$.

2.5 Plane Waves

If $\phi(\vec{x}, t) = \phi(x, t)$ independent of y, z then $\frac{\partial^2\phi}{\partial t^2} = c_0^2\frac{\partial^2\phi}{\partial x^2}$; this is the 1D wave equation with general solution $\phi = f(x - c_0t) + g(x + c_0t)$ for arbitrary functions f, g, i.e. ϕ is the sum of two disturbances which propagate with speed c_0 retaining their shape. The lines $x \pm c_0t$ =constant in the (x, t) plane are called <u>characteristics</u>.

Generalising this slightly, for any fixed vector $\vec{k}$, $\phi = f(\vec{k}\cdot\vec{x} - \omega t)$ satisfies the wave equation for $\omega = c_0 : \vec{k}|$. Such a solution is called a <u>plane wave</u>; it propagates in the $\widehat{\vec{k}}$ direction with speed c_0; note that the speed is independent of $\vec{k}$ and f, this is a <u>non dispersive</u> wave.

We have $\vec{\nabla}\phi = \vec{k}f'(\vec{k}\cdot\vec{x} - \omega t)$ so $\vec{u}$ is parallel to $\vec{k}$; this is a longitudinal wave, unlike the wave on a string.

$\tilde{p} = -\rho_0\frac{\partial\phi}{\partial t} = \rho_0\omega f' = \frac{\rho_0\omega|\vec{u}|}{|\vec{k}|} = \rho_0c_0|\vec{u}|$; the ratio of $\tilde{p}$ to $|\vec{u}|$ is called the <u>acoustic impedance</u>; it is ρ_0c_0 for a plane wave (for $\phi = g(\vec{k}\cdot\vec{x} + \omega t)$ we have $\tilde{p} = -\rho_0c_0|\vec{u}|$).

In linearising the momentum we neglected a $\rho_0 \vec{u} \cdot \vec{\nabla} \vec{u}$ term and kept a $\vec{\nabla} \tilde{p}$ term, so for this to be valid we must have $\rho_0 u^2 << \rho_0 c_0 |\vec{u}| \Rightarrow \frac{|\vec{u}|}{c_0} << 1$; this ratio is called the Mach number.

When is our assumption of incompressibility made in IB valid? In a steady flow with velocity U_0, Bernoulli implies the pressure variables are of order $\rho_0 U_0^2$, so the pressure variations are on the order of $\frac{\rho_0 U_0^2}{c_0^2}$; these are small relative to ρ_0 (and therefore incompressibility is valid) if $M = \frac{u_0}{c_0} << 1$.

Harmonic Plane Waves

The special case $f =$ (the real part of; we will not say this explicitly in general) $Ae^{i(\vec{k}\cdot\vec{x}-\omega t)}$ is called a harmonic plane wave. Recall that we have $\omega = \pm c_0 |\vec{k}|$; this is a specific case of a dispersion relation relating ω to $\vec{k}$. A is called the complex amplitude; ω is the frequency, $\vec{k}$ is the wave vector. The wave number is technically $|\vec{k}|$, but we will also refer to $\vec{k}$ as such. $\theta := \vec{k} \cdot \vec{x} - \omega t$ is the phase; the period is $\frac{2\pi}{\omega}$, and the wavelength is $\frac{2\pi}{|\vec{k}|}$. For a (human-audible) acoustic wave the period lies approximately in the range $10^{-3} - 10^{-1}s$, so the wavelength in air is approximately 0.3-30m.

By Fourier we can decompose any function f into harmonic waves.

For acoustic waves in a stationary medium the material is isotropic, so ω must be independent of the direction of $\vec{k}$, i.e. be a function of $|\vec{k}|$ alone. For a problem where there is no length scale and only a material constant c_0 (with dimensions LT^{-1} [length / time]), by dimensional analysis we find $\frac{\omega}{c_0|\vec{k}|}$ is constant, and so the waves are nondispersive.

2.6 Energy

Recall that from the linearised equations we have $\tilde{p} = c_0^2 \tilde{\rho}, \rho_0 \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} \tilde{p}, \frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{u} = 0$. If we take the second of these equations $\cdot \vec{u}$ and multiply the third equation by $\tilde{\rho} \frac{c_0^2}{\rho_0}$ and add these, we obtain $\frac{\partial}{\partial t}(\frac{1}{2}\rho_0 u^2 + \frac{1}{2}\frac{c_0^2}{\rho_0}\tilde{\rho}^2) + \vec{\nabla} \cdot (\vec{u}\tilde{p}) = 0$; write this as $\frac{\partial}{\partial t}(K+W)+\vec{\nabla}\cdot\vec{I} = 0$. Here K is the density of KE per unit volume; we assert that W is the density of the PE of the fluid due to compression, and $\vec{I}$ is the "acoustic intensity", the flux of energy. This, we assert, is the differential form of energy conservation; integrating it over a volume we have the integral form $\frac{d}{dt}\int_V (K+W) dV = -\int \vec{u} \cdot \vec{n} \tilde{p} dS$; the left hand side is the rate of change of energy in V and the right hand side is the rate of working by $\tilde{p}$ on S.

Although this derivation is all that is required for the tripos, we have not justified that $\frac{1}{2}\frac{c_0^2}{\rho_0}\tilde{\rho}^2$ is an energy density, nor why there is only $\tilde{p}$ and no p_0 on the right hand side of the integral form; a more rigorous derivation answering both of these was given as a handout for this course.

Note that $\vec{I} = \tilde{p}\vec{u}$ has units of watts per meter squared. However, we usually define intensity in decibels, $= 120 + 10\log_{10} I$; 0dB is taken as the limit of hearing, 120dB is the threshold of pain, and 70dB is speaking in full voice.

In terms of ϕ, $K = \frac{1}{2}\rho_0 |\vec{\nabla}\phi|^2, W = \frac{1}{2}\frac{\rho_0}{c_0^2}(\frac{\partial \phi}{\partial t})^2, \vec{I} = -\rho_0 X \frac{\partial \phi}{\partial t}\vec{\nabla}\phi$.

For a plane wave, $\tilde{p} = \rho_0 c_0 |\vec{u}|$ so $K = \frac{1}{2}\rho_0 u^2 = \frac{1}{2}\frac{c_0^2}{\rho_0}\tilde{\rho}^2 = W$, so we have instantaneous energy equipartition (that we should have energy equipartition on average is in some sense unsurprising, but this is surprising): at any point in time and space, the energy is half kinetic and half potential. Also $\vec{I} = (K+W)c_0\widehat{\vec{k}}$; energy is propagated at speed c_0 in the direction of the wave.

For a harmonic plane wave $f = Ae^{i(\vec{k}\cdot\vec{x}-\omega t)}$, K is not $\frac{1}{2}\rho_0 k^2 A^2 i^2 e^{2i(\vec{k}\cdot\vec{x}-\omega t)}$, a schoolboy error; the square of the real part of a complex variable is what we want, and this is not the same as the real part of the square of a complex variable. The actual value of K is quite hard, but if we just want the time average, recall (or proove as an exercise) that for $\phi = Ae^{-i\omega t}, \psi = Be^{-i\omega t}$ then the time average of $\phi\psi$ is $\langle\phi\psi\rangle = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \phi\omega dt = \frac{1}{2}\mathrm{Re}(AB^\star)$, so $\langle K\rangle = \frac{1}{2}\frac{1}{2}\rho_0 k^2 AA^\star = \frac{1}{4}\rho_0 k^2|A|^2$.

2.7 Reflection and Transmission of a harmonic plane wave

Consider a barrier: a "heavy but thin wall" at $x = X(t)$ with equilibrium position 0 and mass M per unit area; suppose both sides are air characterised by ρ_0, c_0. Say we are given a wave incident from the left, and want to calculate the wave reflected away to the left and transmitted away to the right. We have for $x < X$, $\tilde{p} = Ae^{i\omega(t-\frac{x}{c_0})} + Re^{i\omega(t-\frac{x}{c_0})}$, so $W = \frac{1}{\rho_0 c_0}(Ae^{i\omega(t-\frac{x}{c_0})} - Re^{i\omega(t-\frac{x}{c_0})})$. For $x > X$ we have $\tilde{p} = Te^{i\omega(t-\frac{x}{c_0})}, u = \frac{1}{\rho_0 c_0}Te^{i\omega(t-\frac{x}{c_0})}$. Note that mathematically there is no reason there shouldn't be a term representing a wave incident from the right; here it is clear that there shouldn't be such a wave from physical considerations, but when we work in 3D the corresponding condition is less obvious. The right way to see this is to say that we have imposed a <u>radiation condition</u> that for $x > X$ no <u>energy</u> propagates from $x = +\infty$.

We assume $X(t)$ is small so that it suffices to apply our jump conditions at $x = 0$. So the kinematic condition (there is no penetration of the wall) is $u \mid_{x=0} = \dot{X}$, i.e. $\frac{1}{\rho_0 c_0}(A-R)e^{i\omega t} = \frac{1}{\rho_0 c_0}Te^{i\omega t}$, and the dynamic condition (the acceleration of the wall is that due to the pressure difference) is $M\ddot{X} = (A+R-T)e^{i\omega t}$. We can calculate $R = \frac{A}{1-2i\frac{\rho_0 c_0}{\omega M}}, T = \frac{-2i\frac{\rho_0 c_0}{\omega M}A}{1-2i\frac{\rho_0 c_0}{\omega M}}$; notice $R, T \propto A$ as we should expect by linearity. We check the results are sensible in some special cases: if $M = 0, T = A$ and if $M = \infty, T = 0$. The reader may verify $\langle\tilde{p}u\rangle = \frac{1}{\rho_0 c_0}(|A|^2 - |R|^2)$ for $x < 0$ and $\frac{1}{\rho_0 c_0}|T|^2$ for $x > 0$, and these two quantities are equal; thus the time avnerage energy flux is the same $\forall x$, which is as we would expect.

The relevant dimensionless group is $\frac{\rho_0 c_0}{\omega M} = \frac{\frac{\rho_0}{|\vec{k}|}}{M}$; which is the mass of gas per wavelength divided by the mass of the wall (each taken per unit area). Thus high frequency is attenuated more effectively, which tallies with experience of sound.

For example, if we have a wall of thickness $\frac{1}{2}$cm and $\rho_{\mathrm{wall}} \approx 3 \times 10^3$Kg/m, $\rho_{\mathrm{air}} \approx 1\mathrm{Kg}/m^3$, $c_0 = 300m/s$ we find $|T|^2 = \frac{1}{4}|A|^2$ at $\omega = 20s^{-1}$ (acoustic engineers generally work in Hz, $\frac{\omega}{2\pi}$; in this case, $\approx$ 3Hz). At 3000Hz we get a 60dB loss.

2.8 Spherical Waves

For a spherically symmetric function $\phi(r,t)$ $(r = |\vec{x}|)$ we can write $\nabla^2\phi$ as $\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi)$, so we have $\frac{\partial^2}{\partial t^2}(r\phi) = c_0^2\frac{\partial^2}{\partial r^2}(r\phi)$. So $\phi = \frac{f(t-\frac{r}{c_0})}{r} + \frac{g(t+\frac{r}{c_0})}{r}$; clearly the first term represents an outgoing wave and the second an incoming one.

The mathematically inclined may wish to ask about cylindrical waves, and will discover that they give, in the words of the lecturer, a "godawful mess". More generally, we obtain "nice" solutions when solving the wave equation in an odd number of dimensions, and "nasty" ones when solving it in an even number.

Consider a pure outgoing wave, which we choose to write as $\phi = \frac{-q(t-\frac{r}{c_0})}{4\pi\rho_0 r} \Rightarrow \vec{u} = \frac{\hat{\vec{r}}}{4\pi\rho_0}(\frac{\dot{q}(t-\frac{r}{c_0}}{rc_0} + \frac{2(t-\frac{r}{c_0}}{r^2})$. The first term dominates as $r \to \infty$, the second term as $r \to 0$; thus we have $\lim_{r\to 0} 4\pi\rho_0 r^2 u_r = q(t)$; $q(t)$ is the mass flux out of a "point source" at $r = 0$.

Example: consider a "pulsating sphere" of radius $r = a(1 + \epsilon e^{i\omega t})$ for some $\epsilon << 1$, so our boundary condition is $u_r \mid_{r=a} = \frac{\partial\phi}{\partial r} \mid_{r=a} = i\omega\epsilon a e^{i\omega t}$ (applying the boundary condition at $r = a$ which is a valid linearization since ϵ is small. We set a radiation condition such that we have only an outgoing wave; we guess $\phi = -i\omega\epsilon a^3 \frac{A}{r} e^{i\omega(t-\frac{r-a}{c_0})}$ (in general, it is wise to put as much structure as possible in at this stage, since it simplifies later algebra). We find $A = \frac{1}{1+\frac{i\omega a}{c_0}}$, so $u_r = \frac{-i\omega\epsilon a^3}{1+\frac{i\omega a}{c_0}}(\frac{-i\omega}{c_0 r} - \frac{1}{r^2})e^{i\omega(t-\frac{r-a}{c_0})}$, and $\tilde{p} = \frac{-\rho_0\omega^2\epsilon a^3}{1+\frac{i\omega a}{c_0}} \frac{e^{i\omega(t-\frac{r-a}{c_0})}}{r}$. The "far field", where the first term of u_r dominates, is where $\frac{\omega r}{c_0} = kr >> 1$, i.e. many wavelengths away from the sphere. The first term of u_r dominating means the wave "looks like" a plane wave ($\tilde{p} = \rho_0 c_0 u_r$), but with $\tilde{p}, u_r$ decaying like $\frac{1}{r}$. This makes sense, since it means the energy flux (through any large sphere about the origin) is constant. The ration $ka = \frac{\omega a}{c_0}$ is that of sphere size to wavelength. For a large sphere $ka >> 1$, at $r \simeq a$ $\tilde{p}$ is in phase with ω_r, so $\langle I \rangle = \frac{1}{2}\mathrm{Re}(\tilde{p}u_r^\star)$, which is (relatively) big, so sound is radiated effectively. For a small sphere $ka << 1$, called the "acousticly compact limit", at $r \simeq a$ $\tilde{p}$ and u_r are almost out of phase, so we have inefficient radiation. Therefore for a megaphone or the cone of a gramaphone-style speaker, to radiate effectively, we want the size of the cone to be of order k^{-1}.

3 Waves in Solids

We will consider "hard" solids (metals, rocks and similar), where small deformations lead to large stresses; this is known as "linear elasticity" or "classical elasticity".

3.1 Strain

Suppose that a material point is displaced from $\vec{x}$ to $\vec{x} + \vec{u}(\vec{x}, t)$ at time t.

Note that $\vec{u}$ is displacement, rather than velocity as in fluids; this is the standard notation and one "just has to live with it". $\vec{u}$ is <u>small</u> in the sense that $\nabla\vec{u}| << 1$; then velocity is $\frac{\partial\vec{u}}{\partial t}$ and acceleration is $\frac{\partial^2\vec{u}}{\partial t^2}$ ($\frac{D}{Dt}(\frac{\partial\vec{u}}{\partial t} = \frac{\partial^2\vec{u}}{\partial t^2} + \frac{\partial\vec{u}}{\partial t} \cdot \vec{\nabla}\frac{\partial\vec{u}}{\partial t}$, but the second term is a factor of $\vec{\nabla}\vec{u}$ smaller than the first.

At time t, a material line that joins $\vec{x}$ and $\vec{x} + \delta\vec{x}$ becomes δt $\vec{x} + \delta\vec{x} + u(\vec{x} + \delta\vec{x}, t) - \vec{x} - \vec{u}(\vec{x}, t)$, so $\delta\vec{x} \to \delta\vec{x} + \delta\vec{x} + \vec{\nabla}\vec{u}$, so $\vec{\nabla}\vec{u}$ controls the stretch and rotation of material lines $\delta\vec{x}$. If we put $\vec{\nabla}\vec{u} = \frac{1}{2}(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T) + \frac{1}{2}(\vec{\nabla}u - \vec{\nabla}\vec{u}^T)$; the first term is the infinitesimal strain tensor $\underline{\underline{e}}$, and the second is the rotation tensor $\underline{\underline{\Omega}}$. Notice $\delta\vec{x} \cdot \underline{\underline{\Omega}} = \vec{\omega} \times \delta\vec{x}$ (where $\omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}$), which represents a solid body rotation, in which there is no stress. So all the stresses are from $\underline{\underline{e}}$.

3.2 Solids

Mass conservation is given by $\frac{\partial \rho}{\partial t} + \frac{\partial \vec{u}}{\partial t} \cdot \vec{\nabla}\ \rho = -\rho \vec{\nabla} \cdot \frac{\partial \vec{u}}{\partial t}$ as before; (assuming ρ is initially uniform) the second term is negligible, so $\tilde{\rho} = -\rho \vec{\nabla} \cdot \vec{u}$ (note our reference density is here ρ rather than ρ_0 as in fluids); the density variations $\tilde{\rho}$ are small in comparison to the reference density ρ so can be neglected in the momentum equation.

$\vec{\nabla} \cdot \vec{u} = \mathrm{tr}(\underline{\underline{e}})$ is called dilatation; it is volume change per unit volume, and will clearly be $<< 1$.

As per the handout, $\tau = \sigma \cdot \vec{n}, \vec{\sigma} = -p\underline{\underline{\mathcal{I}}}, \rho \ddot{\vec{u}} = \rho \vec{F} + \vec{\nabla} \cdot \underline{\underline{\sigma}}, \sigma = \sigma^T$.

3.3 Constitutive Equation for a Solid

We assume that for a solid, $\underline{\underline{\sigma}}$ depends solely on the local, instantaneous value of $\underline{\underline{e}}$; we can justify this by physical reasoning and experiment. We also assume that the relationship is linear, so $\sigma_{ij}(\vec{x}, t) = A_{ijkl} e_{kl}(\vec{x}, t)$, where A is a material constant (we assume $\underline{\underline{\sigma}} = 0$ if $\underline{\underline{e}} = 0$; there is no "pre-stress"); we also assume the material is isotropic, implying $A_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \mu' \delta_{il} \delta_{jk}$, and uniform, i.e. λ, μ, μ' are independent of $\vec{x}, t$. Since $\sigma = \sigma^T$ we have $\mu = \mu'$ and $\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2 \mp e_{ij}$. λ, μ are called the Lamé moduli of the material, with units of Newtons per metre squared.

Note that $\delta_{kk} = (3\lambda + 2\mu) e_{kk}$ which we can use to write $e_{ij} = \frac{1}{2\mu}(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu} \sigma_{kk} \delta_{ij})$.

The special case $\mu = 0 \Rightarrow \lambda \delta_{ij} e_{kk}$ gives an inviscid fluid $\underline{\underline{\sigma}} = -p\underline{\underline{I}}$ with $p = \lambda \vec{\nabla} \cdot \vec{u}$; this is sometimes called an "elastic liquid".

3.4 Measurement of the Lamé Moduli

3.4.1 Simple Shear

Consider a cuboid of material of height h along the y direction and cross-sectional area A with bottom corner at the origin, to which we apply a force F at the top in the x direction, leading to a displacement Δx of the top in the x direction, so $\vec{u}(\vec{x}) = (\gamma y, 0, 0)$ where $\gamma = \frac{\delta x}{h}$. So $\underline{\underline{e}} = \frac{1}{2}\gamma \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{\underline{\sigma}} = \mu\gamma \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow F = \mu\gamma A \Rightarrow \mu = \frac{\frac{F}{A}}{\frac{\Delta x}{h}}$; μ is called the shear modulus and represents the solid's resistance to shear forces.

3.4.2 Compression

Consider a sphere of material centred on the origin, of original radius a, being forced from all sides with pressure p. $\vec{u}(\vec{x}) = -\epsilon \vec{x}$ by simmetry, so the new radius is $\frac{4}{2}\pi a^3 (1 - \epsilon)^3$; $\frac{\Delta V}{V} = -3\epsilon$ (to linear order in small ϵ) $= \vec{\nabla} \cdot \vec{u}$ as earlier. $\underline{\underline{e}} = -\epsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \underline{\underline{\sigma}} = -\epsilon(3\lambda + 2\mu) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \lambda + \frac{2}{3}\mu = \frac{p}{\frac{\delta V}{V}}$; this value is called the bulk modulus

Both the shear and bulk moduli must be > 0, as otherwise the material would spontaneously deform.

3.4.3 Uniaxial Extension

Say we have a wire of cross sectional area A and apply a force F to both ends pulling them away from the centre, with no "traction" forces on the side, changing its length from L to $L + \Delta L$. $\underline{\underline{\delta}} = \frac{F}{A}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{\underline{e}} = \frac{F}{A}\frac{1}{2\mu}\frac{1}{3\lambda+2\mu}\begin{pmatrix} 2\lambda+2\mu & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \Rightarrow$ in the x direction, $e_{11} = \frac{\Delta L}{L} = \frac{F}{EA}$ where $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$, the Young's modulus; for the y and z directions there is a radial contraction of size $-\nu\frac{\Delta L}{L}$ where $\nu = \frac{\lambda}{2(\lambda+\mu)}$, the Poisson ratio (Thus from this single experiment we can measure both λ and μ).

The reader should verify that our requirement that $\mu > 0, \lambda + \frac{2}{3}\mu > 0 \Rightarrow E > 0, -1 < \nu < \frac{1}{2}$ ($\nu = \frac{1}{2}$ corresponds to $\lambda = \infty$, i.e. an incompressible material. Everyday materials (in fact all known materials up until around 10 years ago) have $\nu > 0$ (i.e. the wire contracts rather than expanding radially), but it has proven possible to construct polymers with $\nu < 0$.

Energy we have relegated to a handout, but beware: it is still entirely examinable. The conclusion is $\frac{d}{dt}\int(K+W)dV = \rho\int\dot{\vec{u}}\cdot\vec{F}dV + \int_S\dot{\vec{u}} + \vec{\tau}dS$; the only "surprising" part is W, which we find to be $\frac{1}{2}\lambda e_{kk}^2 + 2\mu e_{ij}e_{ij}$. In differential form, $\frac{\partial}{\partial t}(K+W) + \vec{\nabla}\cdot\underline{\underline{I}} = 0$, where $\underline{\underline{I}}] - \dot{\vec{u}}\cdot\underline{\underline{\delta}}$.

3.5 Lecturer lost count, or I did

3.6 Equation of motion for an elastic solid

$\rho\frac{\partial^2\vec{u}}{\partial t^2} = \rho\vec{F} + \vec{\nabla}\cdot\underline{\underline{\sigma}}$. So if $\vec{F} = 0$, then for an elastic solid $\rho\frac{\partial^2\vec{u}}{\partial t^2} = (\lambda+\mu)\vec{\nabla}\vec{\nabla}\cdot\vec{u} + \mu\nabla^2\vec{u}$ (This is the equivalent of the Navier-Strokes equations in fluids, but is presently unnamed; when this was told to the lecture audience, it was claimed as Johnson's equation).

For <u>elastostatics</u> the LHS is 0, so we just need to be given $\vec{u}$ or $\underline{\underline{\sigma}}\cdot\vec{n}$ for each point on the boundary; we could prove that then there is a unique solution. For <u>elastodynamics</u> we will additionally need to be given $\vec{u}$ and $\dot{\vec{u}}$ at $t = 0$. On a boundary we take $\underline{\underline{\sigma}}\cdot\vec{n}$ continuous, which is a "physicality" condition meaning that "we cannot have infinite acceleration of the boundary"; taking $\vec{u}$ continuous is defining a "no slip" condition, which of course is not always the case physically. At a <u>free surface</u> $\underline{\underline{\sigma}}\cdot\vec{n} = 0$; at a <u>rigid surface</u> $\vec{u} = 0$.

3.7 Elastic waves

Any vector field $\vec{u}$ may be written $\vec{u} = \nabla\phi + \nabla\times\psi$ with $\nabla\cdot\psi = 0$ [I am dropping $\vec{\nabla}$ because it is effort and wasn't used in lectures; herafter I shall just write ∇] (We can proove this by explicitly constructing them: $\phi(\vec{x}) = -\int\frac{\nabla\cdot\vec{u}(\vec{y})}{4\pi|\vec{x}-\vec{y}|})dV_y, \vec{\psi}(\vec{x}) = \int\frac{\nabla\times\vec{u}(\vec{y})}{4\pi|\vec{x}-\vec{y}|}dV_y)$. ϕ is called the <u>dilatatial potential</u> and $\vec{\psi}$ is the <u>shear potential</u>.

Then $\nabla(\rho\frac{\partial^2\phi}{\partial t^2} - (\lambda+2\mu)\nabla^2\phi) + \nabla\times(\rho\frac{\partial^2\vec{\psi}}{\partial t^2} - \mu\nabla^2\vec{\psi}) = 0$; this is satisfied if (and in fact, though without proof, only if) $\frac{\partial^2\phi}{\partial t^2} = c_p^2\nabla^2\phi$ with $c_p = \sqrt{\frac{\lambda+2\mu}{\rho}}$ and $\frac{\partial^2\vec{\psi}}{\partial t^2} = c_s^2\nabla^2\vec{\psi}$ with $c_s = \sqcup\frac{\mu}{\rho}$. c_p is the dilatational wave speed, c_s is the shear wave speed; from our earlier inequalities involving the moduli, $c_p > c_s$. For steel we have $c_p = 5.9km/s, c_s = 3.2km/s$; for granite these numbers are between $5-7$ and $2.5-4km/s$ respectively.

Summary

[This is all in the above, but it's good to have it in one place - I made a mistake not recording this with fluids.]

$\underline{\underline{e}} = \frac{1}{2}(\nabla\vec{u}+\nabla\vec{u}^T), \underline{\underline{\sigma}} = \lambda\text{tr}\underline{\underline{e}}\underline{\underline{I}} + 2\mu\underline{\underline{e}}, W = \frac{1}{2}(\lambda(\text{tr}\underline{\underline{e}})^2 + 2\mu\underline{\underline{e}} : \underline{\underline{e}}), \frac{\partial}{\partial t}(K+W) + \nabla\cdot\vec{I} = 0, \vec{I} = -\vec{u}\cdot\underline{\underline{\sigma}}, \vec{u} = \nabla\phi + \nabla\times\vec{\psi}, \nabla\cdot\vec{\psi} = 0 \Rightarrow \frac{\partial^2\phi}{\partial t^2} = c_p^2\nabla^2\phi, \frac{\partial^2\vec{\psi}}{\partial t^2} = c_s^2\nabla^2\vec{\psi}, c_p > c_s$ (Note $\underline{\underline{I}} = \delta_{ij}$ is different from $\vec{I}$, the energy flux).

So a general disturbance propagates at two speeds; the first to arrive is the primary wave, which is compressive; the second is the secondary wave, which is shear. For $\mu = 0$ (the "elastic liquid" case) only compressive waves are possible, and these are just the familiar acoustic waves.

In 2D, $\vec{\psi}$ can be replaced by a scalar: $\phi = \phi(x,y,t), \vec{\psi} = (0,0,\psi(x,y,t)) \therefore \vec{u} = (\frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}, \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}, 0)$; it is an exercise for the reader to find $\underline{\underline{e}}$ and $\underline{\underline{\sigma}}$ for this case.

3.8 Plane waves

If $\phi = \phi(t - \frac{\hat{k}\cdot\vec{x}}{c_p}), \vec{\psi} = \vec{0}$ (i.e. a dilatational wave propagating in the direction $\hat{k}$ with speed c_p) then $\vec{u} = \nabla\phi = -\frac{\hat{k}}{c_p}\phi'$, and $\vec{u}$ is parallel to $\hat{k}$.

If $\vec{\psi} = \vec{\psi}(t - \frac{\hat{k}\cdot\vec{x}}{c_s}), \phi = 0$ and $\hat{k}\cdot\vec{\psi}' = 0$ (as $\nabla\cdot\vec{\psi} = 0$) then $\vec{u} = \nabla\times\vec{\psi} = -\hat{k}\times\frac{\vec{\psi}'}{c_s} \therefore \hat{k}\perp\vec{u}$ and we have a transverse wave.

Energy

For a pure P-wave, $\underline{\underline{e}} = \frac{\hat{k}\hat{k}}{c_p^2}\phi'', \sigma_{ij} = (\lambda\delta_{ij} + 2\mu\hat{k}_i\hat{k}_j)\frac{\phi''}{c_p^2}, K = \frac{1}{2}\rho\dot{u}^2 = \frac{1}{2}\rho\frac{\phi''^2}{c_p^2}, W == \frac{1}{2}(\lambda+2\mu)\frac{\phi''^2}{c_p^4} = K, \vec{I} = \frac{\phi''^2\hat{k}}{c_p^3}\cdot(\lambda\underline{\underline{I}} + 2\mu\hat{k}\hat{k}) = (\lambda+2\mu)\hat{k}\frac{\phi''^2}{c_p^3} = \hat{k}c_p(K+W)$, so we have instantaneous energy equipartition and the energy moves with velocity $c\hat{k}$. It is an exercise for the reader to perform the corresponding calculation for a plane S-wave, finding $K = W$ and $\vec{I} = \hat{k}c_s$.

3.9 Harmonic waves, evanescent waves and polarisation

The plane waves $\phi = Ae^{i(\vec{k}\cdot\vec{x}-\omega t)}$ with $\omega = c_p|\hat{k}|, \vec{\psi} = \vec{B}e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \omega = c_s|\hat{k}|$ are harmonic; each is non-dispersive

Evanescent Waves

Near a boundary we can have $\vec{k}$ complex, $\vec{k} = \vec{k}_r + i\vec{k}_i$; then $\phi = Ae^{-\vec{k}_i\cdot\vec{k}}e^{i(\vec{k}_r\cdot\vec{x}-\omega t)}$, so we have attenuation in the direction $\vec{k}_i$ and propagation in the direction $\vec{k}_r$.

To satisfy the wave equation we must have $\omega^2 = c_{p/s}^2(\vec{k}_r + i\vec{k}_i)\cdot(\vec{k}_r + i\vec{k}_i) = c_{p/s}^2(k_r^2 - k_i^2 + 2i\vec{k}_i\cdot\vec{k}_r)$, so for ω to be real we must have $\vec{k}_i\cdot\vec{k}_r = 0$. Notice that the phase speed in the $\vec{k}_r$ direction is $\frac{\omega}{|\vec{k}_r|} = c_{p/s}\sqrt{1-(\frac{k_i}{k_r})^2} < c_{p/s}$; these waves move more slowly than waves in the interior.

Surface waves are more important (by which we mean damaging) in seismology, because they spread in 2D rather than 3D, so are much more damaging [at large distances] (than p or s waves).

Polarization

Suppose we have a plane boundary $y = 0$ (which we take to be horizontal). Consider waves in the (x, y) plane with $\vec{k} = k(\sin\theta, \cos\theta, 0)$; for a p-wave $\vec{u} \propto \vec{k}e^{i(\vec{k}\cdot\vec{k}-\omega t)}$ so there is no z component of $\vec{u}$; for an s-wave $\vec{u} = i\vec{k}\times\vec{B}e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ $\therefore$ $\vec{u} \parallel \vec{k}\times\vec{B}$; the direction of this is called the <u>direction of polarisation</u>. This can be decomposed into a part in the $(0, 0, 1)$ direction alone, $\vec{u} \propto (0, 0, 1)e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, called an <u>SH wave</u> (<u>horizontally</u> polarized), which has $\vec{B} \parallel \vec{k}\times\vec{u} = (\cos\theta, -\sin\theta, 0)$, and the remainder is $\vec{u} \propto (-\cos\theta, \sin\theta, 0)e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, which has no z cpt and is $\perp \vec{k}$, called an <u>SV wave</u> (<u>vertically</u> polarized), which has $B \propto (0, 0, 1)$.

So our possible disturbances are P with x, y components of $\vec{n}$, SH with z components of $\vec{n}$ and SV with x, y components of $\vec{n}$ A general disturbance is a sum of P, SH and SV disturbances. Because u_z, σ_{zy} are continuous at $y = 0$, "SH excites only SH; SV and P excite each other".

3.10 Rayleigh Waves

Consider a solid below $y = 0$, vacuum above this; the boundary condition is a free surface $\underline{\sigma}\cdot\vec{n} = 0$. Is there a propagating surface wave solution ($\vec{u} \to 0$ as $y \to -\infty$) in $y < 0$?

At $y = 0$ we have $\sigma_{xy} = \sigma_{yy} = \sigma_{zy} = 0$; thus we cannot make a single wave solution work; if we try an SH wave, $\sigma_{zy} = 0$ and the wave must be 0. So we try an SV and P wave: put $\omega = ck$ (we can view this as a convenience at this stage, or use our earlier argument that since there is no length scale in the problem this is the only possible dispersion relation), and try a solution $\phi = f(y)e^{ik(x-ct)}, \vec{\psi} = g(y)e^{ik(x-ct)}\vec{e}_z$; k, c must be the same for both waves if we are to have any hope of satisfying the boundary condition, since we must do so on $y = 0$ for all possible x, t.

The wave equations give us that $f = Ae^{\eta_p y}, g = Be^{\eta_s y}$ where $\eta_{p,s} = k\sqrt{1-\frac{c^2}{c_{p,s}^2}}$ (taking the positive square root for decay as $y \to -\infty$) ($\vec{k} = k\vec{e}_x - i\eta_{p,s}\vec{e}_y$)). Then we find $\vec{u} = (ikAe^{\eta_p y} + \eta_s Be^{\eta_s y}, \eta_p Ae^{\eta_p y} - ikBe^{\eta_s y}, 0)e^{i\omega(x-ct)}$. $\sigma_{xy} = \mu(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) = 0$ on $y = 0 \Rightarrow 2ik\eta_p A + k^2(2 - \frac{c^2}{c_s^2})B = 0$, $\sigma_{yy} = \lambda(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}) + 2\mu\frac{\partial u_y}{\partial y} = 0$ on $y = 0 \Rightarrow k^2(2 - \frac{c^2}{c_s^2})A - 2ik\eta_s B = 0$. For a nontrivial solution we must therefore have $(2-\frac{c^2}{c_s^2})^2 - 4(1-\frac{c^2}{c_s^2})^{\frac{1}{2}}(1-\frac{c^2}{c_p^2})^{\frac{1}{2}} = 0$; this is the Rayleigh equation (note that k has disappeared from the problem, so putting $\omega = ck$ was useful, and the wave is non-dispersive). Let $\xi - (\frac{c}{c_s})^2, q = \frac{c_s}{c_p} < 1$, then $f(\xi) := \xi^3 - 8\xi^2 + 8(3 - 2q^2)\xi - 16(1 - q^2) = 0$; $f(0) < 0, f(1) = 1 > 0$ so there is a ξ such that $f(\xi) = 0$. So there is a solution, and in fact we can show this ξ is unique; also note $c < c_s < c_p$. This is the Rayleigh wave.

3.11 Wave reflection and refraction

Consider a boundary at $y = 0$ between two solids, the lower having properties c_s, μ, ρ and the upper c'_s, μ', ρ', and an incident wave at angle θ from the y axis. We shall do the easiest case, an incident SH wave; assume this results in a transmitted SH wave at angle θ' to the y axis and a reflected SH wave at angle $\hat{\theta}$ to the y axis.

Now our incident $\vec{k} = k(\sin\theta, \cos\theta, 0)$ so $\vec{u} = (0,0,1)e^{ik(x\sin\theta + y\cos\theta - c_s t)}$, the reflected wave has $\vec{u} = (0,0,1)Re^{i\hat{k}(x\sin\hat{\theta} - y\cos\hat{\theta} - c_s t)}$ and the transmitted $\vec{u} = (0,0,1)Te^{ik'(x\sin\theta' + y\cos\theta' - c'_s t)}$. At $y = 0$, for all x, t u and σ_{zy} are continuous, so $e^{ik(x\sin\theta - c_s t)}$ must appear in all the disturbances, so $kc_s = \hat{k}c_s = k'c'_s \Rightarrow k = \hat{k}, k' = k\frac{c_s}{c'_s}$ and we have $k\sin\theta = k\sin\hat{\theta} = k'\sin\theta' \Rightarrow \theta = \hat{\theta}, \frac{\sin\theta'}{c'_s} = \frac{\sin\theta}{c_s}$, this last being Snell's law of refraction. So for the reflected wave $\hat{\vec{k}} = k(\sin\theta, -\cos\theta, 0)$ and for the refracted wave $\vec{k'} = k(\sin\theta, \eta', 0)$ where $\eta' = \sqcup\frac{c_s^2}{c_s'^2} - \sin^2\theta$. If $c'_s > c_s$ (i.e. we are moving from a substance in which the wave speed is lower into one in which it is higher) and $\sin\theta > \frac{c_s}{c'_s}$ (i.e. the incident wave is at a large angle to the y axis) then $\sin\theta' > 1$ and η' is pure imaginary, so the transmitted wave is evanescent and we have total internal reflection. In this case $\vec{u} = (0,0,1)Te^{ik(x\sin\theta - c'_s t) - \lambda y}$ where $\lambda = ik\eta'$.

To find T, R we apply our boundary conditions, that $\vec{u}$ and $\sigma_{zy} = \mu\frac{\partial u}{\partial y}$ are continuous on $y = 0$; these give us $1 + R = T, \mu ik\cos\theta(1 = R) = \mu' ik\eta' T$; putting $Z = \frac{\mu'\eta'}{\mu\cos\theta}$ we have $R = \frac{1-Z}{1+Z}, T = \frac{2}{1+Z}$.

The time average energy flux in the y direction per unit area is $\langle I_y \rangle = \langle -\dot{u}\mu\frac{\partial u}{\partial y}\rangle = -\frac{1}{2}\mathrm{Re}(\dot{u}^\star\mu\frac{\partial u}{\partial y})$, so for $y < 0$ this is $\frac{1}{2}\mu kc_s k\cos\theta(1 - |R|^2)$, and for $y > 0$ it is $\frac{1}{2}\mu' kc_s k\eta'|T|^2$ if η' is real, or 0 for η' imaginary. These are equal in either case, so the flux is independent of y (in the total internal reflection case this happens because $|R| = 1$, so there is no net flux of energy in the y direction).

Other cases: for an incident P wave at angle θ we get a transmitted P wave at angle θ' and reflected P wave at angle $\hat{\theta}$ as we would expect, but also a transmitted SV wave at angle $\chi' < \theta'$ and reflected SV wave at angle $\chi < \hat{\theta}$; by Snell's law $\hat{\theta} = \theta, \frac{\sin\theta}{c_p} = \frac{\sin\theta'}{c'_p} = \frac{\sin\chi}{c_s} = \frac{\sin\chi'}{c'_s}$; there are several possibilities which will lead to total internal reflection.

So a single incoming wave gives 4 outcoming waves; however, we can solve this, because we have 4 boundary conditions by continuity of $\vec{u}$ and all three components of $\underline{\underline{\sigma}} \cdot \vec{n}$ at $y = 0$; in general we will need to invert a 4×4 matrix to determine the four complex amplitudes, which puts the problem beyond tripos level. However, the special cases of a "clamped" problem $\vec{u} = 0$ on $y = 0$ or "free" problem $\underline{\underline{\sigma}} \cdot \vec{n} = 0$ on $y = 0$, in which no transmitted wave appears, are quite tractable and the lecturer hints that they are likely to appear on the exam.

In suitable circumstances, an incident P wave can give rise to a pure reflected SV wave, and probably vice versa; this is called mode conversion.

The case of an incident SV wave is almost identical to this.

4 Dispersive Waves

In 1D the phase speed (by which we [informally] mean the speed of a moving crest) of a harmonic plane wave $e^{i(kx-\omega t)}$ is $c = \frac{\omega}{k}$. Suppose c depends on k; then an initial "splash" [which is a superposition of] many k will spread out as t increases; this is dispersion

As we mentioned briefly before, to get dispersion there must be more than a velocity scale in the physics of the problem (e.g. an additional length scale). In 2D or 3D a harmonic wave $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ may be dispersive if $\omega = \omega(\vec{k})$ is anisotropic, i.e. not a function solely of $|\vec{k}|$. This can lead to complicated wave patterns.

We will consider more complicated linear systems than the usual wave equation (also the wave equation with finite boundaries); in this section we consider only media whose properties are independent of x and t, but in section 5 we will relax this restriction.

4.1 1D acoustic wave guide

Consider a "tube" along the x direction; we will cover only the rectangular (cross-section) case, which comes out "neatly" in terms of sin, cos rather than needing Bessel functions and so on. $\frac{\partial^2\phi}{\partial t^2} = c_0\nabla^2\phi$, $\frac{\partial\phi}{\partial\vec{n}} = 0$ on fixed boundaries at $y = 0, a$ and $z = 0, b$. We try a solution of the form $\phi = e^{i(kx-\omega t)}f(y,z) \Rightarrow -\omega^2 f = c_0^2(-k^2 f + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2})$.

We need $\frac{\partial f}{\partial\vec{n}} = 0$ on the walls, so $f = A\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b}$ for m, n integers, and $\omega^2 = c_0^2(k^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}) \therefore \omega = \omega(k)$ is dispersive (the special case $m = n = 0$ is nondispersive, as in section 2)

If we plot ω against k, we find that for m, n not both 0, the curve lies above the line $\omega = c_0 k$ (the $n = m = 0$ case); in particular its y-axis intercept is positive. So for fixed n, m there is a $\min_k \omega = c_0\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}} > 0$, called the cut-off frequency; if we excite this mode at a frequency below the cut-off, our wave is evanescent, i.e. does not propagate down the tube.

Energy

Consider a single mode; define for any $\star$ $\bar{\star} = \frac{1}{ab}\int_0^a\int_0^b \star\, dydz$, the spatial average over area. Then $\langle\bar{K}\rangle = \frac{1}{2}\langle\overline{(\nabla\phi)^2}\rangle = \frac{1}{16}\rho_0 A^2(k^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2})$; $\langle\bar{W}\rangle = \frac{1}{2}\frac{\rho_0}{c_0^2}\langle\overline{(\frac{\partial\phi}{\partial t})^2}\rangle = \langle\bar{K}\rangle$, and we again have equipartition. The flix of energy in the x direction is $\langle\overline{I_x}\rangle = \frac{1}{8}\rho_0 A^2 k\omega = \frac{c_0^2 k}{\omega}\langle\bar{K} + \bar{W}\rangle$, so energy is transported at speed $c_g = \frac{c_0^2 k}{\omega}$, called the group velocity. Note this is $\neq \frac{\omega}{k}$ (note also that $c_g = \frac{\partial\omega}{\partial k}$, though the reason for this is far from aparrent at this point).

It can be useful to visualise our waveguide solution as a superposition of two plane waves "zigzagging" along at angle θ above the $y = 0$ plane in a direction contained in the xz plane, being reflections of each other in the plane along the centre of the tube. Let $\sin\theta = \frac{m\pi c_0}{\omega a}$, then we can express our solution as having $\phi \propto (e^{\frac{i\omega}{c_0}(x\cos\theta + y\sin\theta - c_0 t)} + e^{\frac{i\omega}{c_0}(x\cos\theta - y\sin\theta - c_0 t)})$. Thus the apparent x-phase velocity is $\frac{c_0}{\cos\theta}$, but the energy moves down the tube at velocity $c_0\cos\theta$.

4.2 Love waves (elastic SH waves in a wave guide)

Consider a layer of rock from $y = 0$ to a free surface at $y = h$ ($\bar{\mu}\frac{\partial u}{\partial y} = 0$ on this boundary); we have properties $\bar{\rho}, \bar{\mu}, \bar{c}_s$ in this layer, then below it we have another layer with properties ρ, μ, c_s ($u, \mu\frac{\partial u}{\partial y}$ are continuous across this boundary), in which we want only evanescent disturbances (disturbances $\to 0$ as $y \to -\infty$, no energy transmitted towards $y = -\infty$). For this we need total internal reflection, so $\bar{c}_s < c_s$.

SH waves are given by $\vec{u} = (0, 0, u)$. For $y > 0$ try $u = \cos\alpha(h-y)e^{ik(x-ct)}$, $c = \frac{\omega}{k}$, so that $\bar{\mu}\frac{\partial u}{\partial y} = 0$ on $y = h$. This is ok if $(\frac{\alpha^2}{k^2} + 1)\bar{c}_s^2 = c^2 \therefore c > \bar{c}_s$. For $y < 0$ we try $u = \cos\alpha h e^{ik(x-ct)+\beta y}$; this gives us continuity of u at $y = 0$. Then for the wave equation $c^2 = c_s^2(1 - \frac{\beta^2}{k^2})$; we need β real so we will have evanescence below $y = 0$, so we need $\bar{c}_s^2 < c^2 < c_s^2$. For continuity of σ_{zy} at $y = 0$ we have $\bar{\mu}\alpha\sin(\alpha h) = \mu\beta\cos(\alpha h) \therefore \tan(\sqrt{\frac{c^2}{\bar{c}_s^2} - 1}kh) = \frac{\mu}{\bar{\mu}}\frac{\sqrt{1-\frac{c^2}{c_s^2}}}{\sqrt{\frac{c^2}{\bar{c}_s^2}-1}}$ (this may or may not be the love equation).

Thus we have (admittedly in horribly transcendent form) a way of obtaining a value for c from k, $c(k)$, which gives us $\omega(k)$; thus we have a dispersion relation.

When do we have solutions? Plot both sides of this equation against $\sqrt{\frac{c^2}{\bar{c}_s^2} - 1}$; the left hand side is a tan curve with period $\frac{\pi}{kh}$ (recall the period of tan is usually π, not 2π), while the right hand side comes down from infinity looking like a $\frac{1}{x}$ curve, then runs outwards somewhat before curving downwards again, and meets the axis at $\sqrt{\frac{c^2}{\bar{c}_s^2} - 1} = \sqrt{\frac{c_s^2}{\bar{c}_s^2}}$, i.e. $c = c_s$. So since one curve comes down from infinity to 0 and the other rises from 0 to infinity, we will have at least one real root for any k; there is a second mode iff $\sqrt{\frac{c_s^2}{\bar{c}_s^2} - 1} > \frac{\pi}{kh}$, a third mode iff $\sqrt{\frac{c_s^2}{\bar{c}_s^2} - 1} > \frac{2\pi}{kh}$ and so on.

If we increase k, the tangent curves "squash" to the left, so the value of $x = \sqrt{\frac{c^2}{\bar{c}_s^2} - 1}$ at the intersection of the two curves (i.e. the solution to our equation) decreases, so $c(k)$ decreases (from c_s at maximum to $\bar{c}_s$ at maximum). There is a cutoff frequency for each mode [other than the first].

4.3 Beats and Group Velocity

Consider two waves of equal amplitude and unequal but near ω, k. Set $\phi = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)$, which we can calculate to $= 2\cos(\frac{k_1+k_2}{x} - \frac{\omega_1+\omega_2}{t})\cos(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t)$. At a fixed time t, the signal looks like rapidly oscillating waves of period $\approx \frac{2\pi}{k_1} \approx \frac{2\pi}{k_2}$, the "carrier wave", the peaks and crests of which lie on an "envelope" of much longer period $\frac{2\pi}{k_1-k_2}$, the "modulation".

The crests move at the phase speed $c = \frac{\omega_1+\omega_2}{k_1+k_2} \approx \frac{\omega_1}{k_1} \approx \frac{\omega_2}{k_2}$, but the envelope moves at the group speed $c_g = \frac{\omega_1-\omega_2}{k_1-k_2}$, which may be $> c$ or $< c$; therefore, generally, the crests move relative to the envelope. This suggests that if we have $\omega = \omega(k)$, then taking the limit as $k_1 \to k_2$, $c_g = \frac{\partial\omega}{\partial k}$, which $\neq c$ unless c is constant; we will prove this more thoroughly later.

4.4 Inital Value Problems for Dispersive Waves (1D)

Consider a problem with no finite boundaries, x running from $-\infty$ to ∞, and no disturbances at $\pm\infty$. Assume we have a linear system with $\omega = \omega(k)$; we are given an initial disturbance and want to calculate the disturbance at a later time t. The idea is to express $\phi(x,t) = \int_{-\infty}^{\infty} \hat{\phi}(k,t)e^{ikx}dk$; then $\hat{\phi}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x,t)e^{-ikx}dx$; we split the disturbance into waves, since we can easily calculate the evolution of these, it is just like $e^{-\omega t}$ where $\omega = \omega(k)$.

Example: First order in t: suppose we have $\frac{\partial\phi}{\partial t} = \ldots$, where the right hand side is linear (e.g. $\frac{\partial^n\phi}{\partial x^n}$); we shall need only $\phi(x,0)$ for an initial condition. At $t = 0$, $\hat{\phi}(k,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x,0)e^{-ikx}dx$ (at tripos it can be important to recognise that for ϕ real $\hat{\phi}(-k,0) = \hat{\phi}^\star(k,0)$, but in real life it is usually more useful to just assume ϕ complex), so at time t $\hat{\phi}(k,t) = \hat{\phi}(k,0)e^{-i\omega(k)t}$, so $\phi(x,t) = \int_{-\infty}^{\infty} \hat{\phi}(k,0)e^{ikx-i\omega(k)t}dk$.

Example: Second order in t: suppose $\frac{\partial^2\phi}{\partial t^2} = \ldots$ as before. We now need two initial conditions, usually given as $\phi(x,0)$ and $\frac{\partial\phi}{\partial t}(x,0)$. If $\phi = e^{i(kx-\omega t)}$ then the dispersion relation is $\omega^2 = f(k) \therefore \omega = \pm\sqrt{f(k)}$, so there are two waves $e^{ikx\pm i\omega(k)t}$ for each k. So $\phi(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega t)} + B(k)e^{i(kx+\omega t)}dk$. Then $A+B = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x,0)e^{-ikx}dx$, $-i\omega A+i\omega B = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial\phi}{\partial t}(x,0)e^{-ikx}dx \therefore$ $\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{2}(\hat{\phi}(k,0)\begin{pmatrix} - \\ + \end{pmatrix}\frac{1}{i\omega}\frac{\partial}{\partial t}\hat{\phi}(k,0))$.

Example: The beam equation: $\phi_{tt}+\gamma^2\phi_{xxxx} = 0$, $\phi \to 0$ as $x \to \pm\infty$. This is the equation for a "twanged" ruler, which is a similar problem to that of waves on a string, but there is no tension; rather the forces come from the beam's resistance to bending (and bending problems lead to fourth spatial derivatives). If we had had the time we could in principle have derived this in the solids section, along with an expression for γ, but for our present purposes it suffices to know that $\gamma > 0$ is a material constant with dimensions L^2T^{-1}, i.e. not a velocity.

Putting $\phi = e^{i(kx-\omega t)}$ we find $\omega = \pm\gamma k^2$, our dispersion relation; $c_g = \frac{\partial\omega}{\partial k} = \pm 2\gamma k = 2k$.

Suppose that at time $t = 0$ $\frac{\partial\phi}{\partial t} = 0$ and $\phi = \frac{e^{i\alpha x-x^2}}{\lambda^2}$ for some constants α, λ. We have $A(k) = B(k)$ and find (an exercise which the reader really should do) they are equal to $\frac{\lambda}{4\sqrt{\pi}}e^{-\frac{\lambda^2(k-\alpha)^2}{4}}$.

The graph of $\phi(x,0)$ looks like an envelope of $\pm$ a Gaussian, of width $O(\lambda)$, with carrier wave oscillating inside this of wavelength α^{-1}. For $\lambda >> \alpha^{-1}$ this is called a Gaussian wave packet (or Gaussian wave train) of wavelength α. However, this is a poor choice of language; in fact the graph of A against k looks like a Gaussian centred on α of width $O(\lambda^{-1})$ (which is a version of the uncertainty principle in QM); there are a range of different wavenumbers k and this is crucial to the behaviour, because we have a range of wave speeds of size $\gamma\lambda^{-1}$ within the packet. So $\phi = \phi_+(x,t) + \phi_-(x,t)$, the first of these terms representing waves travelling to the right, the second waves travelling to the left. We have $\phi_\pm = \int_{-\infty}^{\infty} e^{-\frac{\lambda^2(k-\alpha)^2}{4}+ikx\mp i\gamma k^2t}dk$, which we find to be $\frac{\lambda}{2\sqrt{\lambda^2\pm 4i\gamma t}}e^{i\alpha x\mp i\gamma\alpha^2t-\frac{(x\mp 2\gamma\alpha t)^2}{\lambda^2\pm 4i\gamma t}}$.

Consider ϕ_+ alone: there are two limiting behaviours. For $t << \frac{\lambda^2}{4\gamma}$, the variations in distance travelled for waves of different wavelengths] are $<< \lambda$ and dispersion is negligible. We have $\phi \sim \frac{1}{2}e^{i(\alpha x-\omega(\alpha)t)}e^{-\frac{(x-c_g(x)t)^2}{\lambda^2}}$; the factor of $\frac{1}{2}$ comes from the original disturbance splitting into two waves, the first exponential factor is a carrier wave of wavenumber α, with crests moving at velocity $\frac{\omega(\alpha)}{\alpha}$, and the second is a slow Gaussian modulation, of width λ, moving with velocity $c_g(x)$; compare with our earlier work coverage of beats. Since $c_g = 2c$, crests must appear at the front of the packet and disappear at the back.

If $t >> \frac{\lambda^2}{4\gamma}$ dispersion occurs; we have $\frac{1}{4i\gamma t+\lambda^2} = \frac{1}{4i\gamma t}(1-\frac{\lambda^2}{4i\gamma t}+\dots)$ (we need to take two terms in our approximation, as the first is pure imaginary so will give a pure phase term when exponentiated). So $\phi_+ \sim \frac{\lambda}{\sqrt{\gamma t}}e^{-i\frac{\pi}{4}+i\frac{x^2}{4\gamma t}-\lambda^2(x-t)\frac{(\alpha t)^2}{(4\lambda t)^2}}$; we write this as $\frac{\sqrt{\pi}}{\sqrt{\gamma t}}A(\frac{x}{2\gamma t})e^{i\frac{x^2}{4\gamma t}-i\frac{\pi}{4}}$ as $t \to \infty$; this is a stationary phase solution.

For fixed t, the maximum amplitude occurs at the position such that $\frac{x}{2\gamma t} = \alpha$ (recall A has its peak at α), i.e. $\frac{x}{t} = c_g(\alpha)$; the disturbance envelope moves at speed $c_g(\alpha)$. Since the disturbance spreads out over a distance $\propto \gamma t$, conservation of energy (which is $\propto$ amplitude squared) suggests we must have our amplitude $\propto \sqrt{t}$.

4.5 Stationary Phase

Waves as $t \to \infty$.

Suppose $\phi_+(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega(k)t)}dk$. Consider an observer moving with $x = Vt$ as $t \to \infty$ for some fixed V. Then $\phi_(Vt,t) = \int_{-\infty}^{\infty} A(k)e^{i\theta}dk$ where $\theta = (kV-\omega(k))t$. The integrand here looks like many oscillations of width $\propto \frac{1}{t}$ (observe that $\frac{d\theta}{dk} = (V-\omega'(k))t$, so if $\frac{d\theta}{dk} \neq 0$ a typical period is $\propto \frac{1}{t}$) within an envelope of $\pm A(k)$. So as $t \to \infty$, we have lots of cancellation in the integral, since we have positive and negative lobes very close together, which (almost) match up; in fact the integral is exponentially small in k as $t \to \infty$. We can see this informally by integrating by parts n times, which shows that the integral decays faster than t^{-n}, for any n.

If $\frac{d\theta}{dk} = 0$ at $k = k_0(V)$, we have a point of stationary phase and the cancellation is much weaker: near $k = k_0$, $\theta = (k_0V - \omega(k_0))t + (k-k_0)(V - \omega'(k_0))t + \frac{1}{2}(k-k_0)^2(-\omega''(k_0)) + \dots$, but the second term here is 0. So the phase changes by 2π if $k = k_0 + O(t^{-\frac{1}{2}})$, so the oscillations here are of wavelength only $O(t^{-\frac{1}{2}})$. Now the contribution to the integral from k near k_0 is $\phi_+(Vt,t) = A(k_0)e^{i(k_0V-\omega(t_0)t)}\int_{k-k_0=-\infty}^{k-k_0=\infty} e^{-i\omega''(k_0)(k-k_0)^2\frac{t}{2}}dk$; (we are being slightly devious here; in this course we will not justify taking the integral from $-\infty$ to ∞ instead of just for a small region around k_0.

We can find this integral is $\sqrt{\frac{2\pi}{i\omega''(k_0)t}} \therefore \phi_+ = \sqrt{\frac{2\pi}{|\omega''(k_0)|t}}A(k_0)e^{i(k_0V-\omega(k_0))t} - i\frac{\pi}{4}\mathrm{sgn}(\omega''(k_0))$.

Notes: 1) This formula assumes there is a point of stationary phase; if not, the integral is exponentially small. 2) If there are several points of stationary phase, just add up the values for each 3) If $\omega''(k_0) = 0$ then we must include the $(k-k_0)^3\omega'''(k_0)$ term in the above approximation, and find $\phi_+ \propto t^{-\frac{1}{3}}$. There are other special cases, e.g. those where $A(k_0) = 0$, but we shall leave them to the Asymtotic Methods course. 4) This result tells us that when moving at speed V

we see the waves for which $V = \frac{\partial\omega}{\partial k} = c_g \mid_{k_0}$. 5) The best way to rigorously prove all these results is via "steepest descents" in the complex k-plane. 6) Energy: consider a small range of wavenumbers δk near k_0. These have (different) velocities $c_g = \omega'(t)$, so the overall spread of this packet at time t is $\omega''(k_0)\delta kt$ $\therefore$ conservation of energy requires that $A^2\omega''(k_0)t$ is constant, so $\phi_+ \propto t^{-\frac{1}{2}}$. 7) Example: the beam equation $\omega = \gamma k^2$ $\therefore$ $c_g = 2\gamma k, \omega''(k) = 2\gamma$. For stationary phase $2\gamma k_0 = V$ $\therefore$ $k = k_0 = \frac{V}{2\gamma}$ $\therefore$ $\phi_+ \sim A(\frac{V}{2\gamma})\sqrt{\frac{2\pi}{2\gamma t}}e^{i(k_0V-\gamma k_0^2)t-i\frac{\pi}{4}} = \sqrt{\frac{\pi}{\gamma t}}A(\frac{x}{2\gamma t})e^{i\frac{x^2}{4\gamma t}-i\frac{\pi}{4}}$ as before. 8) Extension to 2D or 3D: Suppose $\phi(\vec{x},t) = \int_{-\infty}^{\infty} A(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)}d\vec{k}$. Suppose $\vec{x} = \vec{V}t$ and $t \to \infty$. The calculation is beyond this course, but the same principle applies: the phase $\vec{k}\cdot\vec{x} - \omega(\vec{k})$ must be stationary so $\frac{\partial}{\partial\vec{k}}(\vec{k}\cdot\vec{V} - \omega(\vec{k})) = 0$ $\therefore$ $\vec{c}_g(\vec{k}) = \frac{\partial\omega}{\partial\vec{k}} = \vec{V}$ (an aside: the only possibility was for the waves to move with the group velocity, because the phase speed $\frac{\omega}{k}$ is not a velocity), if any such waves exist. 9) The radiation condition for initial value problems is automatic; $\vec{c}_g$ must be directed away from a source of waves (note it is quite possible for the crests to move towards the soucre).

4.6 Water Waves (Linear)

(This section is mostly revision from part IB)

Consider water with a fixed bottom $z = -h$, a free surface $z = \eta(x,t)$ with equilibrium position $z = 0$, on which $p = p_{\text{atm}}$; in the fluid we have the force $-g$ in the z direction; $\vec{u} = (u,0,w) = \nabla\phi$ and $\nabla^2\phi$. Surface tension at the interface gives that on $z = \eta$, $p = p_{\text{atm}} - \frac{T\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}$; this last term is the surface tension coefficent T multiplied by the curvature at a given point on the surface. For linear waves we consider $\eta << h$ and $|\eta_x| << 1$ - the slope of the surface is small. Then we can apply our boundary conditions of $z = 0$ rather than $z = \eta$; our governing equations are $\nabla^2\phi = 0$ on $-h \le z \le 0$, $\rho\frac{\partial\phi}{\partial t} + p + \rho gz$ is constant by Bernoulli, and the boundary conditions are $\frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} = w$ on $z = \eta$, so linearising $\frac{\partial\eta}{\partial t} = w = \frac{\partial\phi}{\partial t}$ on $z = 0$, and $p = p_{\text{atm}} - T\frac{\partial^2\eta}{\partial x^2}$ on $z = 0$. $\frac{\partial\phi}{\partial z} = 0$ on $z = -h$.

We try a solution $\eta = Ae^{i(kx-\omega t)}, \phi = Be^{i(kx-\omega t)}\cosh(k(z+h))$ (to satisfy $\frac{\partial\phi}{\partial z} = 0$ on $z = -h$ and $\nabla^2\phi = 0$). The dynamic boundary condition implies $\rho\frac{\partial\phi}{\partial t} \mid_{z=0} -T\frac{\partial^2\eta}{\partial x^2} + \rho g\eta$ is constant, but everything on the LHS is $\propto e^{ikx}$ so this constant must be 0. So $-i\omega\rho B\cosh kh + Tk^2A + \rho gA = 0$; the kinematic boundary condition implies $-i\omega A = kB\sinh kh$. So for a nontrivial solution we find our dispersion relation must be $\omega^2 = (\frac{Tk^3}{\rho} + gk)\tanh kh$.

Limiting cases: 1) Shallow water $kh << 1$. $\tanh kh \simeq kh$ $\therefore$ $\omega^2 \simeq \frac{Thk^4}{\rho} + ghk^2$. The disturbance occupies the full depth of the water. If $\frac{\rho g}{Tk^2} >> 1$ this is a <u>long gravity wave</u>, $\omega = \pm k\sqrt{gh}$ $\therefore$ the wave is nondispersive and travels with speed $\sqrt{gh}$ (e.g. the Indian Ocean tsunami was a long gravity wave, despite the very deep (around 1km) water it travelled in, because it had a very long wavelength; it had a speed of around $100ms^{-1}$ and a period of around 10^3s, so $\frac{2\pi}{k} \approx 100km$. Thus it was nondispersive, which is a large part of what made it so destructive). The case $\frac{\rho g}{Tk^2} << 1$ is "impossible" physically; for it to occur in wate we would need $h < 1mm$, which is enormously impractical and any associated phenomena simply don't occur in real physics; of course the equations have a solution, but it is not relevant. 2) Deep water $kh >> 1 \Rightarrow \tanh kh \sim \pm 1$; the disturbance is confined near the surface. For $\frac{\rho g}{Tk^2} >> 1$, an ocean wave, $\omega = \pm\sqrt{g|k|} \therefore c_g = \frac{1}{2}c < c$, so this is dispersive; new crests appear at the back of a group. If $\frac{\rho g}{Tk^2} << 1$, a capillary wave, $\omega = \pm\sqrt{\frac{T|k|^3}{\rho}} \therefore c_g = \frac{3}{2}c > c$; this is again dispersive, but this time new crests appear at the front of a group.

If we plot the complete dispersion relation, we obtain "mirror" images in all four quadrants so the solution looks broadly like an "X" shape. As we increase k from 0, ω initially increases rapidly, and this then curves downwards (the long gravity wave case, looking like a $\sqrt{k}$ plot); at some point in the "middle" general case we reach the point of minimum c_g (i.e. minimum gradient), then the graph curves upwards again to eventually look like a $\omega^{\frac{3}{2}}$ curve (the short capillary wave case); at some point near the start of this region we have the point where we can draw a tangent to the curve passing through the origin and lying below the curve, which gives the minimum phase speed $\frac{\omega}{k}$.

4.7 Moving sources of waves

Consider a wave in a stationary medium with position vector $\vec{x}$, $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$. Suppose a source of waves moves with constant velocity $\vec{U}$; consider a new frame of reference with position vector $\vec{X}$ that moves with the source, $\vec{x} = \vec{X} + \vec{U}t$. Then the wave becomes $e^{i(\vec{k}\cdot\vec{X}-(\omega-\vec{k}\cdot\vec{U})t)}$; the frequency in the source frame is $\omega_S = \omega - \vec{k}\cdot\vec{U}$.

Doppler Effect: let $\vec{k}$ be at angle β to $\vec{U}$. Suppose a source of sound generates waves of frequency ω_S. $\omega = c_0|\vec{k}| \therefore \omega_S = c_0|\vec{k}| - U|\vec{k}|\cos\beta = \omega(1 - M\cos\beta)$ where M is the Mach number $\frac{|\vec{U}|}{c_0}$. So $\omega = \frac{\omega_S}{1-M\cos\beta}$; if $M < 1$ and the source approaches a fixed observer, $\cos\beta > 0$ and $\omega > \omega_S$. If $M < 1$ and the sound retreates from an observer, $\cos\beta < 0$ and $\omega < \omega_S$.

Moving Medium: If a medium at rest has dispersion relation $\omega = \bar{\Omega}(\vec{k})$ and the same medium moves with velocity $-\vec{U}$, we get the new dispersion relation $\omega = \Omega(\vec{k}) = \bar{\Omega}(\vec{k}) - \vec{k}\cdot\vec{U}$ (some books refer to this case as "in a wind $-\vec{U}$"); note that this new dispersion relation is anisotropic (but for the moment we take $\vec{U}$ to be independent of $\vec{x}$ and t).

4.8 Steady waves

In general waves are $[\propto]e^{i\omega t}$ and this is unsteady. But sometimes there are nontrivial solutions (i.e. $\vec{k} \neq 0$) with $\omega = 0$, so the crests do not move.

4.8.1 Capillary Gravity Waves in 1D

Suppose an obstacle moves with velocity $U > 0$ in deep water and creates a steady wave pattern in it's rest frame. We shall see there are short capillary waves in front of it and long gravity waves behind: In this frame, $\omega = \Omega(k) = \pm\sqrt{\frac{T}{\rho}|\vec{k}|^3 + g|\vec{k}|} - Uk$; for steady waves $\omega = 0$. Were $U < c_{\min}$ [the minimum phase velocity for a stationary medium] no steady wave is possible; if $U > c_{\min}$ then we have four possible values for k. $c_g = \frac{\partial\bar{\Omega}}{\partial k} - U$; two negative and two positive. At the smaller of the positive we have $c_g < 0$, at the larger $c_g > 0$, so we have the larger case as $x \to \infty$, and the small (in absolute terms) negative case as $x \to -\infty$, so we have short capillary waves in front and long gravity waves behind as claimed.

4.8.2 Supersonic bang

$\vec{U} = Mc_0(1,0,0)$ for $M > 1$; $\vec{k}$ lies at angle β above $\vec{U}$. In the frame of a moving plane, $\omega = \Omega(\vec{k}) = c_0|\vec{k}| - \vec{U}\cdot\vec{k} = c_0|\vec{k}|(1 - M\cos\beta)$. There exist steady waves if $\beta = \cos^{-1} M^{-1}$, so we can have steady waves for any $|\vec{k}|$, i.e. there are many wavelengths in the steady waves, so we will have a "boom" of indeterminate frequency. For these waves, $\vec{c}_g = c_0\hat{k} - \vec{U} = \cdots = c_0\sqrt{M^2-1}(-\cos\alpha, \sin\alpha)$ where $\sin\alpha = M^{-1}$. This is the direction of propagation of these waves, so there is a steady disturbance on a cone behind the aircraft, of angle $\sin^{-1} M^{-1}$.

Note: to understand the structure of the cone, we need to consider nonlinear effects. There are unsteady waves as well, but they are confined inside the cane (as we can see by calculating the group velocity of a general wave).

4.8.3 Ship Waves

Much of the theory in this section is due to Kelvin. Consider ship waves in deep water (neglect surface tension). In the ship's frame, $\Omega(\vec{k}) = \sqrt{g|\vec{k}|} - \vec{U}\cdot\vec{k}$. Work in two dimensions; again we will have $\vec{U}$ lying along the positive x axis, $\vec{k}$ at angle β above it, and $\vec{c}_g$ at angle α above the negative x axis. For a steady wave, $\sqrt{g|\vec{k}|} - U|\vec{k}|\cos\beta = 0 \therefore |\vec{k}| = \frac{g}{u^2\cos^2\beta}$. $\vec{c}_g = \frac{1}{2}\sqrt{\frac{g}{|\vec{k}|}}\hat{k} - \vec{U} = \frac{1}{2}U\cos\beta(\cos\beta, \sin\beta) - U(1,0)$, so energy is propagated in a direction $\hat{c}_g = (-\cos\alpha, \sin\alpha)$ with $\tan\alpha = \frac{\cos\beta\sin\beta}{2-\cos^2\beta}$. Plotting a graph of this against β we see it rises to a peak before falling through 0 at $\beta = \frac{\pi}{2}$, then curving around and rising up to hit 0 at $\beta = \pi$. Thus there is a maximum value for this, and thus for α; by calculus we find it occurs at $\cos\beta = \sqrt{\frac{2}{3}} \therefore \sin\alpha_{\max} = \frac{1}{3}$; all steady waves are confined to a 19 degree wedge behind the ship.

Steady wave crest pattern: the idea of this is that at $\vec{x} = (x,y) = r(-\cos\alpha, \sin\alpha)$ we know α, so we know $\hat{c}_g$ and hence have β and so $|\vec{k}|$, giving us the complete $\vec{k}$. So we know the phase $\theta = \vec{k}\cdot\vec{x}$ at a point $\vec{x}$. Along a crest, θ is fixed (it will be $0, 2\pi, 4\pi, \ldots$ on different crests) so fixing θ we have $y = y(x)$ an equation for that crest. To perform this actual calculation is hard and tedious; $\theta(\vec{x}) = \vec{k}\cdot\vec{x} = -|\vec{k}|r\cos(\alpha+\beta) = -\frac{rg}{U^2\cos^2\beta}\cos\alpha\cos\beta(1 - \frac{\sin^2\beta}{2-\cos^2\beta} = -\frac{g}{U^2}\frac{x}{\cos\beta}\frac{1}{2-\cos^2\beta} \therefore x = -\frac{U^2\theta}{g}\cos\beta(2-\cos^2\beta), y = x\tan\alpha = -\frac{U^2\theta}{g}\cos^2\beta\sin\beta$ [there were errors in this in lectures; I believe I have fixed them all, but am unsure. However, the bottom line is correct]. So we have a parametric form giving $y(x)$ for fixed θ. We see that the different crests for increasing θ have the same shape, and are just "scaled up". The shape looks like a "dart" with point on the ship all three of whose edges are concave curves; this gives the steady crest pattern.

Notes: 1) Nonlinear terms are important on the edges of the "wedge" 2) Unsteady waves are possible, but all confined within the wedge.

This result is used when advertising the tripos; there are those who argue that this course should be taught with it at the end, so that the culmination of one's three years of mathematics at cambridge is to be able to calculate the angle of waves behind a duck.

4.9 Internal Gravity Waves

Consider a stably stratified atmosphere $\rho = \rho_0(z)$, $\frac{d\rho_0}{dz} < 0$ (i.e. density decreases with height - this is the meaning of stably stratified). A parcel of air which is lifted will fall under gravity $\vec{g}$, then "bounce" upwards because it has fallen into denser air - it will behave like a water wave. Suppose our base state is at rest $\vec{u} = 0$ and the fluid is incompressible; $\frac{D\rho}{Dt} = 0, \nabla \cdot \vec{u} = 0, \rho\frac{D\vec{u}}{Dt} = -\nabla p + \rho\vec{g}$. Let $\vec{u} = (u, 0, w)$ and linearise for small disturbances: $\frac{\partial\tilde{\rho}}{\partial t} + w\frac{\partial\rho_0}{\partial z} = 0, \nabla \cdot \vec{u} = 0, \rho_0\frac{\partial\vec{u}}{\partial t} = -\nabla\tilde{p} + \tilde{\rho}\vec{g}$.

Suppose that ρ_0 and $\frac{d\rho_0}{dz}$ vary slowly compared with the wavelengeth k^{-1} (this case excludes water waves, where we have a very fast change in density between water and air), i.e. $\frac{k^{-1}}{\rho_0}\frac{\partial\rho_0}{\partial z} << 1$; this is called the Boussinesq approximation in this context. Then we may regard ρ_0 and $\frac{d\rho_0}{dz}$ as constants.

Consider a perturbation $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$. We get $-i\omega\tilde{\rho} + w\frac{d\rho_0}{dz} = 0, i\vec{k} \cdot \vec{u} = 0, -i\omega\rho_0\vec{u} = -i\vec{k}\tilde{p} + \tilde{\rho}g$. Taking the last equation $\cdot\vec{k}$ we have $\tilde{p} = \frac{1}{k^2}(-i\vec{k} \cdot \vec{g}\tilde{\rho}) \Rightarrow -i\omega\rho_0\vec{u} = \tilde{\rho}(\vec{g} - \frac{\vec{k}\vec{k}\cdot\vec{g}}{k^2})$. Then taking the z component of this, $w = -\frac{1}{i\omega\rho_0 g}\tilde{\rho}(g^2 - \frac{(\vec{k}\cdot\vec{g})^2}{k^2} \Rightarrow \omega^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dz}\frac{g^2 - \frac{(\vec{k}\cdot\vec{g})^2}{k^2}}{g^2}$.

We define $N^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dz}$; this is > 0 in a stably stratified atmosphere. N is the Brunt-Väisälä frequency (In the atmosphere it coresponds to a period of around 5 minutes). If $\vec{k} = (k, l, m)$ then $\omega^2 = N^2\frac{k^2+l^2}{k^2+l^2+m^2}$; this is a dispersion relation.

Notes: 1) This is anisotropic 2) The phase velocity is $\vec{c} = \frac{\omega}{|\vec{k}|}\hat{k}$; the group velocity is $\vec{c}_g = \frac{\partial\omega}{\partial\vec{k}} = \frac{N^2}{\omega}\frac{m}{(k^2+l^2+m^2)^2}(km, lm, -k^2 - l^2) \therefore \vec{c}_g \cdot \vec{c} = 0$; surprisingly these are perpendicular. 3) $\forall$ real $\vec{k}$, $\omega < N$ - if $\omega > N$ the waves are evanescent. 4) For a fixed value of $\omega < N$, the dispersion relation surprisingly does not fix $|\vec{k}$, only the direction of $\vec{k}$. Suppose $\vec{k} = |\vec{k}|(\sin\beta, 0, \cos\beta)$, then $\sin\beta = \frac{\omega}{N}$; we find this implies $\vec{c}_g \propto (\cos\beta, 0, -\sin\beta)$. So looking at the waves emnating from a fixed source, we see a "st andrew's cross" of lines at angle $\frac{\pi}{2} - \beta$ from the vertical line through the source, with $\vec{c}_g$ moving away from the source, while the crests are in an infinitesimal "wedge" along these four lines moving perpendicular to the lines and towands the horizontal, and no disturbance away from these four lines. 5) In general N is some function of z. In the special case $\rho_0 \propto e^{-\frac{z}{L}}$ for some constant L, N is constant.

5 Ray Theory (Geometric optics)

5.1 Slowly varying media (WKB theory)

The idea behind this is that waves carry energy and momentum over distances $>> k^{-1}$ and times $>> \omega^{-1}$. So, can we use local solutions to find distant behaviour?

Consider a river of slowly varying depth $h(x)$. Let $\epsilon << 1$ be the wavelength divided by the scale on which variations in $h(x)$ occur, so if we use this length as our unit then $k \sim \epsilon^{-1}$ ϕ is free surface displacement; we will expect that the amplitude and wavenumber of disturbances changes slowly.

We look for solutions where $\phi(\vec{x},t)e^{i\frac{\theta(\vec{x},t)}{\epsilon}}$. Let $\vec{k}(\vec{x},t) = \frac{1}{\epsilon}\frac{\partial\theta}{\partial\vec{x}}, \omega(\vec{x},t) = -\frac{1}{\epsilon}\frac{\partial\theta}{\partial t}$. Each of $\vec{k},\omega$ is big but slowly varying. It follows that $\frac{\partial\vec{k}}{\partial t} = -\frac{\partial\omega}{\partial\vec{x}}$ and $d\theta = \epsilon(\vec{k}\cdot d\vec{x} - \omega dt)$ is an exact differential.

Note that $\frac{\partial\phi}{\partial t} = \frac{1}{A}\frac{\partial A}{\partial t}\phi + \frac{i}{\epsilon}\frac{\partial\theta}{\partial t}\phi$ and the first term here is negligible, so this is $\sim i\omega\theta$ by the definition of ω, and similarly $\frac{\partial\phi}{\partial\vec{x}} \sim i\vec{k}\phi$. So at leading order in ϵ, locally, $\phi = Ae^{i(\vec{k}\cdot\vec{x}-\omega t)}$ for constants $A, \vec{k}, \omega$. So a local dispersion relation applies, $\omega = \Omega(\vec{k},\vec{x},t)$. This is right, since at a fixed position we would still expect $\Omega(\vec{k})$ to be paramaterised by material properties, in this case $h(\vec{x})$.

Wave crest kinematics: $\vec{k} = \frac{1}{\epsilon}\frac{\partial\theta}{\partial\vec{x}} \therefore \frac{\partial}{\partial x}\times\vec{k} = 0 \therefore$ at fixed t, [for points $\vec{A},\vec{B}$] $\theta(\vec{B})-\theta(\vec{A}) = 2\pi\times$ the number of crests between $\vec{A}$ and $\vec{B} = \epsilon\int_{\vec{A}}^{\vec{B}}\vec{k}\cdot d\vec{x}$, which is independent of the choice of path between $\vec{A}$ and $\vec{B}$. So wave crests never appear or disappear inside a closed loop.

Evolution of $\vec{k}$ and ω: as a packet of waves moves, $\vec{k}$ and ω must change to satisfy the local dispersion relation. $\frac{\partial k_i}{\partial t} = -\frac{\partial\omega}{\partial x_i} = -\frac{\partial\Omega}{\partial k_j}\frac{\partial k_j}{\partial x_i} - \frac{\partial\Omega}{\partial x_i}$ (using the summation convention). But $\nabla\times\vec{k} = 0 \therefore \frac{\partial k_j}{\partial x_i} = \frac{\partial k_i}{\partial x_j}$. Also $\frac{\partial\Omega}{\partial\vec{k}} = \vec{c}_g$ the local group velocity, so $\frac{\partial k}{\partial t} + \vec{c}_g\cdot\frac{\partial\vec{k}}{\partial\vec{x}} = -\frac{\partial\Omega}{\partial\vec{x}}$ - moving at $\vec{c}_g$, $\vec{k}$ for a packet changes iff the medium varies with $\vec{x}$. Also $\frac{\partial\omega}{\partial t} = \frac{\partial\Omega}{\partial\vec{k}}\cdot\frac{\partial\vec{k}}{\partial t} + \frac{\partial\Omega}{\partial t} \therefore \frac{\partial\omega}{\partial t} + \vec{c}_g\cdot\frac{\partial\omega}{\partial\vec{x}} = \frac{\partial\Omega}{\partial t}$, so moving at $\vec{c}_g$, ω of a packet changes iff the medium varies with t (we don't actually need this second equation to work with, because we can calculate $\omega(\vec{x},t)$ from $k(\vec{x},t)$ and Ω).

We define a <u>ray</u> (or <u>characteristic</u>) as a trajectory $\vec{x} = \vec{x}(t)$ of a wave packet, so that $\frac{d\vec{x}}{dt} = \vec{c}_g$. The time derivative along a ray is $(\frac{d}{dt})_g = \frac{\partial}{\partial t} + \vec{c}_g\cdot\frac{\partial}{\partial\vec{x}}$, and so $(\frac{d\omega}{dt})_g = \frac{\partial\Omega}{\partial t}, (\frac{d\vec{k}}{dt})_g = -\frac{\partial\Omega}{\partial\vec{x}}$ and $\frac{1}{\epsilon}(\frac{d\theta}{dt})_g = -\omega + \vec{c}_g\cdot\vec{k}$; these are called the <u>ray tracing equations</u>.

Given Ω and a wave packet of given $\vec{k}$ and ω at $\vec{x} = 0, t = 0$ we can find $\vec{c}_g = \frac{\partial\Omega}{\partial\vec{k}} = \frac{d\vec{x}}{dt}$ (at $t = 0$), so after a time dt we can find the new position of the packet and the new values of $\vec{k}$ and ω; we can now move forward in t. Then (purely conceptually; of course the calculations are impractical at this level) we could do this for every possible initial value of $\vec{x}$, and thus obtain a complete solution.

Notes: 1) If in fact Ω is independent of $\vec{x}, t$ then $\vec{c}_g = \frac{\partial\Omega}{\partial\vec{k}}$ is independent of $\vec{x}, t$, so rays are straight lines (as we assumed in section 4) 2) (non-examinable) The equations for $\vec{k}$ and $\vec{x}$ are Hamilton's equations for a dynamical system $\vec{q} = \vec{x}, \vec{p} = \vec{k}, H = \Omega(\vec{k},\vec{x},t)$. So wave packets behave as particles.

Non-examinable: Evolution of $|A|$: The energy density of a packet is $(\propto)$ $|A|^2$. In general this is <u>not</u> conserved, but the wave <u>action</u> $\frac{|A|^2}{\omega}$ is conserved (without proof): $\frac{\partial}{\partial t}(\frac{|A|^2}{\omega}) + \frac{\partial}{\partial\vec{x}}\cdot(\vec{c}_g\frac{|A|^2}{\omega}) = 0$. In the special case $\frac{\partial\Omega}{\partial t} = 0$ then $(\frac{\partial\omega}{\partial t})_g = 0$, and then conservation of $\frac{|A|^2}{\omega}$ implies $\frac{\partial}{\partial\vec{x}}\cdot(\vec{c}_g|A|^2) = 0$, so we have constant energy flux.

Rays in a wind revisited: Suppose the dispersion relation for a stationary medium is $\omega = \bar{\Omega}(\vec{k})$ and the medium moves with slowly varying velocity $-\vec{U}(\vec{x})$. Then $\omega = \Omega(\vec{k},\vec{x}) = \bar{\Omega}(\vec{k}) - \vec{k}\cdot\vec{U}(\vec{x})$. A ray is then given by $\frac{\partial\vec{x}}{\partial t} = \frac{\partial\Omega}{\partial\vec{k}} = \frac{\partial\bar{\Omega}}{\partial\vec{k}} - \vec{U}$.

5.2 Waves on a Beach

Consider a beach along the y axis $x = 0$, water of slowly varying depth $h(x)$, and an incoming wave train with $\vec{k} = (k, l)$. Neglect surface tension, then $\omega = \Omega(\vec{k}, \vec{x}, t)$ where $\Omega^2 = g|\vec{k}| \tanh(|\vec{k}|h)$. This is isotropic, so $\vec{c}_g \parallel \hat{k}$. Assume that far from the shore, $h \to \infty, k \to k_\infty, l \to l_\infty \therefore \omega \sim \sqrt{g|\vec{k}|} \therefore \omega_\infty = \sqrt{g\sqrt{k_\infty^2 + l_\infty^2}}$. Now $\frac{\partial \Omega}{\partial t} = 0 \therefore \omega$ is constant on rays, so $\omega = \omega_\infty$ everywhere. $\frac{\partial \Omega}{\partial y} = 0 \therefore l$ is constant on rays, so $l = l_\infty$ everywhere. But $\frac{\partial \Omega}{\partial x} \neq 0$, so k varies. In theory we should calculate the rays here, but there is no need in this case (and indeed, cases where we would need to are beyond the tripos level); we can infer $k(x)$ directly from Ω here. $\omega_\infty^2 = g\sqrt{k^2 + l_\infty^2} \tanh(\sqrt{k^2 + l_\infty^2} h(x))$ gives an implicit equation for $k(x)$. We can't solve this equation in general, but as $h \to 0$, the $\tanh(a)$ term $\to a$, and so $k^2 \sim \frac{\sqrt{k_\infty^2 + l_\infty^2}}{h} >> l_\infty^2$. So $\vec{k}$ becomes perpendicular to the shoreline as $h \to 0$; the crests are parallel to the shoreline, and further apart than they were in deep water.

Note: In practice, as $h \to 0$ eventually our slowly varying approximation fails. An asymptotic methods course would show that we can get reflected waves, which have been ignored in this analysis. Also, the wave amplitude increases; in practice the waves break and the problem becomes nonlinear.

5.3 Snell's Law and Fermat's Principle

Consider a nondispersive, steady, isotropic, slowly varying medium, for which $\omega = c(\vec{x})|\vec{k}| = \Omega(|\vec{k}|, \vec{x})$. $\frac{\partial \Omega}{\partial t} = 0$ so ω is constant on rays, and $\vec{c}_g = c(\vec{x})\hat{k}$, so the ray equation is $\frac{d\vec{x}}{dt} = \vec{c}_g = c(\vec{x})\hat{k}$.

Snell: consider the <u>stratified</u> case $c = c(z)$, $\vec{k} = (k, 0, m)$ at angle ψ from the z axis, i.e. $= \sqrt{k^2 + m^2}(\sin\psi, 0, \cos\psi)$. Then $\Omega = \sqrt{k^2 + m^2} c(z)$ and $\frac{\partial \Omega}{\partial x} = 0 \therefore k$ is constant so $\frac{\omega}{k} = c(z)\frac{\sqrt{k^2+m^2}}{k} = \frac{c(z)}{\sin\psi}$, so ψ is some function $\psi(z)$ and $\frac{c(z)}{\sin\psi(z)}$ is constant; this is Snell's law as before. On a ray, $\frac{dx}{dt} = \frac{ck}{\sqrt{k^2+m^2}}$, $\frac{dz}{dt} = \frac{cm}{\sqrt{k^2+m^2}} \therefore \frac{dz}{dx} = \frac{m}{k} = \pm\sqrt{\frac{\omega^2}{c^2 k^2} - 1}$; for a given $c(z)$ we can find the equation for each ray.

Example: Suppose $c(z) = \alpha z$ for some $\alpha > 0$. Then each ray is an arc of a circle; it is this kind of relation which leads to a "mirage", as a ray initially heading upwards will curve around and hit the x axis again heading downwards.

Fermat: In such a medium ($\Omega = c(\vec{x})|\vec{k}|$), a ray (locally) minimizes the time of travel between two points $\vec{A}$ and $\vec{B}$, i.e. $T = \int_{\vec{A}}^{\vec{B}} \frac{ds}{c(\vec{x})} = \int_{\vec{A}}^{\vec{B}} \frac{|\dot{\vec{x}}|dt}{c(\vec{x})}$ is minimized on the ray: we will show that the Euler-Lagrange equation $\frac{d}{dt}(\frac{\dot{\vec{x}}}{c(\vec{x})|\dot{\vec{x}}|}) - |\dot{\vec{x}}|\frac{\partial}{\partial \vec{x}}(\frac{1}{c}) = 0$ is satisfied on rays. A ray is given by $\frac{d\vec{x}}{dt} = c(\vec{x})\hat{k} \therefore |\dot{\vec{x}}| = c(\vec{x})$, so the EL equation is $(\frac{d}{dt})_g(\frac{\hat{k}}{c}) + \frac{1}{c}\frac{\partial c}{\partial \vec{x}} = 0$. But $\frac{\hat{k}}{c} = \frac{\vec{k}}{\omega}$ and $(\frac{d}{dt})_g \omega = 0$ and $(\frac{d\vec{k}}{dt})_g = -\frac{\partial \Omega}{\partial \vec{x}} = -|\vec{k}|\frac{\partial c}{\partial \vec{x}}$ so we have the result; the reader is referred to the Feynmann lectures on physics for more detail on these results.

Note: 1) There are other, more complex variational statements for more general dispersion relations $\Omega(\vec{k}, \vec{x}, t)$, which involve minimizing <u>phase</u> rather than time. 2) Near the minimizing path, we get constructive interference; cf stationary phase, earlier. 3) Fermat implies Snell: e.g. for the case of two distinct media with speeds c_1, c_2 if $\vec{A}$ lies h_1 below the boundary and $\vec{B}$ lies h_2

above the boundary, the two being l apart in the "horizontal" direction, then for a ray which goes in a straight line to the point $(x,0)$ on the boundary and thence in a straight line to $\vec{B}$, $T = \frac{\sqrt{x^2+h_1^2}}{c_1} + \frac{\sqrt{(l-x)^2+h_2^2}}{c_2}$; for a ray $\frac{\partial T}{\partial x} = 0 \Rightarrow \frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$.

6 Nonlinear 1D waves

In this section we relax the small amplitude assumption; remain 1D in x, t: $\vec{u}(\vec{x}, t) = (u(x,t), 0, 0)$. We will cover perfect gases and water waves.

6.1 1D waves in a perfect gas

$p = p_0(\frac{\rho}{\rho_0})^\gamma$ (at constant entropy). Let $c^2(\rho) = \frac{dp}{p\rho}$ (this corresponds to our earlier, linear c if $\rho = \rho_0$) $= \gamma\frac{p_0}{\rho_0}(\frac{\rho}{\rho_0})^{\gamma-1} = c_0^2(\frac{\rho}{\rho_0})^{\gamma-1}$. Mass conservation is $\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$, momentum $\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}) + c^2\frac{\partial\rho}{\partial x} = 0$. Take the first equation $\times\frac{c}{\rho}$ and the second equation $\times\frac{1}{\rho}$ and add and subtract; we obtain $\frac{\partial u}{\partial t} + (u \pm c)\frac{\partial u}{\partial x} \pm \frac{c}{\rho}(\frac{\partial\rho}{\partial t} + (u \pm c)\frac{\partial\rho}{\partial x}) = 0$. Now let $Q = \int_{\rho_0}^{\rho}\frac{c}{\hat\rho}d\hat\rho$ [I have added the $\hat{}$ here] $= \frac{2}{\gamma-1}(c - c_0)$. Then our equations become $(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x})(u \pm Q) = 0$, so along a path C_-, called a characteristic, given by $\frac{dx}{xt} = u - c$, the <u>Riemann invariant</u> $u - Q$ is constant; similarly for a characteristic C_+ given by $\frac{dx}{xt} = u + c$, the Riemann invariant $u + Q$ is constant; this derivation is a very frequent tripos question.

Example: Suppose that u and c are given at $t = 0$, for all x, an initial value problem or Cauchy problem. We want to calculate $u(x,t)$ and $c(x,t)$. Conventionally, we draw the diagram "the wrong way around", with the x axis horizontal and the t axis vertical. If we can somehow find the characteristics, then at any $P = (x,t)$ we know $u + Q$ from its value at the bottom A of the C_+ through P and similarly $u - Q$, so we know u and Q (and hence c) at P. If we can do this for all P at a given t, then since we know the slopes of the $C_\pm$ at P (they are $u \pm t$ we can proceed forward in time to $t + dt$.

Note: a) This is difficult and nonlinear b) The characteristics $C_\pm$ are not [generally] straight lines in the (x,t) plane c) the information at P originates only from a finite region AB at $t = 0$; of course we have no way to say how large this region is without calculating the characteristics. d) There is trouble if two C_+ intersect, since we have two different values $u + Q$ should take at the point of intersection; this will give shocks, see later e) In practice, further analytic progress is only possible in the weakly nonlinear case or where either the C_+ or the C_- originate in fluid at rest; we will cover only this second case.

6.2 Simple waves

Suppose the initial conditions are such that for all relevant x_0 at $t = 0$, $u(x_0, 0) = Q(x_0, 0) = \frac{2}{\gamma-1}(c(x_0,0) - c_0)$. This includes the special case $u = 0, c = c_0$. Then on every C_-, $u - Q = 0 \therefore u = Q$ everywhere ($\forall x$ and $\forall t$), so $c = c_0 + \frac{\gamma-1}{2}u$. Now on a C_+, $\frac{dx}{dt} = u + c = c_0 + \frac{\gamma+1}{2}u$ and $u + Q$ is constant, so u is constant on each C_+, $= u(x_0, 0)$ where x_0 is the value of x corresponding to $t = 0$ on the relevant C_+. So $\frac{dx}{dt} = c_0 + \frac{\gamma+1}{2}u(x_0, 0)$ and this is constant; each C_+ is a straight line with equation $x = x_0 + (c_0 + \frac{\gamma+1}{2}u(x_0, 0))t$. So given (x, t) we solve this nonlinear equation to find x_0, then $u(x, t) = u(x_0, 0)$ and $c(x, t) = c_0 + \frac{\gamma-1}{2}u(x_0, 0)$.

Example: Cauchy problem (revisited), special case: $u = 0, c = c_0$ (i.e. undisturbed gas) everywhere outside a compact region CD at $t = 0$. There are then six regions to consider: I to the left of and belof the C_- from C, II to the right of and below the C_+ from D, III above both the C_- from D and the C_+ from C, IV between the C_+s from C and D and above the C_- from D, V between the C_-s from C and D and above the C_+ from C, and VI below the C_- from D and the C_+ from C.

In regions I and II, every $C_\pm$ originates from undisturbed fluid, so $u = 0, c = c_0$; this is also true in III. In IV we have a simple wave (because the C_- came from undisturbed fluid), so the C_+s are straight but not necessarily parallel. In V we have another kind of simple wave, slightly different but analagous to that which we have covered (the C_+ all originate in undisturbed fluid, the C_- will be straight). The behaviour in region VI is very complex.

6.3 Shock formation in a simple wave

If two C_+ characteristics intersect then we have two predictions for u and c at a single point (x, t), which is impossible. So the physics must change; some assumption breaks down. Recall that $x = x_0 + (c_0 + \frac{\gamma+1}{2}u(x_0, 0))t$, so x is no longer monotonic in x_0. This will occur when $\frac{\partial x}{\partial x_0}|_t = 0 \Rightarrow 1 + \frac{\gamma+1}{2}\frac{du}{dx_0}(x_0, 0)t = 0$, so the shock first forms when $t = \frac{2}{\gamma+1}\frac{1}{\max(-\frac{du}{dx_0}(x_0,0))}$; it occurs at the value of x corresponding to these values of x_0 and t.

We have "wave steepening" as t increases. Were this to continue past the shock, the wave would curve over itself to look like an S shape, but before that we have $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_0}\frac{\partial x_0}{\partial x} = \infty$, so our inviscid assumption breaks down.

6.4 Piston problems

Suppose we have a piston at position $x = X(t)$ in a tube of gas. Suppose for $t \leq 0$ $u = 0, c = c_0 \forall x$, and for $t > 0$ $X(t)$ is given; without loss of generality take $X(0) = 0$¿ What is the motion in the gas for $t > 0$?

Expansion wave (rarefaction wave): suppose $\ddot{X}(t) < 0 \forall t$, the piston accelerates out of the gas. All C_- originate in undisturbed fluid ($x > 0, t = 0$), where $u = 0, c = c_0$, so the flow is everywhere a simple wave and every C_+ is a straight line. $\frac{dx}{dt} = c_0 + \frac{1}{2}(\gamma + 1)u$ and u is constant on each C_+. There are two regions to consider, I below the line $x = c_0 t$ and II above it. In I, each C_+ intersects the x axis at some $x > 0, t = 0$ so $u = 0, c = c_0$. In II, the C_+ intersects the piston path, at say $t = \tau > 0$, when $x = X(\tau) < 0$ and $u = \dot{X}(\tau)$, so a C_+ has equation $x = X(\tau) + (c_0 + \frac{\gamma+1}{2}\dot{X}(\tau))(t - \tau)$. Given (x, t) this is an implicit equation for $\tau < t$ and then $u(x, t) = \dot{X}(\tau), c = c_0 + \frac{\gamma-1}{2}\dot{X}(\tau)$. Note that $c = 0$ ($\Rightarrow \rho = 0$) if $\dot{X}(\tau) = \frac{-2c_0}{\gamma-1}$ (i.e. $M = 5$ (recall $\gamma = 1.4$) (this may be called hypersonic flow). This is the maximum possible velocity of the piston if a vacuum is not formed (which suggests that the speed of gas advancing into a vacuum will be $M = 5$, though it would be improper to conclude that that is the case just from this because e.g. our equation of state assumptions break down when our gas becomes too rarefied).

Notes: 1) Slopes $(c_0 + \frac{1}{2}(\gamma+1)\dot{X}(\tau))^{-1}$ of successive C_+ increase (since $\ddot{X} < 0$), so there are no intersections and hence no shocks. Eventually the slope increases through infinity and becomes negative. 2) (Relevant to the example sheet): if $\dot{X}(0) < 0$ the motion is impulsively started, and we have many C_+ originating at the origin. This phenomenon is called an expansion fan.

It is very rare that we can solve these problems exactly, but it does happen in some special cases:

Example: Suppose $X(\tau) = \frac{1}{2}f\tau^2$ for some constant $f < 0$, so $\tau = \sqrt{\frac{2X}{f}}$. There are in fact three regions on the diagram; the two mentioned before, plus region III, the vacuum, which is the region to the left of the C_+ which is "moving" at speed $5M$ in the negative direction.

In region II, the equation of characteristics is $\frac{1}{2}\gamma\tau^2 + (\frac{c_0}{f} - \frac{1}{2}(\gamma+1)t)\tau + \frac{x-c_0t}{f} = 0$, so we can find τ. We have two roots, but one is unphysical, corresponding to a piston moving inwards in negative time. The solution with $\tau > 0$ is $\tau = \frac{1}{\gamma}(\frac{1}{2}(\gamma+1)t - \frac{c_0}{f} + \sqrt{\frac{1}{4}(\gamma+1)^2t^2 + \frac{c_0^2}{f^2} + \frac{(\gamma-1)c_0t}{f} - \frac{2\gamma x}{f}}$. Then $u(x,t) = ft, c(x,t) = c_0 + \frac{\gamma-1}{2}ft$. (Nice as it is to have an analytic solution, it is of course almost entirely useless in practice, since it's too complicated to have any kind of "natural" understanding of). This is valid provided that $x > -\frac{2c_0}{\gamma-1}(t + \frac{3c_0}{(\gamma-1)f})$; otherwise, a vacuum forms.

Shock formation: Suppose $\ddot{X}(\tau) > 0$ for some τ. Then the slopes of C_+ decrease, so some C_+ intersect and a shock forms. As before, the equation of a C_+ is $x = (c_0 + \frac{1}{2}(\gamma+1)\dot{X}(\tau))(t-\tau) + X(\tau)$, so we have a shock if $\frac{\partial x}{\partial \tau} \mid_t = 0$. We find $t = t_s = \min_\tau \tau + \frac{c_0 + \frac{1}{2}(\gamma-1)\dot{X}(\tau)}{\frac{1}{2}(\gamma+1)\ddot{X}(\tau)}$, and $x = x_s$ is given by the equation of a C_+ with $t = t_s$ and the minimizing value of τ.

Example: Suppose $X(\tau) = \frac{1}{2}f\tau^2, f > 0$. Then $t_s = \min_\tau \frac{2(c_0+\gamma f\tau}{(\gamma+1)f}$; the minimum occurs at $\tau = 0$, so $t_s = \frac{2c_0}{(\gamma+1)f}, x_s = \frac{2c_0^2}{(\gamma+1)f}$. So the shock occurs on the C_+ passing through 0 (since $\tau = 0$. It is actually possible to solve this entire example explicitly, and one can then find the shock by calculating $\frac{\partial}{\partial x}$ and seeing where it becomes infinite, but this is of course a lot more work.

6.5 Nonlinear shallow water waves in 1D

Consider a wavelength much larger than the water depth, and no restriction of amplitude - there is density ρ, a body force of g downwards, and a free surface of height $h(x,t)$ on which $p = 0$. Then the water velocity $u(x,t)$ is (assumed to be) the same throughout the depth of the water, independent of z, and the pressure is hydrostatic (i.e. we neglect vertical accelerations in comparison to g). Conservation of mass implies $\rho\frac{\partial h}{\partial t} + \rho\frac{\partial}{\partial x}(uh) = 0$. $p = \rho g(h-z) \therefore \int_0^h p\,dz = \frac{1}{2}\rho g h^2 =: -F(x)$, the sideways force integrated over the fluid depth, so the x-momentum equation is $\rho\frac{\partial}{\partial t}(uh) + \rho\frac{\partial}{\partial x}(hu^2) = \frac{\partial F}{\partial x} = -\frac{\partial}{\partial x}(\frac{1}{2}\rho g h^2)$. Rearranging, $\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$. We could have written the full equations, taken appropriate asymptotic assumption, and had these equations droup out, but it is easier and quicker to use this "sideways force" trick. If we set $\frac{\partial}{\partial t} = 0$ then these are just the equations of hydraulics.

Compare this pair of equations with the perfect gas equations; if we substitute $h \to \rho, u \to u, g \to \frac{c^2}{\rho}$ they are the same. But for full equivalence we must have the equation of state; we have $c = \sqrt{g\rho}$ rather than $c = c_0(\frac{\rho}{\rho_0})^{\frac{\gamma-1}{2}}$. So we must set $\gamma = 2$ and $g = \frac{c_0^2}{\rho_0}$. So "water is a perfect gas with $\gamma = 2$"; all phenomena from perfect gases occur equivalently for shallow water waves, and indeed wave steepening is a far more intuitive effect in this setting. We can solve problems by $C_\pm$ characteristics as before.

6.6 1D shock waves in a perfect gas

Idea: In a simple wave $c = c_0 + \frac{\gamma-1}{2}u$ and so $(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x})(u + \frac{2}{\gamma-1}(c-c_0)) = 0 \Rightarrow \frac{\partial u}{\partial t} + (c_0 + \frac{1}{2}(\gamma+1)u)\frac{\partial}{\partial x} = 0$; this is a differential form of the C_+ equation. This gives wave steepening; if we imagine the graph of u against x to initially look like one period of a sine wave, then as t increases the peak and trough move closer in x and the part between them becomes steeper and steeper, becoming vertical at $t = t_s$. If we let the same equation continue, the graph becomes something like a back-to-front S shape, the steep part having become so steep it is now slanted like / rather than
, and there are three values of u for some values of x, which is unphysical.

We have neglected e.g. viscosity. Informally, this would give a term $\nu\frac{\partial^2 u}{\partial x^2}$ on the right hand side of our equation, and without proof our grapth then becomes one with a steepening upwards slope, then a sharp peak and a near-vertical drop downwards, a sharp low and a steep upwards slope which gradually becomes shallower. On the drop, our $\nu\frac{\partial^2 u}{\partial x^2}$ term balances $u\frac{\partial u}{\partial x}$. The viscous effects are local, confined to the shock, the region of the drop, of size of order ν; there is a jump in pressure, density and velocity across the shock.

We want to treat the shock as a discontinuity (for small ν) and use the inviscid theory elsewhere, so we will need to find the jump conditions. But first, we need to know where to put the shock.

If we assume no entropy change, C_+ theory tells us that we put the vertical line of the shock in the position on our original unphysical graph such that the two "lobes" that it cuts off are of equal area; this must be the case so that mass is conserved. This technique is called "shock fitting", and works well if the entropy change is small - so-called "weak shock theory".

Rankine-Hugoniot conditions: Assume that the shock separates two uniform regions. Across the shock mass, momentum and (total) energy are conserved (i.e. there are no chemical reactions or overall heat loss), but entropy is not conserved. The lecturer strongly advises students to approach these problems by working in a frame in which the shock is at rest.

Suppose we have a shock at $x = 0$; on the left we have u_1, p_1, ρ_1 and on the right u_2, p_2, ρ_2. Then mass flux gives $\rho_1 u_1 = \rho_2 u_2$ (1), momentum flux gives $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$ (2) as in IB. Energy flux, as per the earlier handout in this course (we need to use the full energy flux, not our simplified "acoustic energy flux"), is $p_1 u_1 + (\frac{1}{2}\rho_1 u_1^2 + w_1)u_1 = p_2 u_2 + (\frac{1}{2}\rho_2 u_2^2 + w_2)u_2$ where $w_i = \frac{1}{\gamma-1}(p_i - \frac{p_0}{\rho_0}\rho_i)$; rearranging this and using the mass flux equation we obtain $\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 = \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{1}{2}u_2^2$ (3). These three equations (1),(2),(3) are called the Rankine-Hugoniot equations.

Example: Suppose on the right we have undisturbed gas $u = 0, \rho = \rho_0, p = p_0$, and on the left we increase the pressure to $p = p_0(1+\beta)$ for some $\beta > 0$, so we will have some $u = u_1, \rho = \rho_1$ and a shock will propagate to the right at some speed V; given p_0, ρ_0, V we want to find u_1, ρ_1, V (Aside: $\beta << 1$ is a weak shock, $\beta >> 1$ is a strong shock).

We will work in a frame with the shock at rest, so on the right $u = -V, \rho = \rho_0, p = p_0$ and on the left $u = u_1 - V, \rho = \rho_1, p = p_0(1+\beta)$. Then mass gives $\rho_1(u_1 - V) = \rho_0(-V)$, momentum gives $p_0(1+\beta) + \rho_1(u_1 - V)^2 = p_0 + \rho_0 V^2$ and energy gives $\frac{\gamma}{\gamma-1}\frac{p_0(1+\beta)}{\rho_1} + \frac{1}{2}(u_1 - V)^2 = \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0} + \frac{1}{2}(-V)^2$. The first two equations imply $V^2 = \frac{p_0}{\rho_0}\frac{\beta\rho_1}{\rho_1-\rho_0}, (V-u_1)^2 = \frac{\beta p_0}{\rho_1-\rho_0}\frac{\rho_0}{\rho_1}$, then substituting this into the last we get an equation in density only, which allows us to find $\frac{\rho_1}{\rho_0} = \frac{1+\frac{\gamma+1}{2\gamma}\beta}{1+\frac{\gamma-1}{2\gamma}\beta}$.

Note that even for $\beta \to \infty$, $\frac{\rho_1}{\rho_0} \to \frac{\gamma+1}{\gamma-1}$ remains finite. Also, $\frac{\rho_1}{\rho_0} < (\frac{p_1}{p_0})^{\frac{1}{\gamma}} = (1+\beta)^{\frac{1}{\gamma}}$, so entropy must change across the shock. In fact if $\beta << 1$ $\frac{\rho_1}{\rho_0} - (1+\beta)^{\frac{1}{\gamma}} = O(\beta^3)$, so the entropy change is small. In practice weak shock theory works well up to $\beta \approx 1$.

Also $V^2 = \frac{p_0}{2\rho_0}(2\gamma + \beta\gamma + \beta) = \frac{c_0^2}{2\gamma}(2\gamma + \beta\gamma + \beta)$, so $V > c_0$ if $\beta > 0$; the shock moves supersonically into undisturbed gas. Notice $V \to \infty$ as $\beta \to \infty$.

$(V - u_1)^2 = \frac{p}{2\rho_0}\frac{(2\gamma+\beta\gamma-\beta)^2}{2\gamma+\beta\gamma+\beta}$. But $c_1^2 = \gamma\frac{p_0(1+\beta)}{\rho_1}$ and so $(\frac{V-u_1}{c})^2 = \frac{1+\frac{\gamma-1}{2\gamma}\beta}{1+\beta} < 1$, so gas emerges from the shock subsonically.

Those readers who enjoy an algebraic challenge should consider the following problem: a shock moves to the right into undisturbed gas $p = p_0, \rho = \rho_0$ next to a wall, then reflects from the wall and heads towards the left. The gas left behind the reflected shock must have $u = 0$, but what is $p = p_2$? We find that as $\beta \to \infty, p_2 = \frac{3\gamma-1}{\gamma-1}p_0(1+\beta) \sim 8p_0(1+\beta)$.

6.7 Hydraulic Jumps

Shallow water theory gives us "$\gamma = 2$ behaviour, so we will also have shocks; however, after the shock (herafter called a bore or hydraulic jump) the physics differs from the perfect gas theory.

From the above, we expect discontinuity of h and u. We can conserve mass and momentum but not energy. In the rest frame of the bore, suppose we have water density ρ (the same on both sides), height h_1 and velocity u_1 on the left and h_2, u_2 respectively on the right. Mass flux gives $\rho u_1 h_1 = \rho u_2 h_2$, momentum flux $-F_1 + \rho h_1 u_1^2 = -F_2 + \rho h_2 u_2^2$. The energy flux is $-Fu + uh(\frac{1}{2}\rho u^2 + \frac{1}{2}\rho g h) = uh(\frac{1}{2}\rho u^2 + \rho g h)$ and this will be discontinuous across the bore.

Example: say we have initially on the right water at height h_0 with $u = 0$, and on the left we raise the height to $h_0(1+\beta)$ (β is called the strength of the bore), giving some speed $u = u_1$; a bore will move with velocity V. In the frame of reference of the bore, on the left $u = u_1 - V$ and on the right $u = -V$. So $h_1(1+\beta)(u_i - V) = h_0(-V)$ and $\frac{1}{2}gh_0^2(1+\beta)^2 + h_0(1+\beta)(u_1 - V)^2 = \frac{1}{2}gh_0^2 + h_0(-V)^2$, so $V^2 = (1+\frac{\beta}{2})(1+\beta)gh_0$. This implies $V > \sqrt{gh_0}$, so the bore moves supersonically, but as before the fluid emerging moves subsonically: $u_1 = \frac{\beta V}{1+\beta} \therefore \frac{V-u_1}{c_1} = \frac{V}{1+\beta}\frac{1}{\sqrt{gh_1}} = \frac{1+\frac{\beta}{2}}{(1+\beta)^{\frac{3}{2}}} < 1$.

Energy: the flux behind the jump is $(u_1 - V)(1+\beta)h_0\rho(\frac{1}{2}(u_1 - V)^2 + gh_0(1+\beta))$, and in front is $(-V)h_0\rho(\frac{1}{2}V^2 + gh_0)$; they differ by $\rho V h_0^2 g\frac{\beta^3}{4(1+\beta)}$; this energy must be lost.

There are two mechanisms for dissipation: in a strong jump $\beta \gtrsim 0.5$, there are turbulent losses. In a weak jump $\beta \lesssim 0.5$ we get the formation of steep waves (not covered by long wave theory) behind the jump, which carry energy away from the jump.

This is the end of this course.