# Riemann Surfaces

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May 14, 2008

This course requires the IB Cplx Anal course for concepts of cx anal fns, isolated singularities, the argument prinzip, open mapping T etc.

A problem in cplx anal is that many "ordinary" real functions suddenly become multivariate when we extend them to the cplx plane, for example $\sqrt{}$, log. One possible response is to say that these functions are "not really" multivariate but rather are defined on the wrong domain; for a given cplx analytic fn we can consider its "best" or "natural" domain, which in this course we take to be not necessarily an open subset of the cplx plane but often a slightly more general object called a Riemann Surface - one definition of which is that it is a surface on which we can do cplx anal. (We will not formally define a RS at this stage; rather we will build up our defn as we go along). Conversely, we can also study RS by considering those cplx analytic functions defined on it.

The recommended books for this course are listed in the schedules; sadly no single book covers all the material (and all the books together cover much more than is in this course); printed notes will be made available.

## 1 Mini-revision of cplx anal

For $U \subset \mathbb{C}$ open we define $f : U \to \mathbb{C}$ is holomorphic or complex analytic if $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists (and is finite) $\forall z_0 \in U$, or equivalently $\forall D(a,r) \subset U$, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n \forall z \in D(a,r)$ (conventionally this defines a complex analytic function while the first definition is what it means to be holomorphic, but the two are equivalent in the spaces we are working with). By a theorem it is also equivalent that $f$ has a Cauchy Integral Formula: for any $\overline{D}(a,r) \subset U$ then $f(z) = \frac{1}{2\pi i}\oint_{|z-a|=r} \frac{f(w)}{w-z} dw \forall z$ w/ $|z-a| < r$; as an excercise the reader should try and find other equivalent definitions (if you find 3 you're doing well; if you find more than 4, tell me).

For a general set $S \subset \mathbb{C}$ (particularly the case where $S$ is a single point), if we say $f$ is "holomorphic on $S$" we mean it is holomorphic on some open set containing $S$.

### 1.1 Isolated zeroes

If $f : U \to \mathbb{C}$ for $U$ open is holomorphic. and $f \not\equiv 0$ but $f(a) = 0$ then $\exists \epsilon > 0$ s.t. $f$ has no zeroes on $D^\star(a,\epsilon)$; then $a$ will be a zero of order $n$ if $f(z) = c_n(z-a)^n$+higher order terms.

## 1.2 Finiteness principle

If $K \subset U$ compact and $f \not\equiv 0$ then $f$ has only finitely many zeroes in $K$; let $z_1, dots, z_N$ be these zeroes and $n_1, \ldots, n_N$ their corresponding orders, then $f(z) = (z-z_1)^{n_1} \ldots (z-z_n)^{n_N} g(z)$ where $g$ is holomorphic on $K$ and never zero on $K$.

As an example, $\sin(\pi z)$ on $D(0, N+\frac{1}{2}) = \pi z(z-1)(z+1)\ldots(z-N)(z+N)g_N(z)$; in fact it turns out that as Euler showed in 1742 we can write $\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$.

For $f : D^\star(a,r) \to \mathbb{C}$ holomorphic with an isolated singularity at $a$ we have a Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ where $c_n = \frac{1}{2\pi i}\int_{|z-a|=s} \frac{f(z)}{(z-a)^{n+1}} dz$ (for $s < r$); this expansion is valid for $z \in D^\star(a,s)$. $\sum_{n=-\infty}^{-1} c_n(z-a)^n$ is called the principal part; the singularity is a removable singularity if $c_n = 0 \forall n < 0$, a pole of order $N$ if $c_{-N} \neq 0$ but $c_n = 0 \forall n < -N$, and an essential singularity otherwise.

We define $f$ is meromorphic on an open $U \subset \mathbb{C}$ if at each $z \in U$ $f$ is holomorphic or has a pole. If $K \subset U$ compact and $f$ meromorphic then $f$ has only finitely many poles in $K$.

"partial fractions": if $f$ is meromorphic on an open $U \subset \mathbb{C}$, $K \subset U$ is compact and $w_1, \ldots, w_M$ are all the poles of $f$ in $K$, then $f(z) = P_1(\frac{1}{z-w_1}) + \cdots + P_M(\frac{1}{z-w_M}) + \tilde{g}(z)$ where each $P_j$ is a polynomial without constant terms and $\tilde{g}$ is holomorphic on $K$. For example, $\frac{\pi^2}{\sin^2(\pi z)}$ on $D(0, M+\frac{1}{2}) = \sum_{n=-M}^{M} \frac{1}{(z-n)^2} + g_M(z)$; in fact it can be shown that $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

Factorization: if $f : U \to \mathbb{C}$ is meromorphic and $K \subset U$ compact, $z_1, \ldots, z_N$ are the zeroes in $K$ with respective orders $n_1, \ldots, n_N$ and similarly $w_1, \ldots, w_M$ the poles of orders $m_1, \ldots, m_M$ then $f(z) = \frac{(z-z_1)^{n_1} \ldots (z-z_N)^{n_N}}{(z-w_1)^{m_1} \ldots (z-w_M)^{m_M}} \breve{g}(z)$ where $\breve{g}$ is holomorphic and never zero on $K$.

We let $f(a) = \infty$ mean $f$ has a pole at $a$; this is just notation.

## 1.3 Proposition

Let $f$ have an isolated singularity at $a$, then $f$ has a pole at $a$ of order $\leq m$ iff $|f(z)| \leq \frac{M}{|z-a|^m}$ for $z$ near $a$.

Zero is in some sense "not really special"; we could replace zero with any constant in all our theorems, and as long as we did this consistently they would all remain valid; this leads us into:

## 1.4 Local Structure of Holomorphic Maps

For $f : D(z_0, r) \to \mathbb{C}$ holomorphic and $f(z_0) = w_0$ we define that $f$ takes the value $w_0$ with multiplicity $n$ at $z_0$ iff $f(z - w_0)$ has a zero of order $n$ at $z_0$; we write $v_f(z_0) = n$ [this $v$ should look funny, like -v-], the valency of $f$ at $z$. It turns out that this is in some sense preserved if we slightly perturb the point.

Notation: For $a \in \mathbb{C}, r > 0$, $D(a,r)$ is the open disc, $D^\star(a,r)$ the punctured (open) disc, $\overline{D(a,r)}$ the closed disc, and $\gamma(a,r)$ the contour $t \mapsto a + re^{it}, 0 \leq t \leq 2\pi$.

## 1.5 Local Mapping Theorem

Let $f : D(z_0, r) \to \mathbb{C}$ be holomorphic with $f(z_0) = w_0, v_f(z_0) = n$. Then $\exists \epsilon > 0, \delta > 0$ such that for $w \in D^\star(w_0, \delta)$ the equation $f(z) = 0$ has exactly $n$ solutions $z$ in $D^\star(z_0, \epsilon)$, and the valency is 1 at each of these.

Corollary: We get the converse "for free": for $f : D(z_0, r) \to \mathbb{C}$ holomorphic with $f(z_0) = w_0$, if $\exists \epsilon > 0, \delta > 0$ such that for $w \in D^\star(w_0, \delta)$ the equation $f(z) = 0$ has exactly $n$ solutions $z$ in $D^\star(z_0, \epsilon)$, and the valency is 1 at each of these, then the valency of $f$ at $z_0$ must be $n$ because this is the only value it could take.

The proof of the theorem: we take $\epsilon > 0$ such that on $D^\star(z_0, 2\epsilon), f(z) \neq w_0$ and $f'(z \neq 0)$; we satisfy the first condition by isolated zeroes applied to $f(z) - w_0$, and the second by continuity if $f'(z_0) \neq 0$ or isolated zeroes applied to $f'(z)$ if not. Set $\delta = \min_{|z - z_0 = \epsilon} |f(z) - w_0$; we have $\delta > 0$ by isolated zeroes. Let $w \in D^\star(w_0, \delta)$, then $f(z) - w = (f(z) - w_0) + (w_0 - w)$; if $|z - z_0| = \epsilon$ then $|w - w_0| < \delta \leq |f(z) - w_0|$.

Now we use Rouché's theorem: $f(z) - w_0$ and $f(z) - w$ have the same number of zeroes (counted with multiplicity) in $D(z_0, \epsilon)$, but since $f'(z) > 0$ these are all simple zeroes so there are $n$ distinct zeroes.

We can also generalize this result to poles; see questions 1 and 5 on the first example sheet for this course.

Recall the Weierstrass-Casorati Theorem (without proof in this course): If $f : D^\star(a, r) \to \mathbb{C}$ is holomorphic and $a$ an essential singularity, then $\forall w \in \mathbb{C} \exists$ a sequence $z_n \to a$ such that $f(z_n) \to w$; this result is useful for the example sheet.

We define that for open $U \subset \mathbb{C}$ the map $f : U \to \mathbb{C}$ is biholomorphic or a conformal equivalence if $f$ is holomorphic and has a holomorphic inverse $f^{-1} : f(U) \to U$.

## 1.6 Inverse Function Theorem

Suppose $f : U \to \mathbb{C}$ is holomorphic, $z_0 \in U, f'(z_0) \neq 0$, then $\exists \epsilon > 0$ such that $f : D(z_0, \epsilon) \to f(D(z_0, \epsilon))$ is a conformal equivalence: sine $f'(z_0) \neq 0, v_f(z_0) = 1 \therefore f$ is 1:1 on $D(z_0, \epsilon)$ for some $\epsilon > 0$; on this disc $f^{-1}$ is well defined, and it is furthermore continuous by the open mapping theorem (use the topological definition of continuity). If we assume the result from question 3 on the first example sheet for this course, then $f^{-1}(w) = \frac{1}{2\pi i} \oint_{\gamma(z_0, \epsilon)} \frac{z f'(z)}{f(z) - w} dz$. Then we change variables: let $\zeta = f(z)$, then $d\zeta = f'(z) dz, z = f^{-1}(\zeta)$; then the above is $\frac{1}{2\pi i} \int_{f(\gamma(z_0, \epsilon))} \frac{f^{-1}(\zeta)}{\zeta - w} d\zeta$, so $f^{-1}$ has a Cauchy Integral Formula, and is therefore holomorphic as required.

## 1.7 Remarks

$(f^{-1})'(w_0) = \frac{1}{f'(z_0)}$

## 1.8 Corollary

If $\phi : U \to \phi(U)$ is holomorphic and 1:1 then $\phi^{-1} : \phi(U \to U$ is holomorphic, by the broof of the inverse mapping theorem or the local mapping theorem. If $U \subset \mathbb{C}$ is open we can equally well use $z$ or $\phi(z)$ coordinates for the points of

$U$; any holomorphic function on $U$ corresponds to one on $\pi(U)$ by composition with $\phi^{-1}$, and vice versa.

So, if we have a holomorphic $f : U \to V$ and conformal equivalences $u : U \to \tilde{U}, v : V \to \tilde{V}$ then $\tilde{f} = v \circ f \circ u^{-1}$ is an alternative expression for the same map; $\tilde{f}$ is holomorphic if and only if $f$ is. This raises a couple of questions:

The numerical values of the function differ between $f, \tilde{f}$; what properties are preserved?

If we have many representations for a function $f$, is there a "best" or canonical one?

From the proof of the local mapping theorem, valency is one invariant; in fact more is true, valency gives us a complete set of invariants:

### 1.9 Theorem: Local Structure of Holomorphic Maps

Assume $f : D(z_0, r) \to \mathbb{C}, f(z_0) = w_0, v_f(z_0) = n$, then $\exists$ conformal equivalences $u, v$ near $z_0, w_0$ respectively such that $u(z_0) = 0, v(w_0) = 0$ and $\tilde{f}(\tilde{z}) = \tilde{z}^n$ - locally there are only countably many distinct holomorphic functions.

Write $f(z) - w = (z - z_0)^n g(z)$ with $g(z_0) \neq 0, g$ holomorphic. By continuity of $g$ take $\epsilon > 0$ such that $|g(z) - g(z_0)| < |g(z_0)| \forall |z - z_0| < \epsilon$. Then $|\frac{g(z)}{g(z_0)} - 1| < 1$ i.e. $\frac{g(z)}{g(z_0)} \in D(1, 1)$ for $z$ in this region, so $\log \frac{g(z)}{g(z_0)}$ can be taken to be a single-valued holomorphic function on $D(z_0, \epsilon)$. Let $h(z) = \exp(\frac{1}{n} \log \frac{g(z)}{g(z_0)} + \frac{1}{n} \log g(z_0))$, then $(h(z))^n = g(z)$. Now write $f(z) = w_0 + (u(z))^n$ where $u(z) = (z - z_0)h(z)$ so $u(z_0) = 0, u'(z_0) = h(z_0) \neq 0$ (since $(h(z_0))^n \neq 0$) so $u(z)$ is a conformal equivalence near $z_0$; then put $v(w) = w - w_0$ which is clearly a conformas equivalence. Then $\tilde{f}(\tilde{z}) = v \circ f \circ u^{-1}(\tilde{z}) = (f \circ u^{-1})(\tilde{z}) - w_0 = (u(u^{-1}(\tilde{z})))^n = \tilde{z}^n$.

# 2 Holomorphic Maps on the Riemann Sphere

$S^2 = \{X^2 + Y^2 + Z^2 = 1\} \subset \mathbb{R}^3_{X,Y,Z}$; we write $N = (0, 0, 1), S = (0, 0, -1), S_0 = S^2 \setminus \{N\}, S_\infty = S^2 \setminus \{S\}$. We define $\phi : (X, Y, Z) \in S_0 \mapsto \frac{X+iY}{1-Z} \in \mathbb{C}, \psi : (X, Y, Z) \in S_\infty \mapsto \frac{X-iY}{1+Z} \in \mathbb{C}$; these are stereographic projections from the north and south pole, both homeomorphisms onto $\mathbb{C}$. For $P \in S^2 \setminus \{N, S\}, \psi(P) = \frac{X-iY}{1-Z} = \frac{X^2+Y^2}{(1+Z)(X+iY)} = \frac{1-Z^2}{(!+Z)(X+iY)} = \frac{1-Z}{X+iY} = \frac{1}{\phi(P)}$. If we let $0 \neq z = \phi(P)$, then $P = \phi^{-1}(z), \psi \circ \phi^{-1}(z) = \frac{1}{z}$ and this is a holomorphic function; similarly $\phi \circ \psi^{-1}(z) = \frac{1}{z}$.

We define $S^2$ with $\psi, \phi$ is the Riemann Sphere. We can use $\phi$ to informally identify $S^2 = \mathbb{C} \cup \{\infty\}$ but we need to make this mean the right things - when is a function holomorphic at $\infty$? When does it take the value $\infty$ holomorphicly?

Definition: a continuous map $F : U \subset S^2 \to S^2$ (for $U$ open) is a holomorphic map if $\phi \circ F \circ \phi^{-1}, \phi \circ F \circ \psi^{-1}, \psi \circ F \circ \phi^{-1}, \psi \circ F \circ \psi^{-1}$ are all holomorphic functions where they are defined (e.g. $\phi \circ F \circ \phi^{-1}$ is defined on $\phi(S_0 \cap U \cap F^{-1}(S_0)) \subset \mathbb{C}$; the reader should find a similar expresson for the other three functions). Note that since $F$ is continuous, $\phi(S_0 \cap U \cap F^{-1}(S_0)) \subset \mathbb{C}$ and similar are open. Where everything is defined, if $f(z) = \phi \circ F \circ \phi^{-1}(z)$ then $\phi \circ F \circ \psi^{-1}(z) = (\phi \circ F \circ \phi^1) \circ (\phi \circ \psi^{-1})(z) = f(\frac{1}{z})$; similarly $\psi \circ F \circ \phi^{-1}(z) = \frac{1}{f(z)}, \psi \circ F \circ \psi^{-1}(z) = \frac{1}{f(\frac{1}{z})}$; thus it makes sense to ask that all four of these maps are holomorphic.

Proposition: if $F : S^2 \to S^2 \setminus \{N\}$ is holomorphic then $F$ is constant: $F$ is determined by $f(z) = \phi \circ F \circ \phi^{-1}(z), h(z) = \phi \circ F \circ \psi^{-1}(z)$; $f$ and $h$ are holomorphic on $\mathbb{C}, h(z) = f(\frac{1}{z})$. $h(z)$ is bounded on $\{|z| \leq 1\}$ so $f(z)$ is bounded on $\{|z| \geq 1\}$ and also on $\{|z| \leq 1\}$, so $f$ is a bounded entire function and so constant by Liouville's theorem; similarly $h$ is constant so $F$ is constant.

Proposition: let $U \subset S^2$ be open, $N \notin U$, $F$ not $\equiv N$. Then $f : U \to S^2$ is holomorphic iff $f = \phi \circ F \circ \phi^{-1}$ is meromorphic on $\phi(U)$ with poles at $\phi(F^{-1}(N))$; thus if $z = \phi(P)$, $F(P) = N$ iff $f(z) = \infty$ (which recall just means $f$ has a pole at $z$): for the forward implication let $z_0 \in \phi(U)$, put $g(z) = \psi \circ F \circ \phi^{-1}(z)$, then TFAE: i) $f(z_0)$ is not defined (i.e. not a point $\in \mathbb{C}$), ii) $F(\phi^{-1}(z_0) = N$, iii) $g(z_0) = 0$. As $F \not\equiv N \Rightarrow g \not\equiv 0$, if $g(z_0) = 0$ then $z_0$ is an isolated zero, so $z_0$ is a pole of $f(z) = \frac{1}{g(z)}$. For the reverse implication, assume $f$ meromorphic on $\phi(U)$; since $N \notin U$, STP $f, g$ are holomorphic where defined: $f$ is holomorphic away from poles; $g = \frac{1}{f}$, and if $f(z_0) = \infty$ then $g$ has a removable singularity at $z_0$, so it can be made holomorphic on its domain.

Proposition: For $V \subset S^2$ open, $N \in V \subset S_\infty$, a non-constant continuous map $F : V \to S^2$ is holomorphic iff firstly $F$ is holomorphic on $V \setminus \{N\}$ (i.e. $f = \phi \circ F \circ \phi^{-1}$ is meromorphic on $\phi(V \setminus \{N\})$ and secondly $f(z) = \sum_{k=-\infty}^{n} c_k z^k$ for some $n \in \mathbb{Z}, c_n \neq 0$ is valid for $|z| >$ some $R$; we shall only proove the forward direction but the arguments are reversible. $F : V \to S^2$ is holomorphic, so apply the previous proposition to $U = V \setminus \{N\}$ and we have the first part; for the second part $S \notin V \Rightarrow F$ is determined by $F \circ \psi^{-1} : \psi(V) \subset \mathbb{C} \to S^2$; write $h(z) = \phi \circ F \circ \psi^{-1}(z)$; this is meromorphic on $\psi(V)$ by proposition 2 with $\phi$ replaced by $\psi$ and $N$ replaced by $S$. $N \in V \Leftrightarrow 0 \in \psi(V)$; $h(z) = \tilde{c}_m z^m + \tilde{c}_{m+1} z^{m+1}$ is a Laurent series with finite principal part valid for $|z| <$ some $r$, for some $m \in \mathbb{Z}$; then $f(z) = \phi \circ F \circ \phi^{-1}(z) = h(\frac{1}{z}) = \cdots + \tilde{c}_{m+1} z^{-1-m} + \tilde{c}_m z^{-m}$ is valid for $|z| \geq \frac{1}{r}$; putting $n = -m, R = \frac{1}{r}$ we have the second part.

So, meromorphic functions are equivalent to holomorphic maps $\to S^2$, so in a sense "poles are not really singularities either"; we've just used the wrong target space.

If $f$, non-constant, has a zero at $a$, then $\frac{1}{f}$ has a pole. For $f(z)$ to be holomorphic at $\infty$ means $g(z) = f(\frac{1}{z})$ is holomorphic at 0 (i.e. has a removable singularity or pole there). We can then define the "multiplicity of $\infty$"; $f(\infty) = a \in \mathbb{C}$ with multiplicity $n$ iff $g(0) = a, v_g(0) = n$ (This definition still works for $f(\infty) = \infty$; this is with multiplicity $n \Leftrightarrow g$ has a pole of order $n$ at 0.

Example: $f(z) = \frac{p(z)}{q(z)}$ where $p, q$ are polynomials with no common factor is certainly meromorphic on $\mathbb{C}$, so corresponds to a meromorphic map $\to S^2$; $f(\frac{1}{z})$ is also meromorphic in $\mathbb{C}$. In particular, it has a Laurent series at 0 with finite principal part, so by the above $f = \phi \circ F \circ \phi^{-1}$ for some holomorphic $F : S^2 \to S^2$.

Theorem: Every holomorphic $F : S^2 \to S^2$ is defined by some rational function $f$ as in the above example; if $F$ is non-constant then it is surjective: assume $F$ is non-constant, then $f = \phi \circ F \circ \phi^{-1}$ is a meromorphic function on $\mathbb{C}$ and $f(z) = c_m z^m + c_{m-1} z^{m+1} + \ldots$ is valid $\forall |z| > R$ for some $R$; let this series be $P_0(z) + c_0 + c_{-1} z^{-1} + \ldots$, so $P_0(z)$ (possibly 0) consists of positize powers on $z$. So $f(z)$ has no poles on $\{|z| > R\}$ and since $\{|z| \leq R\}$ is a compact set $f$ can only have finitely many poles, say $z_0, \ldots, z_N$, which must lie in this set. Let $P_j(\frac{1}{z - z_j})$ be the principal part of the Laurent expansion at $z_j$ (so $P_j$ is a polynomial). Now let $f(z) = P_0(z) + P_1(\frac{1}{z - z_1}) + \cdots + P_N(\frac{1}{z - z_N}) + g(z)$. Then $g$ must be holomorphic on $\{|z| > R\}$. $(f - P_0)(\frac{1}{z}$ has a removable singularity at 0 so is bounded near $z = 0$, so $(f - P_0)$ is bounded on $\{|z| > R\}$; likewise $P_j(\frac{1}{z-z_j})$ is bounded on $\{|z| > R\}$ so $g$ is bounded on this set; $g$ is holomorphic on $\{|z| \leq R\}$ (recall the "partial fractions" result earlier), so $g$ is a bounded entire function and so constant, so $f$ is a rational function. For surjectivity, let $F = \phi^{-1} \circ f \circ \phi$ with $f(z) = \frac{p(z)}{q(z)}, \deg p = d, \deg q = l, d > 0$ or $l > 0$. Suppose $d > l$ and consider $f(z) = a \in \mathbb{C}$; we need $p(z) = aq(z)$ but writing this as $p(z) - aq(z) = 0$ this is a polynomial of degree $d$ so has $d$ roots. For $a = \infty$, $f(\infty) = \infty$ (i.e $g(z) = f(\frac{1}{z})$ has a pole at 0); this is with multiplicity $d - l$, and $f(z) = \infty$ at the $l$ roots of $q(z)$. So there are actually $d$ inverses for each $z$. The $d = l$ and $d < l$ cases are left as an excercise for the reader, but we will find that there are $\max\{d, l\}$ inverses for each point.

Therefore, despite initilly appearing "worse", rational functions are in fact the "nicest" class of functions to consider in practice, since although they have poles, entire functions may have genuine singularities at $\infty$ (and this is the "interesting part" of any study of entire functions).

Remarks: A rational function $f(z) = \frac{p(z)}{q(z)} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ has $f(z) = \lim_{z\to\infty} \frac{p(z)}{q(z)}$ for $z = \infty$, $\infty$ if $q(z) = 0$, and $\frac{p(z)}{q(z)}$ otherwise; this is a fully rigorous use of $\infty$. Also, the number of solutions $P \in S^2$ of $F(z) = w$ is an invariant, $\max\{\deg p, \deg q\}$; it is the same $\forall w \in S^2$. It is known as the <u>degree of $F : S^2 \to S^2$</u>. Finally, if $\deg F = 1$ then $F$ is a bijection and we can show $F^{-1}$ is holomorphic; $F = \frac{az+b}{cz+d}$, a Möbius transformation. So $\mathrm{Aut}(S^2)$ is the group of Möbius transformations; see question 6 on the first example sheet for this course.

Theorem: suppose $P_1, \ldots, P_K, Q_1, \ldots, Q_K$ all $\in S^2$ with $P_i \neq Q_j \forall i, j$ (but perhaps $P_i = P_j$ and/or $Q_i = Q_j$ for some $i, j$, then $\exists$ holomorphic $F : S^2 \to S^2$ such that $F(P_i) = 0, F(Q_i) = \infty$ with multiplicity given by repitition, and there are no other zeroes or poles, and if $G$ is another such map then $G = cF$ for some constant $c \neq 0 \in \mathbb{C}$: Suppose all $P_i, Q_j \neq \infty$, then just set $F(z) = \frac{(z-P_1)\ldots(z-P_K)}{(z-Q_1)\ldots(z-Q_K)}$, then $F(\infty) = 1 \neq 0, \infty$ and we have the result; if $P_{r+1} = \cdots = P_K = \infty$ then $Q_i \neq \infty \forall i$ and setting $F = \frac{(z-P_1)\ldots(z-P_r)}{(z-Q_1)\ldots(z-Q_K)}$ sufficies; the similar final case is left as an exercise for the reader. If $G$ is another such map then $\frac{G}{F}$ is meromorphic on $\mathbb{C}\setminus$ the finitely many points $P_i, Q_j$, and has removable singularities at all of these by considering Laurent expansions. So $\frac{G}{F}$ is a holomorphic map $S^2 \to S^2 \setminus \{0\}$, and therefore some nonzero constant.

What we have shown here is a special case of the general result that holomorphic maps on a compact space are <u>algebraic</u> maps, which leads into the area of algebraic geometry.

From 1715, some of the best mathematicians were interested in integrals of the form $\int_0^\lambda \frac{dx}{Q(X)}$ where $Q(x)$ is a cubic or quartic without multiple roots. These are called <u>elliptic integrals</u> and cannot be solved with "ordinary" functions; they lead into <u>elliptic functions</u>.

For $Q(X)$ a quadratic, e.g. $\int_0^\lambda \frac{dx}{\sqrt{1-x^2}} = I(\lambda)$, the inverse of $I(\lambda)$ is $\lambda(\alpha)$ such that $\int_0^{\lambda(\alpha)} \frac{dx}{\sqrt{1-x^2}}$; we can find $\lambda(\alpha) = \sin\alpha$. This is very surprising since it is periodic, but we can explain this using complex analysis; the periodicity comes from taking different paths around the poles of the function in the integrals. The inverse for e.g. $Q(x) = (1 - mx^2)(1 - x^2)$ for $0 < m < 1$ was discovered in c. 1820 by Abel and Jacobi to have two independent periods, one in $\mathbb{R}$ and one in $i\mathbb{R}$. We shall look at periodic functions over $\mathbb{C}$.

Assume $\exists \lambda \neq 0$ such that $f(z + \lambda) = f(z) \forall z$; wlog we can take $\lambda = 1$ (by considering $g(z) = f(\lambda z)$. Assume $f : S \to \mathbb{C}$ where $S = \{z | \alpha < \Im z < \beta\}$. Define $\mathbf{e}(z) = e^{2\pi i z}$; we have $\mathbf{e}(z + 1) = \mathbf{e}(z)$; $\mathbf{e}(S) = A := \{e^{-2\pi\beta} < |z| < e^{-2\pi\alpha}\}$, an annulus. Set $F(w) = F(\mathbf{e}(z)) = F(e^{2\pi i z}) = f(z) \forall w \in A$; this is well defined as $f$ is periodic and holomorphic since $\mathbf{e}(z)$ has a local holomorphic inverse near each $z \in S$ by the IFT. So $F \circ \mathbf{e} = f \Rightarrow F = f \circ$ the local inverse of $\mathbf{e}$. Then we have a Laurent expansion $F(w) = \sum_{n=-\infty}^{\infty} c_n w^n$ valid for $w \in A$ where $c_n = \frac{1}{2\pi i} \oint_\gamma \frac{F(w)dw}{w^{n+1}}$ where $\gamma$ is some loop winding once (anticlockwise) about the hole in the middle of $A$; it will be the image of some path from $a$ to $a + 1$ [direction uncertain] in $S$. Changing variables by $w = \mathbf{e}(z)$ this is $\int_a^{a+1} f(z)\mathbf{e}(-nz)dz$, since $dw = 2\pi i \mathbf{e}(z)dz$. So $f(z) = \sum_{n=-\infty}^{\infty} c_n \mathbf{e}(nz)$, a Fourier Series convergent on $S$. Therefore, "singly periodic" functions are "nothing new".

## 2.1 Doubly-periodic functions

Take $\lambda_1, \lambda_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$, then the lattice $\Lambda$ is $\{n\lambda_1 + m\lambda_2 : m, n \in \mathbb{Z}\} \subset \mathbb{C}$, an additive subgroup of $\mathbb{C}$.

Define: $f(z)$ is doubly-periodic or $\Lambda$-periodic if $f(z + \lambda) = f(z) \forall \lambda \in \Lambda$. We may wlog take $\lambda_1 = 1$ so that $\Lambda = \{n + m\tau | n, m \in \mathbb{Z}\}$; furthermore we wlog take $\Im\tau > 0$. Then we have the fundamental parallelogram $P_\xi$, the parallelogram with corners $\xi, \xi + 1, \xi + \tau, \xi + 1 + \tau$. We can join the top and bottom sides to map this into a tube, and thence into the torus or elliptic curve $E_\Lambda := \frac{\mathbb{C}}{\Lambda}$. The set of continuous $\Lambda$-periodic functions is then equivalent to that of continuous functions on $E_\Lambda$. We define that Elliptic Functions are meromorphic (see later for why we do not consider only holomorphic functions) doubly-periodic functions on $\mathbb{C}$.

Let $P = P_\xi$.

Theorem: constraints on elliptic functions. Assume $F$ is meromorphic and $\Lambda$-periodic on $\mathbb{C}$ and $\xi \in \mathbb{C}$ such that $F$ has no zeroes or poles on the boundary of $P_\xi$; we can do this by drawing a slightly larger region than the fundamental parallelogram; this will be a compact set so contain only finitely many zeroes and poles, so we can then draw a fundamental parallelogram missing all of them. Let $a_1, \ldots, a_k$ be (all) the zeroes of $F$ on the compact set $P_\xi$, repeated by multiplicities; similarly $b_1, \ldots, b_l$ poles. Then firstly if $F$ is holomorphic then it is constant (which is why we are considering meromorphic functions), secondly the sum of residues of $F$ at poles in $P_\xi$, not repeated by multiplicities, is 0, thirdly $k = l$, and finally $\sum_{i=1}^k a_i - \sum_{j=1}^l b_j \in \Lambda$. For the first part if $F$ is holomorphic then it is continuous on $P_\xi$ so bounded on $P_\xi$ therefore bounded on $\mathbb{C}$ so constant by Liouville's theorem, for the second by the residue theorem the sum of residues is $\frac{1}{2\pi i} \oint_{\partial P_\xi} F(z)dz$, but by the periodicity the integral along the bottom left to right cancels the integral along the top right to left and similarly, so the total integral is 0. For the third part, by the argument principle (logarithmic residues), $k - l = \frac{1}{2\pi i} \oint_{\partial P_\xi} \frac{F'(z)}{F(z)} dz = 0$ similarly. For the last, let $I = \frac{1}{2\pi i} \int_{\partial P_\xi} \frac{zF'(z)}{F(z)} dz$;

the horizontal sides part of this is $-\tau\frac{1}{2\pi i}\int_{\xi}^{\xi+1}\frac{F'(z)}{F(z)}dz$ (as all the rest of it cancels, apart from the difference of $\tau$ in $z$). Then if we let $u = F(z)$ this becomes $\frac{-\tau}{2\pi i}\int_{F\circ\gamma}\frac{du}{u}$, but by the periodicity $F\circ\gamma$ is a closed loop, so this is $-\tau n(F\circ\gamma, 0)$ where $n(F\circ\gamma, 0)$ is a winding number and so an integer; similarly the part from the sloping sides ins an integer so the overall integral is $n\tau + m$. But by the residue theorem $I = \sum_{z\in P}\text{res}\frac{zF'}{F}$, where e.g. $\text{res}_{a_i}\frac{zF'}{F} = a_i\times$ the multiplicity of this zero, so $I = \sum a_i - \sum b_j$ as required.

Remarks: 1) For $F$ elliptic, # zeroes of $F$ in $P$ = # poles of $P$ in $P$. Then $\forall c\in\mathbb{C}$, $F - c$ is elliptic with the same poles as $F$, so any value $c$ is taken the same number of times modulo $\Lambda$ (counted with multiplicity). We call this number the degree of an elliptic function $F$. 2) $F$ cannot have degree 1 as then $\sum_{z\in P}\text{res}F \neq 0$ (assuming $F$ is non-constant).

## 2.2 $\theta$-functions

Define $\theta(z) = \theta(z,\tau) = \sum_{n=-\infty}^{\infty}\mathbf{e}(\frac{1}{2}n^2\tau + nz)$ (where $\mathbf{e}(t) = \exp(2\pi iz)$)where $\tau$ is a fixed paramater with $\Im\tau > 0$. We shall write $\sum$ to mean $\sum_{n=-\infty}^{\infty}$.

Properties: 1) $\theta$ is holomorphic; we shall show this by proving it is holomorphic on $S_R = \{z : |\Im z| \leq R\}$ for general $R$. We shall show uniform convergence on $S_R$ using Weierstrass' M-test: let $\tau = \tau_1 + i\tau_2, z = x + iy$; we have $\tau_2 > 0$. Then $|\mathbf{e}(\frac{1}{2}n^2\tau + nz)| = \exp(-\pi\tau_2 n^2 - 2\pi ny) \leq \exp(-\pi\tau_2 n^2 + 2\pi|n|R) = \exp(\pi\tau_2(|n| - \frac{R}{\tau_2})^2 + \frac{\pi R^2}{\tau_2}) \leq M_R e^{-\pi\tau_2(|n|-\frac{R}{\tau_2})^2} \leq M_R(e^{-\pi\tau_2})^{|n|}$ for sufficiently large $|n|$, so the series converges (in fact better than a geometric series) uniformly on $S_R$ and also absolutely; this is true $\forall R > 0$ so $\theta$ is holomorphic on $\mathbb{C}$.

2) $\theta(z)) = 1 + \sum_{n=1}^{\infty}\mathbf{e}(\frac{1}{2}\tau n^2 + nz) + \mathbf{e}(\frac{1}{2}n^2\tau - nz)$ (this is valid by absolute convergence) so $\theta(-z) = \theta(z)$.

3) $\theta(z+1) = \theta(z)$ since all terms are periodic - we have a Fourier series.

4) $\theta \not\equiv 0$ as the coefficients of the FS are nontrivial.

Since $\theta$ is holomorphic it cannot be doubly-periodic, but it is "as close as possible": $\theta(z+\tau,\tau) = \sum\mathbf{e}(\frac{1}{2}n^2\tau + n\tau + nz) = \sum\mathbf{e}(\frac{1}{2}(n+1)^2\tau - \frac{\tau}{2} + (n+1)z - z) = \mathbf{e}(-\frac{\tau}{2} - z)\sum\mathbf{e}(\frac{1}{2}(n+1)^2\tau + (n+1)z = \mathbf{e}(-\frac{\tau}{2} - z)\theta(z,\tau)(\star)$, since we can replace $n+1$ with $n$ and the series is the same.

Proposition: Let $P$ be such that $\theta$ has no zeroes on $\partial P$ (which we can do in the same way as above), then $\theta$ has exactly one zero in $P$ and this is a simple zero (i.e. a zero of order 1): by the argument principle, $2\pi i\times$ the number of zeroes in $P = \oint_{\partial P}\frac{\theta'(z)}{\theta(z)}dz$; the integrals along the sloped sides cancel since $\theta$ has period 1. $\theta'(z+\tau) = \mathbf{e}(-\frac{\tau}{2} - z)\theta'(z) - 2\pi i\mathbf{e}(-\frac{\tau}{2} - z)\theta(z) \therefore \frac{\theta'(z+\tau)}{\theta(z+\tau)} = -2\pi i + \frac{\theta'(z)}{\theta(z)}$, so the integral along the horizontal edges is $2\pi i$, so we have one zero counted with multiplicity, i.e. one simple zero, as required.

Proposition: $\theta(\frac{1}{2} + \frac{\tau}{2}) = 0$ is the unique zero modulo $\Lambda$: from $(\star)$, $\theta(z - \frac{1}{2} - \frac{\tau}{2})\mathbf{e}(-\frac{\tau}{2} - (z - \frac{1}{2} - \frac{\tau}{2})) = \theta((z - \frac{1}{2} - \frac{\tau}{2}) + \tau)$. Define $f(z) := \theta(z - \frac{1}{2} - \frac{\tau}{2})$, then we need $f(0) = 0$; we have $f(z)\mathbf{e}(-z + \frac{1}{2}) = \theta(z - \frac{1}{2} + \frac{\tau}{2}) = \theta(z + \frac{1}{2} + \frac{\tau}{2})$ since $\theta$ is even, $= \theta(-z - \frac{1}{2} - \frac{\tau}{2})$ by evenness, $= f(-z)$, so putting $z = 0$ we have $\mathbf{e}(\frac{1}{2}) = -1$ so $f(0) = -f(0)$ and $f(0) = 0$ as required.

Theorem: Construction of Elliptic Functions: suppose we have $a_1, \ldots, a_k, b_1, \ldots, b_k \in P \subset \mathbb{C}$, with possibly some repitions but never $a_i = b_j$, such that $\sum a_i - \sum b_j \in \Lambda$ (note this automatically implies $k \geq 2$). Then there is an elliptic function with zeroes in $P$ exactly at the $a_i$, poles exactly at the $b_j$, and this is unique up to multiplcation by a nonzero constant: Uniqueness is easy, if $F_1, F_2$ are two such functions then $\frac{F_1}{F_2}$ is $\Lambda$-periodic with only removable singularities in $P$ so constant. For existence, let $f(z) = \theta(z - \frac{1}{2} - \frac{\tau}{2})$, and $F(z) = \frac{f(z-a_1)...f(z-a_k)}{f(z-b_1)...f(z-b_k)}$; this has the correct zeroes and poles and period 1. Now $f(z+\tau-a) = \mathbf{e}(-z+a+\frac{1}{2})f(z)\forall a \in \mathbb{C}$ by ($\star$), so $F(z+\tau) = \mathbf{e}(\sum_{i=1}^{k}(a_i - b_i))F(z)) = \mathbf{e}(m\tau)F(z)$ for some $m \in \mathbb{Z}$, so $\mathbf{e}(-mz)F(z)$ is a function as required.

Elliptic functions are important in many areas of research; some of them have names in their own right. We want to consider a simple example, so we shall look at a function of elliptic degree 2 (recall there are none of degree 1), which will have two poles; for many years mathematicians thought the easiest of these to study would be those with two simple poles, but in 1862 Weierstrass found that a simpler theory could be developed for an elliptic function with one double pole.

Definition: The Weierstrass $\wp$-function is a meromorphic $\Lambda$-periodic function 1) with $\deg \wp = 2$ and a double pole at $z = 0$ 2) such that the Laurent expansion of $\wp$ at $z = 0$ has leading term $\frac{1}{z^2}$ and no constant term; this does in fact define $\wp$ uniquely.

Proposition: 1) $\wp$ is uniquely determined if it exists 2) $\wp(-z) = \wp(z)\forall z \notin \Lambda$: for 1), suppose $\wp_1(z), \wp_2(z)$ satisfy the definition. Then $g(z) = \wp_1(z) - \wp_2(z)$ is an elliptic function with no poles not on $\Lambda$, but by the Laurent expansions it has at worst a simple pole at $z = 0$, so it is of degree $\leq 1$ so must be constant, but by the Laurent expansion at 0 this constant must be 0, so $\wp_1 \equiv \wp_2$. For 2), $\wp(-z)$ also satisfies the definition of $\wp$, so must be $\equiv \wp(z)$.

Theorem: $\wp$ exists: for suiatable $A \in \mathbb{C}$, $\wp(z) = A - \frac{d}{dz}\log f(z)$ (with $f$ as defined above) satisfies the definition; equally well, $\wp(z) = B + C\mathbf{e}(z)\left(\frac{\theta(z)}{\theta(z) - \frac{1}{2} - \frac{\tau}{t}}\right)^2$ for suitable $B, C \in \mathbb{C}$:

For the first definition, this is clearly meromorphic and periodic with period 1. Now as above we can compute $\frac{f'(z+\tau)}{f(z+\tau)} = 2\pi i + \frac{f'(z)}{f(z)}$ so $\frac{d}{dz}\frac{f'}{f}\mid_{z+\tau}= \frac{d}{dz}\frac{f'}{f}\mid_z$ and this is elliptic; $\frac{f'}{f}$ is holomorphic away from $f = 0$ and has simple poles on $\Lambda$; write $f(z) = zg(z)$ so $\frac{f'}{f} = \frac{1}{z} + \frac{g'}{g}$, then $\left(\frac{f'}{f}\right)' = -\frac{1}{z^2}+$ some function holomorphic at 0, so $A - \left(\frac{f'}{f}\right)' = \frac{1}{z^2} + \tilde{g}(z)$ with $\tilde{g}$ holomorphic, and by a suitable choice of $A$ we can have a function as required.

For the second definition, we will use the construction of elliptic functions from $\theta$ as above; $\left(\frac{\theta(z)}{\theta(z-\frac{1}{2}-\frac{\tau}{2})}\right)^2 = \left(\frac{f(z+\frac{1}{2}+\frac{\tau}{2})}{f(z)}\right)^2$; using the same notation as above, $a_1 = a_2 = -\frac{1}{2} - \frac{\tau}{2}, b_1 = b_2 = 0$; then for $a_1 + a_2 - b_1 - b_2 = n + m\tau$ we have $m = -1$ so $F(z) = \mathbf{e}(-mz)\left(\frac{f(z+\frac{1}{2}+\frac{\tau}{2})}{f(z)}\right)^2 = \mathbf{e}(z)\left(\frac{\theta(z)}{\theta(z-\frac{1}{2}-\frac{\tau}{2})}\right)^2$ is elliptic with the correct poles, so we can choose $C$ such that $F(z) = \frac{1}{z^2} + \ldots$ and $B$ such that the constant term vanishes.

$\wp'$ witll be elliptic of degree 3 and even, so it has three zeroes:

Theorem $\wp'(\frac{1}{2}) = 0 = \wp'(\frac{\tau}{2}) = \wp'(\frac{1+\tau}{2})$ (as $\wp'$ has degree 3, a trivial corollary is that these are all simple zeroes and the only zeroes of $\wp' \mod \Lambda$): $\wp'(\frac{1}{2}) = -\wp'(-\frac{1}{2})$ since $\wp'$ is odd, but this $= -\wp'(\frac{1}{2})$ by periodicity, so $\wp'(\frac{1}{2}) = 0$; the other two are found similarly.

Corollary: $v_\wp(z) = 2$ for $2z \in \Lambda$, 1 otherwise: $v_\wp$ is always 1 or 2 since $\deg \wp = 2$, and (for $z_0$ not a lattice point), by differentiating, $v_\wp(z) = 2 \Leftrightarrow \wp'(z) = 0$, which is the case iff $2z \in \Lambda$; for $z \in \Lambda$ we have a double pole so $v_\wp(z) = 2$, and we have the result.

Remarks, without proof: since $\wp(z)$ is even its Laurent expansion at $z = 0$ is $\wp(z) = \frac{1}{z^2} + 3E_4z^2 + 5E_6z^4 + \dots$ with terms of the form $(2k+1)E_{2k+2}z^{2k}$, where $E_{2k}| = \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^{2k}}$, the Eisenstein series for $\Lambda$. We could define $\wp, \wp'$ explicitly by $\wp'(z) = -2\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3}, \wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$, but in practise this definition is actually less useful than the one we have given.

Let $f_j(z)$ denote "some holomorphic function of $z$"; then we have $\wp(z) = \frac{1}{z^2} + 3E_4z^2 + 5E_6z^4 + z^6 f_1(z) \therefore \wp'(z) = -\frac{2}{z^3} + 6E_4z + 20E_6z^3 + z^5 f_2(z) \therefore (\wp'(z))^2 = \frac{4}{z^6} - \frac{24E_4}{z^2} - 80E_6 + z^2 f_3(z)$; also $4p(z)^3 = \frac{4}{z^6} + \frac{36E_4}{z^2} + 60E_6 + z^2 f_4(z)$ so $\wp'(z)^2 - 4\wp(z)^3 + 60E_6\wp(z) + 140E_6 = z^2 f_5(z)$; the left hand side is elliptic without poles so constant, so the right hand side is also constant; it must be 0, so we have an ODE for the $\wp$-function: $\wp'^2 = 4\wp^3 - 60E_4\wp - 140E_6$, where $E_4, E_6$ are complex constants.

[no, I'm not following this either]

If $Q(p) = (\wp')^2$ we have $Q(z) = 4z^3 - 60E_4z - 140E_6$; $g_2 = 60E_4 = 60\sum_{\omega\neq 0\in\Lambda} \frac{1}{\omega^4}$, $g_3 = 140E_6 = 140\sum_{\omega\neq 0\in\Lambda} \frac{1}{\omega^6}$. Using the local inverse, $\left((\wp^{-1})'(z)\right)^2 = \frac{1}{Q(z)}$; let $z = \wp(w)$. Then $\wp^{-1}(z) - \wp^{-1}(z_0) = \int_{z_0}^{z} \frac{1}{\sqrt{Q(z)}}$, an elliptic integral. This is an equality as sets - the LHS is a multivalued function, the RHS is an integral dependent on the path and which value of the $\sqrt{\ }$ we take.

Let $e_1, e_2, e_3 := \wp(\frac{1}{2}), \wp(\frac{\tau}{2}), \wp(\frac{1+\tau}{2})$. Then $\Gamma = \{2\int_{e_i}^{e_j} \frac{dz}{\sqrt{Q(z)}}\}$; since $Q(z) = 4(z-e_1)(z-e_2)(z-e_3)$ this is $\{\int_{e_i}^{e_j} \frac{dz}{\sqrt{(z-e_1)(z-e_2)(z-e_3)}}\}$. So $\wp$ "sees" its domain $\frac{\mathbb{C}}{\Lambda}$; we can recover the value of $\tau$ from this integral.

If $f, g$ are elliptic with the same periods then so are $f \pm g, fg, \frac{f}{g}$ for $g \not\equiv 0$; the meromorphic $\Lambda$-periodic functions form a field (aside: this is another reason for using meromorphic rather than holomorphic functions, since the field structure is far nicer than the ring structure we would have for holomorphic functions).

Theorem: 1) If $f$ is even and elliptic then $f(z) = R(\wp(z))$ for some rational function $R$. We will not proove this part, but it is not difficult, and involves no new ideas; we write $f(z) = c\frac{(\wp(z)-\wp(a_1))\dots(\wp(z)-\wp(a_k))}{(\wp(z)-\wp(b_1))\dots(\wp(z)-\wp(b_l))}$ which has the correct poles, and then argue that it is in fact the required function. 2) For $f$ elliptic, $f(z) = R_1(\wp(z)) + \wp'(z)R_2(\wp(z))$: write f as a sum of even and odd parts ($\frac{1}{2}(f(z) + f(-z))$ etc.), call the even part $R_1(\wp(z))$ by 1), then $\frac{1}{\wp'(z)}\times$ the odd part is even, so $= R_2(\wp(z))$ and we are done.

# Riemann Surfaces and holomorphic maps

Recall the definition of a topological space; since we have a notion of open sets it makes sense to speak of continuous maps and homeomorphisms.

Definition: a topological surface $X$ is a Hausdorff topological space so that $\forall p \in X \exists U_p \ni p$ open such that $U_p \subset X$ is homeomorphic to an open disc in $\mathbb{C}$.

Remarks: 1) dimension is actually a topological property (though we shall only see this in advanced topology) so homeomorphic spaces have the same dimension; thus this definition is reasonable. 2) We need not have $X \subset \mathbb{R}^3$.

The homeomorphisms $\phi_p : U_p \to \phi(U_p) \subset \mathbb{C}$ are called coordinate charts; $U_p$ is called a coordinate neighbourhood in $X$. $z = \phi_p(q)$ is a local complex coordinate. For $U_p \cap U_{\tilde{p}} \neq \emptyset$, $\phi_{\tilde{p}} \circ \phi_p^{-1} : \phi_p(U_p \cap U_{\tilde{p}}) \to \mathbb{C}$ is called a transition function; we write $\tau_{ij}$ for $\phi_j \circ \phi_i^{-1}$.

Definition: A complex structure on a topological surface $X$ is a collection of charts $(\phi_\alpha, U_\alpha)$ such that $\bigcup_\alpha U_\alpha = X$ and all the $\tau_{\alpha\beta}$ are holomorphic where defined; since $\tau_{\alpha\beta}^{-1} = \tau_{\beta\alpha}$ this means they are biholomorphic.

A Riemann surface is a topological surface with a complex structure. (Some books call this an abstract Riemann surface, to which the astute reader may reply "So what about concrete Riemann surfaces, then?" The concrete Riemann surfaces are algebraic curves; see later in the course).

Examples: any open $U \subset \mathbb{C}$, with chart the identity on $U$; more generally, for any $X$ a Riemann surface, any open $U \subset X$ is also a RS by the restriction of the charts of $X$ to the intersections of their domains with $U$. $S^2$ with the stereographic projections $\phi, \psi$ is a Riemann surface; so is any elliptic curve, formally $E = \frac{\mathbb{C}}{\Lambda}$, with points $z + \Lambda$, but we have not shown this yet:

Proof: define the quotient map $\pi : z \in \mathbb{C} \mapsto z + \Lambda \in E$; $\pi(z) = \pi(z') \Leftrightarrow z - z' \in \Lambda$. Define $W \subset E$ is open iff $\pi^{-1}(W) \subset C$ is, a quotient topology on $E$. $E$ is Hausdorff, since for any two points we can put them in the fundamental open parallelogram of $\Lambda$ in $\mathbb{C}$ and use the Hausdorffness of $\mathbb{C}$.

Let $D_i \subset \mathbb{C}$ be a family of open discs in $\mathbb{C}$ with diameter $< \max\{\frac{1}{2}, \Im \frac{\tau}{2}\}$. Then if $z, z' \in D_i$ then $z = z'$, and $\pi(D_1) \cap \pi(D_j)$ is connected and open in $E$. Define $U_i = \pi(D_i)$, then $(\pi \mid_{D_i}) : D_i \subset \mathbb{C} \to U_i \subset E$ is a homeomorphis, and $\phi_i$ defined by $(\pi \mid_{D_i})^{-1}$ is a chart. Then $\tau_{ij} = \phi_j \circ \phi_i^{-1}$ has $\tau_{ij}(z) - z \in \Lambda$, so $\tau_{ij}(z) - z$ is continuous on a connected set $\phi_i(U_i \cap U_j)$, taking values in a discrete set, so it is constant; call this constant $\lambda_{ij} \in \Lambda$. Then $\tau_{ij}(z) = z + \lambda_{ij}$, which is certainly holomorphic, so we have a complex structure and the elliptic curve is a RS.

Without detail, a similar example is $\frac{\mathbb{C}}{\mathbb{Z}}$, a Riemann Surface which is a topological cyclider.

Definition: For $R, S$ two Riemann surfaces, let $\phi_i : U_i \subset R \to \mathbb{C}, \psi_\alpha : W_\alpha \subset S \to \mathbb{C}$ be their respecive charts (complex structure), then a continuous map $f : R \to S$ is holomorphic if $\psi_\alpha \circ f \circ \phi_i^{-1}$ is holomorphic (as a function $\mathbb{C} \to \mathbb{C}$) where defined $\forall i, \alpha$.

Examples: of course any holomorphic function from an open $U \subset \mathbb{C} \to \mathbb{C}$; we have proven already that the non-constant meromorphic functions on an open $U \subset \mathbb{C}$ are precisely the non-constant holomorphic maps $U \to S^2$.

Proposition: Let $R$ be an RS with charts $\psi_j : U_j \subset R \to \mathbb{C}$, (wlog) all the $U_j$ connected. Take $\phi : S^2 \setminus \{N\} \to \mathbb{C}$ stereographic projection as before. Then a continuous $f : R \to S^2$ is holomorphic iff $\forall \phi \circ f \circ \psi_j^{-1}$ such that $f \circ \psi_j^{-1}$ is non-constant, $\phi \circ f \circ \psi_j^{-1}$ is a meromorphic function with poles precisely at $\psi_j(U_j \cap f^{-1}(N))$. The proof is the same as that of the last example above, just with domain $R$ rather than (some subset of) $S^2$ and $\psi_j$ instead of $\phi$ for the chart applied [inverted] before $f$.

More examples: rational functions on $\mathbb{C}$ correspond precisely to holomorphic maps $S^2 \to S^2$, as proven earlier. Elliptic functions descend to [or, less formally, are] holomorphic maps $\frac{\mathbb{C}}{\Lambda} \to S^2$, and the quotient map $\mathbb{C} \to \frac{\mathbb{C}}{\Lambda}$ is holomorphic: for any chart $\phi_\alpha$, $z \in D \mapsto z + \Lambda$ by $\pi$, then $\mapsto z + \lambda_\alpha$ by $\phi_\alpha$ for some $\lambda_\alpha \in \Lambda$, so $\phi_\alpha \circ \pi$ is certainly holomorphic.

Singly-periodic functions similarly descend to holomorphic maps on $\frac{\mathbb{C}}{\mathbb{Z}}$ and the quotient map is holomorphic; consider these exercises.

Definition: $f : R \to S$ is biholomorphic or a conformal equivalence if it is holomorphic with a well defined holomorphic inverse $f^{-1} : S \to R$. We say $R, S$ are biholomorphic or conformally equivalent if a surjective biholomorphic such $f$ exists. This is the most rigid way of defining when surfaces are equivalent; in geometry we consider the somewhat weaker notion of being diffeomorphic, and in topology the far weaker idea of being homeomorphic; see questions 7 and 8 on the second example sheet for this course.

# A closer look at holomorphic maps

For $R, S$ connected Riemann surfaces, $f : R \to S$ a holomorphic map, $p \in R, q = f(p) \in S$, $\exists$ charts $\phi, \psi$ near $p, q$ respectively and in fact centred at them (i.e. $\phi(p) = 0 = \psi(q)$). Then $\hat{f}(z) = \psi \circ f \circ \phi^{-1}(z)$ is holomorphic near $z = 0$ (note that $\hat{f}$ is not uniquely determined; for different choices of chart $\phi', \psi'$ we obtain $\hat{f}' = v \circ \hat{f} \circ u^{-1}$ where $u = \phi \circ \phi'^{-1}, v = \psi \circ \psi'^{-1}$ are local conformal equivalences near 0.

## Isolated Values Principle

If $f \not\equiv$ a constant and $R$ connected, then $f^{-1}(q)$ will be a discrete set in $R$ $\forall q \in S$: let $X = \{x \in R : \exists \text{ open } U \subset R, x \in U \text{ such that } f\mid_U \equiv \text{ a constant}\}$, $Y = \{y \in R : \exists \text{ open } V \subset R, y \in V \text{ such that } f(y) \neq f(y') \forall y' \in V \setminus \{y\}\}$. We have $X \cap Y = \emptyset, X$ open in $R$. For $p, q, \hat{f}$ as before, $\hat{f}(0) = 0$, so either $\hat{f} \equiv 0$ so $p \in X$, or $z = 0$ is an isolated zero of $\hat{f}$; this means $p \in Y$; moreover, $\forall z_0$ near 0, $\hat{f}(z) - \hat{f}(z_0)$ has $z_0$ an isolated zero, so $Y$ is open, so $Y$ open; since $f$ is non-constant $Y$ is non-empty, so since $R$ is connected $R = Y$ and we have the result.

Note that Riemann Surfaces are connected iff they are path-connected, since this is true of open subsets of $\mathbb{C}$.

We define: if $\hat{f}(z) \not\equiv 0$ then let the order of the zero at $\hat{f}(0)$ be $n$, then $v_f(p) := n$, the branching order of $f$ at $p$. By the local mapping theorem this is well defined, since $\hat{f}$, and hence also $f$, are locally $n : 1$ near 0 (or respectively $p$), so for any charts $\tilde{\phi}, \tilde{\psi}$ near $p, q$, $v_f(p) = v_{\tilde{\psi}\circ f\circ\tilde{\phi}^{-1}}(z)$ where $z = \phi(p)$. If $v_f(p) > 1$ we say $p$ is a ramification point and $q$ is the branch point.

If $f$ is non-constant and $R$ is connected then $\{p \in R : v_f(p) > 1\}$ is discrete, since $v_f(p) > 1 \Leftrightarrow v_{\hat{f}}(0) > 1 \Leftrightarrow \hat{f}(z) = z^2 g(z)$ for some holomorphic $g \Leftrightarrow \hat{f}'(0) = 0$, but this must be an isolated zero of $\hat{f}'$ as otherwise $\hat{f}' \equiv 0$ and $f$ is constant, by the isolated value principle.

We have local structure of holomorphic maps: $v_f(p) = n \Rightarrow \exists$ open $U \ni p, V \ni q = f(p)$ such that $\forall q' \in V \setminus \{q\}, f^{-1}(q') \cap U$ consists of $n$ distinct points, and $v_f = 1$ at each of these; we have proven this for $\hat{f}(z)$ so it holds for $f$ since the charts are bijections. As an "exercise", the readers should see that we have already proven we can choose charts $\phi, \psi$ such that $\hat{f}(z) = z^n$.

The open mapping theorem also holds: if $f$ is non-constant holomorphic and $R$ connected then $f(U)$ is open $\forall$ open $U \subset R$; this is true for a "small" open set (i.e. one $\subset U \cap f^{-1}(V)$ for $U, V$ coordinate domains) as it is true for $\hat{f}$, and any open subset of $R$ is a union of such small open sets.

The inverse mapping theorem: $v_f(p) = 1 \Rightarrow \exists$ open $U \ni p, V \ni f(p)$ and holomorphic $g : V \to U$ such that $g \circ f \mid_U = \mathrm{Id}_U, f \circ g \mid_V = \mathrm{Id}_V$. Corollary: if $f : R \to S$ is a holomorphic bijection then $f^{-1} : S \to R$ is holomorphic.

Theorem and definition: degree: for $f : R \to S$ non-constant holomorphic, $R, S$ compact and connected, $k(q) := \sum_{p \in f^{-1}(q)} v_f(p)$ is finite and independent of $q \in S$; $\deg f := k$: $f^{-1}(q)$ is discrete and $R$ compact, so $f^{-1}(q)$ is finite and $k(q)$ is finite (note that compact of a RS is equivalent to saying the BWT holds). Let $q \in S$, then $\forall p \in f^{-1}(q) \exists$ open $N_p \ni p, V_p \ni q$ such that $\forall q' \in V_p \setminus \{q\}$, $f^{-1}(q') \cap N_p$ consists of $v_f(p)$ points (surjectivity of $f$ wil be a consequence of our argument[, and so was not necessary as a hypothesis]). Define $V = \bigcap_{p \in f^{-1}(q)} V_p, U_p := f^{-1}(V) \cap N_p$; we can wlog take the $U_p$ all distinct by reducing $V$ if necessary. We now claim we can further reduce $V$ to $\tilde{V} \subset V$ with $q \in \tilde{V}$ such that $f^{-1}(\tilde{V}) = \bigcup_{p \in f^{-1}(q)} \tilde{U}_p$ where $\tilde{U}_p := U_p \cap f^{-1}(\tilde{V})$; if not, we must have a sequence $q_n \to q$ such that $\forall q_n \exists p_n : f(p_n) = q_n, p_n \notin U_p \forall p \in f^{-1}(q)$. So since $R$ is compact we have a subsequence of the $p_n$ $p_{n_k} \to$ some $\check{p}$ (as $k \to \infty$), but then $f(\check{p}) = f(\lim_{k\to\infty} p_{n_k} = \lim_{k\to\infty} f(p_{n_k}) = \lim q_{n_k} = q$, but $p_n \notin p_{\check{p}} \forall$ large $n$, a contradiction. If $q' \in \tilde{V} \setminus \{q\}$ then $\# f^{-1}(q') = \sum_{p \in f^{-1}(q)} v_f(p)$. $\forall x \in f^{-1}(q'), v_f(x) = 1$ so $k(q') = k(q) \forall q'$ close to $q$. So $y \in S \mapsto k(y) \in S$ is a continuous function on the connected $S$ taking values in a discrete set so a constant map. Note that although the assumption of compactness of $S$ may seem unnecessary, it is actually needed; see question 9 on the second example sheet for this course.

Corollary: under the hypotheses of the above theorem, $f$ is surjective.

Note that degree as defined here recovers the degree of a rational function for $f : S^2 \to S^2$, and the degree of an elliptic function for $f : \frac{\mathbb{C}}{\Lambda} \to S^2$.

# Euler Characteristics

We shall quote several results which are proven in the algebraic topology course: for $S$ a compact connected Riemann surface, we shall ignore its complex structure and treat it as a topological surface; $S$ is orientable: for the transition functions $\tau_{ij}(z) = \phi_j \circ \phi_i^{-1}(z)$, wlog take $\tau_{ij}(0) = 0$, then $\tau_{ij}(z) = Az + O(z^2)$; write $z \cong \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ and consider $A : \mathbb{R}^2 \to \mathbb{R}^2$ as a matrix; it will be $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$ ,so $\det A = a^2 + b^2 > 0 \forall A \in \mathbb{C}[\setminus\{0\}]$ so transition functions preserve orientation and $S$ is orientable. From topology we have that all compact oriented connected surfaces are classified by the Euler characteristic $\chi$, i.e. two such surfaces have the same $\chi$ iff they are homeomorphic.

$\chi$ can be defined by triangulation: Let <u>edges</u> mean images of homeomorphisms $[0, 1] \to S$ where $S$ is our topological surface, <u>verticies</u> be the images of 0 and 1 under these. We require that $S\setminus$ the set of edges has finitely many connected components, each homeomorphic to an open disc; then <u>faces</u> are the closures of these components. We require that any two faces share at most one edge, any two edges meet at one vertex or not at all, and for any vertex there is a neighbourhood homeomorphic to a disc [centred at 0] with the vertex corresponding to 0, radii corresponding to edges and sectors corresponding to different faces (note this implies at least 3 edges meet at each vertex). Then sets of edges, verticies and faces satisfying all these form a triangulation $T$ (this is not the best definition of a triangulation, but works for our purposes).

Let $E(T,S)$ be the number of edges, similarly $V(T,S), F(T,S)$ for verticies and faces; now from topology we have $V(T,S)-E(T,S)+F(T,S) = \chi(S)$ independently of the specific triangulation $T$. Now we have for the genus $g(S)$ that $2-2g(S) = \chi(S)$; informally, $g$ is the number of "handles" we must attach to $S^2$ to obtain $S$. By considering the projection of a tetrahedron we can find $\chi(S^2) = 2, g(S^2) = 0$; we can find that $\chi$ for a torus is 0 and $g = 1$, and so on.

A fact, which is a theorem in topology, is that every compact connected $S$ has a triangulation; this comes down to the fact that $S$ is "second countable" i.e. has a countable basis.

Theorem (Riemann-Hurwitz formula): for $R, S$ compact connected Riemann surfaces, $f : R \to S$ non-constant holomorphic and $k = \deg f$, $\chi(R) = k\chi(S) - \sum_{p\in R}(v_f(p)-1)$, or equivalently $g(R)-1 = k(g(S)-1) + \frac{1}{2}\sum_{p\in R} v_f(p) - 1$. This is possibly the most important theorem in the cours; it combines a holomorphic quantity (using power series) $v_f$, an algebraic quantity $k$, and a topological quantity $\chi$ or $g$. We assume the above fact, so take $T_S$ a triangulation of $S$. Suppose first $v_f(p) = 1 \forall p \in R$, then $\forall q \in S \exists$ open $V_q \ni q$ such that $f^{-1}(V_q) = \bigsqcup$ [i.e. a disjoint union of] sets, each of which is mapped by $f$ biholomorphicly ont $V_q$, by local structure of holomorphic maps.

We have $S = \bigcup_{q\in S} V_q$; by compactness take a finite subcover $S = \bigcup_{i=1}^n V_i$. Refine $T_S$ such that each face is a subset of some $V_i$, by baricentric subdifision: take a point inside a face, and join it to each vertex of the face and the midpoint of each side of the face. So we have a triangulation $T_k$ by the above $f^{-1}(V_i)$ with $k = \deg f$ copies of everything in $T_S$, so $\chi(R) = k\chi(S)$.

If there is a unique $p \in R$ with $v_f(p) > 1$, say $v_f(p) = r$, let $q = f(p)$ and make $q$ a vertex in $T_S$ (by baricentric subdivision if necessary). We have $r \leq k$ by the definition of $k$. Locally near $p$, $\hat{f}(z) = z^r$ for suitable charts, so we again have $k$ copies of everything except that $f^{-1}(q) = k-(r-1)$ points so $\chi(R) = k\chi(S)-(r-1)$ as there are $r-1$ fewer verticies in our triangulation of $R$. Now since there are only finitely many $p$ with $v_f(p) > 1$, considering them one at a time we have the general result.

Notice that the $\sum_{p\in R}(v_f(p)-1)$ looked like an infinite sum, but actually only has finitely many nonzero terms. We have now proven the first form of the result; we then have $2-2g(R) = k(2-2g(S)) - \sum_{p\in R}(v_f(p)-1)$ and the second form follows. Since $\chi = 2-2g$ we have $\chi$ is always even for a Riemann surface (this is not the case for a general topological surface, but rather follows from orientability).

Example: consider $\wp : \frac{\mathbb{C}}{\Lambda} \to S^2$; $\deg \wp = 2, \chi(\frac{\mathbb{C}}{\Lambda}) = 0, \chi(S^2) = 2$. Recall that the points where $v_\wp = 2$ are $\frac{1}{2}+\Lambda, \frac{\tau}{2}+\Lambda, \frac{1+\tau}{2}+\Lambda$, and also the pole of order 2 at $0+\Lambda$, so the e formula says $0 = 2\times 2 - 4\times 1$ which is indeed the case.

$g(\frac{\mathbb{C}}{\Lambda}) = 1, g(S^2) = 0$ so if we have $f : S^2 \to \gamma$ for some topological surface $\gamma$ then in the second form of the theorem the LHS is $\leq 0$, so the RHS must be also, implying we must have $g(\gamma) = 0$ and $\gamma$ is a topological sphere.

## Algebraic Curves

Definition: a complex algebraic curve in $\mathbb{C}^2$ (sometimes called an affine [algebraic] curve) is $C = \{(s,t) \in \mathbb{C}^2 : P(s,t) = 0\}$ for some complex polynomial of two variables $P$. $C$ is non-singular if $\forall(s_0, t_0 \in C$ with $P(s_0,t_0) \in C$, $(\frac{\partial P}{\partial s}(s_0,t_0), \frac{\partial P}{\partial t}(s_0,t_0)) \neq (0,0)$. An example is a complex line $\{\lambda s + \mu t + \nu = 0\}$ for $(\lambda,\mu) \neq (0,0)$; this is biholomorphic to $\mathbb{C}$.

Theorem: A non-singular algebraic curve $C = \{P(s,t) = 0\}$ has a natural complex structure (so is a Riemann Surface): for example, $P(s,t) = t - q(s)$ for some polynomial $q$ is a graph of $q$, and the projection map $(s,t) \in C \mapsto s \in \mathbb{C}$ is a bijection and $C$ can be identified with $\mathbb{C}$.

To proove the above theorem we need the following:

Implicit Function Theorem: Suppose $P(s,t)$ is a polynomial, $P(s_0,t_0) = 0$, $\frac{\partial P}{\partial t}(s_0,t_0) \neq 0$, then $\exists! h : D(s_0,\epsilon) \to D(t_0,\delta)$ for some $\epsilon, \delta > 0$ such that $h(s_0) = t_0$ and $t = h(s)$ iff $P(s,t) = 0$, for $(s,t) \in D(s_0,\epsilon) \times D(t_0,\delta)$; also $h$ is holomorphic. The proof of this is nonexaminable; as a sketch, let $f(t) = P(s_0,t)$, then $f(t_0) = 0, f'(t_0) \neq 0$ so $t_0$ is a zero of order 1. So we have $\delta > 0$ such that $f(t) \neq 0$ on $D^\star(t_0, 2\delta)$; let $\gamma(u) = t_0 + \delta e^{iu}, \delta \in [0,2\pi]$; define $\Gamma_s(u) = P(s,\gamma(u))$. By the argument principle, $1 = \frac{1}{2\pi i}\oint_\gamma \frac{f'(t)}{f(t)}dt = n(\Gamma_{s_0}, 0)$, a winding number. Now we claim ($\star$) $n(\Gamma_s, 0)$ is holomorphic (as a function of $s$) on $D(s_0,\epsilon)$ for some $\epsilon > 0$; given this, we then have $n(\Gamma_s,0) \equiv 1 \forall |s - s_0| < \epsilon$, i.e. $h$ is unique as a map $D(s_0,\epsilon) \to D(t_0,\delta)$. Now to show it is holomorphic, we have that $h(s) = \frac{1}{2\pi i}\int_\gamma \frac{t\frac{\partial}{\partial t}P(s,t)}{P(s,t)}dt$; compare this with the formula in question 3 on the first example sheet for this course, which is this formula for the special case $P(s,t) = t - q(s)$; now we claim ($\star\star$) that this is holomorphic.

Both our claims were of the form that if $Q(s,t)$ is holomorphic then $\int_\gamma Q(s,t)dt$ is holomorphic; for ($\star$) this was the case $Q = \frac{f'}{f} = \frac{\frac{\partial}{\partial t}P}{P}$, for ($\star\star$) this was the case $Q = \frac{t\frac{\partial}{\partial t}P}{P}$. We shall proove the result in general: write $Q(s,t) = Q(s_0,t) + (s-s_0)Q_1(s_0,t) + (s-s_0)^2R(s,t)$, so $\left|\frac{\int_\gamma Q(s,t)dt - \int_\gamma Q(s_0,t))dt}{s-s_0} - \int_\gamma \frac{\partial Q}{\partial s}(s_0,t)dt\right|$, using $Q_1 = \frac{\partial}{\partial s}Q(s_0,t)$, is $|\int_\gamma (s-s_0)R(s,t)dt| = (s-s_0)\int_\gamma R(s,t)dt = M|s-s_0| \to 0$ as $s \to s_0$, as required.

In fact we have proven the result for any $P(s,t)$ which is holomorphic (i.e. holomorphic in both variables), not just polynomials. Also, note that the inverse function theorem as seen in part IB is just the special case of this where $P(s,t) = t - q(s)$.

Now, the proof of the main theorem, above: let $(s_0,t_0) \in C$; wlog take $\frac{\partial P}{\partial t}(s_0,t_0) \neq 0$ (recall that one of $\frac{\partial}{\partial t}, \frac{\partial}{\partial s}$ must be $\neq 0$). Define $\pi_1 : (s,t) \in C \mapsto s \in \mathbb{C}, \pi_2 : (s,t) \in C \mapsto t \in \mathbb{C}$, projection maps. By the implicit function theorem there is a unique holomorphic $h(s)$ such that $C \cap (D(s_0,\epsilon) \times D(t_0,\delta)) = U$ is the graph of $h$, $= \{(s,h(s)) : |s - s_0| < \epsilon\}$. So $\pi_1 |_U$: $U \to \mathbb{C}$ is a bijection onto its image, and a homeomorphism (where we have the obvious topology on $\mathbb{C}$). Since $C \subset \mathbb{C}^2$, a metric space, it has a metric so is Hausdorff; thus $C$ is a topological surface.

The transition functions between charts ar just the identity when both charts are restrictions of the same $\pi_i$. Otherwise say our charts are $\phi : (s,h(s)) \mapsto s, \hat{\phi}$ :

$(s, h(s)) \mapsto h(s)$, then the transition function is just $h(s)$ which is holomorphic by the IFT; similarly its inverse is also holomorphic. So we have a complex structure and $C$ is a Riemann surface.

Example: $\{(s,t) : s^n + t^n = a^n\}$ for some parameter $a \neq 0 \in \mathbb{C}$; this is sometimes called a Fermat curve.

Algebraic curves in $\mathbb{C}^2$ are never compact; e.g. $\{\lambda s + \mu t = \nu\}$ for $(\lambda, \mu) \neq (0, )$ is a complex line $\cong \mathbb{C}$ so not compact; $\{s^2 + t^2 = 1\}$ is unbounded, as we can take $s = ix, t = y$, then the equation is satisfied whenever $y^2 - x^2 = 1$ and this is unbounded in $\mathbb{R}^2$; we can use the fundamental theorem of algebra to show these surfaces are not compact in general, as is done in the first question on the third example sheet.

We can sometimes "compactify" an (affine) algebraic curve $C$, i.e. find a compact Riemann surface $\bar{C}$ such that $C$ is an open subset of $\bar{C}$ and $\bar{C} \setminus C$ is a finite set of points:

## Projective Curves

Definition: complex projective space $\mathbb{P}^n$ or $\mathbb{C}P^n$ is the set of all 1-dimensional complex subspaces of $\mathbb{C}^{n+1}$ (i.e. lines through 0); we write $z_0 : z_1 : \cdots : z_n \in \mathbb{P}^n$ for the span of a cpmlx vector $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. These give homogenous complex coordinates on $\mathbb{P}^n$, i.e. $\lambda z_0 : \lambda z_1 : \ldots | \lambda z_n = z_0 : \cdots : z_n \forall \lambda \neq 0 \in \mathbb{C}$.

Define $\Pi : (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\} \mapsto z_0 : \cdots : z_n \in \mathbb{P}^n$; then we define a topology on $\mathbb{P}^n$ by the quotient topology under this map. Then $\Pi$ is continuous, and $\mathbb{P}^n$ is compact, since it is the continuous image under $\Pi$ of the compact set $\{\vec{z} \in \mathbb{C}^{n+1} : |\vec{z}| = 1\}$.

$\mathbb{P}^n$ is Hausdorff: for $\vec{z}, \vec{w} \in \mathbb{C}^n \setminus \{0\}$ linearly independent (i.e. representing distinct points of $\mathbb{P}^n$, wlog take $\vec{z} = (1, 0, \ldots, 0), \vec{w} = (1, 1, 0, \ldots, 0)$ by complex linear isomorphisms from $GL(n+1, \mathbb{C})$; these are homeomorphisms of $\mathbb{C}$ preserving sets $\Pi^{-1}(\ldots)$. (Consider $A^n = \{z_0 = 1\} \subset \mathbb{C}^{n+1}$, affine space. Then $(\mathbb{C}\vec{z}) \cap A^n = \{\vec{z}\}, (\mathbb{C}\vec{w}) \cap A^n = \{\vec{w}\}$ [it seems like the lecturer was going to use this in this proof, then didn't - but the reader should still know the definition). Put $U_z = \{1 : z_2 : \cdots : z_n \mid |z_j| < \frac{1}{3} \forall 1 \leq j \leq n\}, U_w = \{1 | 1 + w_1 : w_2 : \cdots : w_n \mid |w_j| < \frac{1}{3} \forall 1 \leq j \leq n\}$; these are both subsets of $\mathbb{P}^n$. Then $\Pi^{-1}(U_z), \Pi^{-1}(U_w)$ are open in $\mathbb{C}^{n+1} \setminus \{0\}$ (just by writing what they are) and these are disjoint sets, so we have the result.

Note that any $f : \mathbb{P}^n \to X$ is continuous iff $f \circ \Pi : \mathbb{C}^{n+1} \setminus \{0\} \to X$ is.

Consider $(z_1, \ldots, z_n) \in \mathbb{C}^n \mapsto (1, z_1, \ldots, z_n) \in A^n \mapsto 1 : z_1 : \cdots : z_n \in \mathbb{P}^n$; this is a homeomorphism (as we can check by a straightforward argument along the lines of the previous result) onto its image, $\Pi(A^n)$, which is open in $\mathbb{P}^n$ (as $\Pi^{-1}(\Pi(A_n)) = \{z_0 \neq 0\} \subset \mathbb{C}^{n+1} \setminus 0$ (and therefore also an open subset of $\subset \mathbb{C}^{n+1}$, should this ever be written instead by mistake as $\mathbb{C}^{n+1} \setminus 0$ is open. So $\mathbb{P}^n$ has an open subset homeomorphic to $\mathbb{C}^n$; identifying this with $\mathbb{C}^n$ we have a decomposition of $\mathbb{P}^n$ into $\mathbb{C}^n$ and $\mathbb{P}^{n-1}$; informally we form $\mathbb{P}^n$ by adding a copy of $\mathbb{P}^{n-1}$ at infinity to $\mathbb{C}^n$.

Herafter we take $n = 2$; by the above we have an open cover $\mathbb{P}^2 = A_0^2 + A_1^2 + A_2^2$ where $A_i^2 = \{z_i \neq 0\} \subset \mathbb{P}^2$; $\mathbb{P}^2 \setminus A_0^2 \cong \mathbb{P}^1$, so informally $\mathbb{P}^2$ is a compactification of $\mathbb{C}^2$ by adding the projective line at infinity. We shall use the notation $X : Y : Z$ for $z_0 : z_1 : z_2$.

Let $P(X, Y, Z)$ be a homogenous polynomial of degree $d$; that is, $P(\lambda X, \lambda Y, \lambda Z) = \lambda^d P(X, Y, Z) \forall \lambda \in \mathbb{C} \setminus \{0\}$. Then a (complex) projective curve in $\mathbb{P}^2$ is $C = \{X : Y : Z \mid P(X, Y, Z)) = 0\}$; for some such $P$; this is a well defined subset of $\mathbb{P}^2$. Note $C$ is closed: $P$ is continuous on $\mathbb{C}^3$ so $\mathbb{C}^3 \setminus P^{-1}(0)$ is open, so $\Pi^{-1}(\mathbb{P}^2 \setminus C)$ is open so $P^2 \setminus C$ open so $C$ closed; since this makes $C$ a closed subset of a compact space it is compact.

We define $a : b : c \in C$ is a singular point of a projective curve $C$ if $\frac{\partial P}{\partial X}(a, b, c) = \frac{\partial P}{\partial Y}(a, b, c) = \frac{\partial P}{\partial Z}(a, b, c) = 0$. We say $C$ is non-singular if it has no singular points; some examples of non-singular projective curves are a projective line $\{\alpha X + \beta Y + \gamma Z = 0\}$ (for not $\alpha = \beta = \gamma = 0$), (projective) quadric curve $\{X^2 + Y^2 + Z^2 = 0\}$, (projective) Fermat curve $\{X^n + Y^n + Z^n = 0\}$.

Theorem: every non-singular projective curve has the natural structure of a (compact) Riemann surface; will proove below.

Lemma 1 (a special case of Euler's formula): if $P(X, Y, Z)$ is a homogenous polynomial of degree $m$ then $X\frac{\partial P}{\partial X} + Y\frac{\partial P}{\partial Y} + Z\frac{\partial P}{\partial Z} = mP \forall X, Y, Z$: this is easy for polynomials, wlog we can take $P = X^i Y^j Z^k$ as $P$ will be a linear combination of such terms and both sides are linear, but then just apply the Leibnitz formula.

Lemma 2: For $a : b : c \in C$, $a \neq 0$, $a : b : c$ is a non-singular point in $C$ iff $(\frac{b}{a}, \frac{c}{a}) \in \mathbb{C}^2$ is a non-singular point in $C_0 = \{P(1, s, t) = 0\}$, since $(\frac{b}{a}, \frac{c}{a})$ is a singular point in $C - 0$ iff $P(1, \frac{b}{a}, \frac{c}{a}) = 0 = \frac{\partial P}{\partial Y}(1, \frac{b}{a}, \frac{c}{a}) = \frac{\partial P}{\partial Z}(1, \frac{b}{a}, \frac{c}{a}) \Leftrightarrow P(a, b, c) = 0 = \frac{\partial P}{\partial Y}(a, b, c) = \frac{\partial P}{\partial Z}(a, b, c) = 0$ but by Lemma 1 this is the case iff $\frac{\partial P}{\partial X}(a, b, c) = 0$ as well, as $a \neq 0$, i.e. this is the case iff $a : b : c$ is a singular point in $C$.

Now, the proof of the main theorem: let $C = \{P(X, Y, Z) = 0$ be a projective curve in $\mathbb{P}^2$. Then $C = C_0 \cup C_1 \cup C_2$ where $C_i = C \cap A_i^2$; then $C_i$ is an algebraic curve in $\mathbb{C}^2$, e.g. $C_0 = \{(s, t) \in \mathbb{C}^2 : P(1, s, t) = 0\}$; these $C_i$ are called the affine pieces of $C$. By Lemma 2 these are non-singular so have complex structures; putting all these together, we have an open cover of $C$ by coordinate domains. The transition functions between charts of the same $C_i$ are holomorphic; if we have a transition function between charts of different $C_i$, say we have charts $\phi_0, \phi_1$ on $C_0, C_1$ respectively near $a : b : c \in C_0 \cap C_1$, then $a \neq 0 \neq b$. $C_0 = \{P(1, s, t) = 0\}$; say $\phi_0(s, t) \in U_0 \subset C_0 \subset C \mapsto s$, then $\phi_0^{-1}(s) = (s, h(s))$ for some holomorphic $h$. Now $s = \frac{Y}{X}, t = \frac{Z}{X}$, so $\phi_0(X : Y : Z) = \frac{Y}{X}$; similarly $\phi_1(X : Y : Z) = \frac{X}{Y}$ or $\frac{Z}{Y}$. Then we find $\phi_1 \circ \phi_0^{-1}(s) = \phi_1(1 : s : h(s)) = \frac{1}{s}$ or $\frac{h(s)}{s}$, and either of these is holomorphic where defined, so transition functions between charts on $C_0, C_1$, and similarly all transition functions, are holomorphic and we have the result.

An algebraic curve $C \subset \mathbb{C}^2$ has projections $\pi_1, \pi_2 : C \to \mathbb{C}$ which are well defined and holomorphic (whether or not they are charts); by contrast the maps e.g. $X : Y : Z \mapsto \frac{Y}{Z}$ are only defined on open subsets of our projective curve $C$ ($\{Z \neq 0\} \cap C$ in this case).

If we have an algebraic curve in $\mathbb{C}^2$ given by $\{q(s, t) = 0\}$ with $\deg q = k$, to extend it to a projective curve in $\mathbb{P}^2$ we define $C = \{X : Y : Z \mid X^k q(\frac{Y}{X}, \frac{Z}{X} = 0\}$; we have $C_0 \cong \{q(s, t) = 0\}$, and $C \setminus C_0$ is a finite set of points, induced by the roots of $q_k(1, t) = 0, q_k(s, 1) = 0$ where $q_k$ is the terms of degree $k$ in $q(s, t)$. Note we have no guarantee that the curve is nonsingular at points of $C \setminus C_0$.

Without proof, it is a fact that any non-singular projective curve in $\mathbb{P}^2$ is connected.

$\mathbb{P}^1 = \{X : Y : 0\} \subset \mathbb{P}^2$ is a Riemann Surface, as we can write it as a non-singular projective curve by $P(X, Y, Z) = Z$.

Proposition: $\mathbb{P}^1$ is biholomorphic to $S^2$ (which recall we informally identified with $\mathbb{C} \cup \{\infty\}$; we can identify this with $\mathbb{P}^1$ by $z : 1 \leftrightarrow z, 1 : 0 \leftrightarrow \infty$): put $\Psi : s : t \in \mathbb{P}^1 \mapsto (\frac{2\Re(s\bar{t})}{|s|^2+|t|^2}, \frac{2\Im(s\bar{t})}{|s|^2+|t|^2}, \frac{|s|^2-|t|^2}{|s|^2+|t|^2}) \in S^2 \subset \mathbb{R}^3$ (we come up with this map by inverting stereographic projection). This is continuous, since the RHS is continuous as a map $\mathbb{C}^2 \setminus \{0\} \to \mathbb{R}^3$, and of the form $\psi \circ \Pi$.

We claim $\Psi$ is holomorphic; we use the usual stereographic projection charts $\phi, \psi$ on $S^2$, and charts on $\mathbb{P}^1$ by $s : t \mapsto \frac{s}{t}$ (valid for $t \neq 0$) and $s : t \mapsto \frac{t}{s}$ (valid for $s \neq 0$); their respective inverses are $s \mapsto s : 1, t \mapsto 1 : t$. Then we can calculate that coordinate expressions of $\Psi$ will be $s \mapsto s : 1 \mapsto \ldots \mapsto s$ or $\frac{1}{s}$ (depending whether we use $\phi, \psi$ respectively), similarly $t \mapsto \frac{1}{t}$ or $t$ (again corresponding to $\phi, \psi$); these are all holomorphic where defined, so $\Psi$ is holomorphic. Now we just need to proove $\Psi$ is bijective, then the inverse mapping theorem gives the result.

$\Psi^{-1}(u, v, w) = (u + iv) : (1 - w)$ for $w \neq 1$ and $(1 + w) : (u - iv)$ for $w \neq -1$; note that these agree for $w \neq \pm 1$, as then $(u+iv) : (1-w) = (u+iv)(u-iv) : (1-w)(u-iv)$ (note $u - iv \neq 0$), which is $(u^2 + v^2) : (1 - w)(u - iv) = (1 - w^2) : (1 - w)(u - iv) = (1 + w) : (u - iv)$.

## Branched Covers

Consider $F : R \to S$ a non-constant holomorphic surjection onto a connected $S$. Let $B = \{x \in S : x = F(y) \text{ with } v_F(y) > 1\}$, the <u>branch locus of $F$</u>. We say $F$ is a <u>cover of $S$ branched over $B$</u>. If $R, S$ are compact, then $\forall x \in S \setminus B \exists$ an open neighbourhood $U \ni x$ such that $F^{-1}$ is a $\bigsqcup$ [notation: disjoint union] of neighbourhoods biholomorphic to $U$, and $\forall x \in B$ where $F(y) = x$, $F$ is locally $z \mapsto z^n$ where $n = v_F(y)$.

We shall only consider the cases $S = S^2$ or $\mathbb{C}$.

Proposition: Let $C = \{P(s, t) = 0\}$ be an algebraic curve in $\mathbb{C}^2$, define $\phi : (s, t) \in C \mapsto s \in \mathbb{C}$ (recall this is holomorphic), and let $(s_0, t_0) \in C$. Then $v_\phi(s_0, t_0) > 1 \Leftrightarrow \frac{\partial P}{\partial t}(s_0, t_0) = 0$; moreover $v_\phi(s_0, t_0) = n \Leftrightarrow \frac{\partial P}{\partial t}(s_0, t_0) = \cdots = \frac{\partial^{n-1} P}{\partial t^{n-1}}(s_0, t_0) = 0$ but $\frac{\partial^n P}{\partial t^n}(s_0, t_0) \neq 0$. It sufficies to show the reverse implication; we wlog take $s_0 = t_0 = 0$. Now if $\frac{\partial P}{\partial t}(0, 0) \neq 0$ then $\phi$ is a chart of $C$ near $(0, 0)$ and $v_\phi(0, 0) = 1$; if $\frac{\partial P}{\partial t}(0, 0) = 0$ then $\frac{\partial P}{\partial s}(0, 0) \neq 0$ as $C$ is non-singular; then we have a chart near $(0, 0)$ by second projection $(s, t) \mapsto t$ with inverse $t \mapsto (h(t), t)$ for some $h(t)$ holomorphic near 0 with $h(0) = 0$.

We must have $P(0, t) = t^n q_1(t)$ with $q_1(0) \neq 0$ by hypothesis; let $q_i$ denote holomorphic functions which don't vanish at 0. Then write $P(s, 0) = sq_2(t)$, then $P(s, t) = t^n q_1(t) + sq_3(s, t)$; let $h(t) = t^l q_4(t)$; recale $0 \equiv P(h(t), t) = t^n q_1(t) + t^l q_4(t) q_5(t)$, so $n = l = v_h(0) = v_\phi(0, 0)$, since $h$ is a local coordinate expression for $\phi$.

Theorem: There exist Riemann surfaces of any genus $g \geq 0$; given such a $g$, let $h(s)$ be a polynomial of degree $2g + 2$ with no multiple roots; then we have that the algebraic curve $C = \{(s, t) \in \mathbb{C} : t^2 - h(s) = 0\}$ is nonsingular. We have an open $C_0 = \{(s, t) \in C : s \neq 0\}$ with $C \setminus C_0$ being 1 or 2 points.

Let $k(z) = z^{2g+2} h(\frac{1}{z})$; this is a polynomial with no multiple roots. Then $Y = \{(z, w) \in \mathbb{C}^2 : w^2 - k(z) = 0\}$ is a non-singular algebraic curve and we have an open $Y_0 = \{(z, w) \in Y : z \neq 0\} \subset Y$; this time we have $Y \setminus Y_0$ is precisely two points, as $k(0) \neq 0$.

Now $F : (s,t) \in C_0 \mapsto \frac{1}{s}, \frac{t}{s^{g+1}} \in Y_0$ is a valid function, as $k(z = \frac{1}{s}) = \frac{1}{s^{2g+2}}h(s) = \frac{t^2}{s^{2g+2}} = (\frac{t}{s^{g+1}} = w)^2$. $F$ is holomorphic: we have charts $(s,t) \mapsto s$ or $t$ on $C_0$, $(z,w) \mapsto z$ or $w$ on $Y_0$; take $\phi : (s,t) \mapsto s, \psi : (z,w) \mapsto w$, then $s$ maps by $\phi^{-1}$ to $(s, a(s))$ which maps by $F$ to $(\frac{1}{s}, \frac{a(s)}{s^{g+1}})$, which maps by $\psi$ to $\frac{a(s)}{s^{g+1}}$, so this composition $\hat{f}$ is holomorphic; the result for other choices of charts is similar.

$F$ is bijective: $F^{-1}(z,w) = (\frac{1}{z}, \frac{w}{z^{g+1}}) \in C_0$ for any $(z,w) \in Y_0$; thus $F$ is biholomorphic.

Define $X$ to be the quotient of $C \sqcup Y$ by the relation $x \sim F(x) \forall x \in C_0$; define $W \subset X$ is open iff both $W \cap C, W \cap Y$ are (regarding $C, Y$ as subsets of $X$). This $X$ is Hausdorff; the only nontrivial check is for $X \in X \setminus Y, y \in X \setminus C$ (since $C, Y$ are both Hausdorff). In this case $x \in C \setminus C_0, y \in Y \setminus Y_0$; put $u_x = \{(s,t) \in C : |s| < \frac{1}{2}\}, u_y = \{(z,w) \in Y : |z| < \frac{1}{2}\}$; these are disjoint in $X$ as $|s| < \frac{1}{2} \Rightarrow |\frac{1}{s} = z| > 2$.

We use the charts from $C$ and $Y$, which together cover $X$; we only need to check the transition functions for when we have $\phi_C : U \subset C \to \mathbb{C}, \phi_Y | V \subset Y \to \mathbb{C}$ charts on $C, Y$ respectively. The transitions can be computed using $F$ as $(\phi_Y \circ F), \circ \phi_C^{-1}, (\phi_C \circ F^{-1}) \circ \phi_Y^{-1}$; these are both holomorphic since $\phi_Y \circ F$ is a chart on $C$ and $\phi_C \circ F^{-1}$ is a chart on $Y$, as $F$ is biholomorphic.

Further we have that $X = \{(s,t) \in C : |s| \leq 1\} \cup \{(z,w) \in Y : |z| \leq 1\}$, a union of two compact sets (these are compact subsets of $C, Y$ respectively as they are closed and bounded in $\mathbb{C}^2$) so compact. Also (without proof, since such is much more an exercise in topology than part of this course) $X$ is connected.

Define $f : X \to \mathbb{P}^1 (= S^2)$ by $f(x) = s : 1$ if $x = (s,t) \in C, 1 : z$ if $x = (z,w) \in Y$; this is valid since $s = \frac{1}{z} \forall x \in C \cap Y$. Then $f$ is holomorphic as $f\mid_C, f\mid_Y$ are (since they are first projections of algebraic curves); we claim $\deg f = 2$; for this it is sufficient to show there are infinitely many $p \in \mathbb{P}^1$ with $f^{-1}(p) = 2$, since $X$ is compact so the set of $x \in X$ with $v_f(x) > 1$ is discrete so finite - consider $C \subset X$.

[I'm unclear about what's happening here; hope this is right]

$f\mid_C$ is first projection $(s,t) \mapsto s$ of an algebraic curve, so $f^{-1}(x)$ is two points iff $h(s) \neq 0$. Now $p \in X$ has $v_f(p) > 1$ iff $f^{-1}(f(p))$ is a single point $(p)$, so there are no ramifications on $X \setminus C = Y \setminus Y_0$, since $P \setminus Y_0$ is two points, but $f(Y \setminus Y_0)$ is the single point $1 : 0$.

Using a proposition from above we can find the ramification points in $C \subset X$; they are the $(s,t)$ such that $P(s,t) = 0 = \frac{\partial P}{\partial t}(s,t) \star$, where $P(s,t) = t^2 - h(s)$. We have $\frac{\partial^2 P}{\partial t^2}(s,t) \equiv 2 \neq 0$ so the condition $\star$ becomes that $t = 0$ and $h(s) = 0$, so we have $2g + 2$ ramification points with $v_f = 2$. Now Riemann-Hurwitz applied to $f : X \to \mathbb{P}^1$ gives $\chi(X) = 2\chi(\mathbb{P}^1) - (2g+2) = 2 - 2g$, since $\chi(\mathbb{P}^1) = 2$. So the genus of $X$ is $g$, as required.

Some remarks on this construction: $X$ is, informally, a $2 : 1$ branched cover of $\mathbb{P}^1$ branched over $2g + 2$ points. $X$ has a meromorphic function of degree 2 $(f)$; such a Riemann surface is called hyperelliptic. Also note that $X \setminus C$ is finite so $X$ is a compactification of $C$, but $X$ is not a projective curve in $\mathbb{P}^2$.

# Meromorphic Differentials on Riemann surfaces

Let $S$ be a connected RS. Then holomorphic maps $F : S \to \mathbb{P}^1$ are equivalent to meromorphic functions $f = \psi \circ F : S \setminus F^{-1}(1 : 0) \to \mathbb{C}$ (where $\psi$ is the chart $z : 1 \mapsto z$ on $\mathbb{P}^1$. We want to consider the derivative of this meromorphic function, but to do this we must take a chart on $S$. But then our derivative depends on the choice of chart: by the chain rule, $(f\circ\phi_\beta)' = ((f\circ\phi_\alpha)\circ(\phi_\alpha^{-1}\phi_\beta))' = (f\circ\phi_\alpha)'\circ\tau'_{\alpha\beta}$; [which is not generally $= (f\circ\phi_\alpha)'$]; thus writing $f_\alpha$ for $f\circ\phi_\alpha^{-1}$, $f'_\alpha(z) = f'_\beta(\tau_{\alpha\beta}(z))\tau'_{\alpha\beta}(z)(\star)$. Therefore, we cannot just take $f'$; rather:

Definition: a <u>(meromorphic) differential</u> $\eta$ on a RS $S$ is a collection of meromorphic functions $\eta_\alpha$ on $\phi_\alpha(V_\alpha)\subset\mathbb{C}$ corresponding to the charts $(\phi_\alpha, V_\alpha)$ of $S$ satisfying $\eta_\alpha(z) = \eta_\beta(\tau_{\alpha\beta}(z))\tau'_{\alpha\beta}(z)(\star\star)\forall z\in\phi_\alpha(V_\alpha\cap V_\beta)$ for any pair of charts $\phi_\alpha,\phi_\beta$ where $\tau_{\alpha\beta} = \phi_\beta\circ\phi_\alpha^{-1}$. Each $\eta_\alpha$ is a <u>local expression</u> or <u>formula</u> for $\eta$ in the chart $\phi_\alpha$. It is notationally convenient to write $\eta_\alpha(z)dz$ rather than $\eta_\alpha$; then if $w = \tau_{\alpha\beta}(z)$ is another local coordinate then $(\star\star)$ becomes $\eta_\alpha(z)dz = (\eta_\beta(w)\circ\tau'_{\alpha\beta}(z))dz = \eta_\beta(w)dw(\dagger)$. $(\star)$ is a special case of a meromorphic differential; put $\eta_\alpha = f'_\alpha$, then $(\star\star)$ holds, and $(\dagger)$ becomes $f'_\alpha(z)dz = f'_\beta(w)dw$. We will write $df$ for this example of a differential.

More generally, for $f, g$ meromorphic functions on $S$ put $\eta_\alpha(z) = g_\alpha(z)\frac{df_\alpha(z)}{dz}(z)$; then $(\star\star)$ holds: $g_\alpha(z)\frac{df_\alpha}{dz}(z) = g_\beta(w)\frac{df_\beta(w)}{dw}(w)\frac{dw}{dz}$, where $w = \tau_{\alpha\beta}(z)$, but this is $\eta_\beta(w)\tau'_{\alpha\beta}(z)$ as required, so this is a valid differential, denoted $gdf$. [Similarly] for $\eta$ a differential and $g$ a meromorphic function, $g\eta$ is a differential; conversely, suppose $\eta,\zeta$ differentials, and $\zeta_\alpha\not\equiv 0$ for some $\alpha$ (which by connectedness implies $\zeta_\alpha\not\equiv 0\forall\alpha$). Then $(\star\star)\Rightarrow\frac{\eta_\alpha}{\zeta_\alpha} = \frac{\eta_\beta\tau'_{\alpha\beta}}{\zeta_\beta\tau'_{\alpha\beta}} = \frac{\eta_\beta}{\zeta_\beta}$ so there is a well defined meromorphic function $f$ on $S$ such that $\forall\gamma\eta_\gamma = f\zeta_\gamma$. Then write $\eta = f\zeta$, and we have proven:

Proposition: Suppose $S$ is a connected RS and $\zeta$ a nonzero differential on $S$, then any meromorphic differential $\eta$ on $S$ is $f\zeta$ for some meromorphic $f$ on $S$.

Suppose $h$ is a non-constant meromorphic function on $S$. Then every differential on $S$ is $fdh$ for some meromorphic $f$.

Fact (this is in fact a deep theorem): 1. every RS carries a non-constant meromorphic function, 2. Every compact RS is "algebraic", which informally means it is biholomorphic to some $\{z_0:\cdots:z_n\in\mathbb{P}^n|P_1(z_0,\ldots,z_n) = 0,\ldots,P_N(z_0,\ldots,z_n) = 0\}$ for some homogenous polynomials $P_i$. This result was conjectured by Riemann in around 1850 and proven over the next 50 years.

Note that if $S$ is a non-singular projective curve in $\mathbb{P}^2$ and $0:0:1\in S$, and $S$ not a projective line $\{\alpha X+\beta Y = 0\}$ for some $\alpha,\beta\in\mathbb{C}$, then $X:Y:Z\in S\mapsto X:Y\in\mathbb{P}^1$ is a non-constant meromorphic function.

Definition: a meromorphic differential $\eta$ has a zero/pole of order $m$ at $p\in S$ if for some local expression $\eta_\alpha$ with $p\in V_\alpha$, $\eta_\alpha$ has a zero/pole of order $m$ at $\phi_\alpha(p)\in\mathbb{C}$; note that this is well defined since if $p\in V_\beta$ then $\eta_\beta = \eta_\alpha\tau'_{\alpha\beta}$ and $\tau'_{\alpha\beta}$ is holomorphic and never 0. A differential is <u>holomorphic</u> if it has no poles.

An example is $S = S^2$ with the stereographic projection charts $\alpha,\beta$ and $f_\beta(w) = w^2+1$, $g_\beta(w) = \frac{1}{w^2-1}$. Then $(gdf) = \frac{2w}{w^2-1}dw$; let $z$ be the coordinate under $\alpha$, $w = \frac{1}{z} = \tau(z) = \tau_{\alpha\beta}(z)$, then $(gdf)_\alpha = \frac{2(\frac{1}{z})}{(\frac{1}{z})^2-1}(-\frac{1}{z^2})dz = \frac{2}{z(z^2-1)}dz$; we have a zero at $w = 0$ ($z$ undefined or $\infty$) and poles $w = \pm1$ (corresponding to $z = \pm1$) and $z = 0$ (at which $w$ is undefined); these are all of order 1, so we have 3 simple poles and 1 simple zero.

We want to write the zeroes and poles conveniently; for $gdf$ as in the example, we could write these as $1"w = 0" + (-1)"w = 1" + (-1)"w = -1" + (-1)"z = 0"$. A better way to handle this is to use divisors:

## Divisors

Assume $S$ is a compact connected Riemann surface.

Definition: A divisor on $S$ is a formal sum $D = \sum_{P_i \in S} n_i P_i$ where $n_i \in \mathbb{Z}$ and only finitely many $n_i$ are nonzero. If $n_i = 1$ we write $P_i$ for $1P_i$. Divisors form a group under addition, using the distributive law $kP + lP = (k+l)P$.

Define the degree $\deg D = \sum_i n_i$; this gives a homomorphism $\deg : \mathrm{Div}(S) \to \mathbb{Z}$ from the group of divisors on $S$.

We say a divisor $D \in \mathrm{Div}(S)$ is effective, $D \geq 0$, if $n_i \geq 0 \forall i$; thus we write $D \geq D'$ to mean $D - D' \geq 0$.

The divisor of a meromorphic function $f \not\equiv 0$ on $S$ is $(f) = \sum_i k_i A_i - \sum_j l_j B_j$, where $A_i$ are the zeroes of $f$, $k_i$ their corresponding orders, and similarly $B_j$ poles, $l_j$ orders.

We say $D$ is a principal divisor if $D = (f)$ for some meromorphic $f \not\equiv 0$. We say $D \sim D'$, $D$ and $D'$ are linearly equivalent, if $D - D'$ is a principal divisor.

If $\omega$ is a meromorphic differential on $S$ then we can write $(\omega) = \sum \text{zeroes} - \sum \text{poles}$ as for a meromorphic function; any such divisor $(\omega)$ is called a canonical divisor.

Properties: $\deg(f) = 0$, since the sum of orders of zeroes must equal the degree of the meromorphic function $f$, but this also equals the sum of the orders of poles. $(fg) = (f) + (g)$; if $g \not\equiv 0$, $(\frac{f}{g}) = (f) - (g)$. For any two canonical divisors $(\omega), (\omega')$ we have $(\omega) \sim (\omega')$, since there exists a meromorphic $f$ such that $\omega' = f\omega$, so $(\omega') = (f) + (\omega)$. A consequence of this is that $\deg(\omega)$ is independent of the particular choice of meromorphic differential $\omega$, but is rather a property of the surface $S$ (Note we assume $S$ has non-constant meromorphic functions, which is a true fact but not proven in this course).

Given $D \in \mathrm{Div}(S)$, consider the complex vector space $\mathcal{L}(D) = \{f \text{meromorphic on } S : D + (f) \geq 0\} \cup \{0\}$; the reader may verify this is in fact a vector space. Define $l(D) = \dim \mathcal{L}(D)$.

Lemma: $\mathcal{L}(D)$ is finite dimensional: let $D = \sum_i n_i P_i - \sum_j m_j Q_j$ with $n_i, m_j > 0$; then $(f) + D \geq 0 \Rightarrow f$ has poles only at $P_i$, with orders $\leq n_i$. Choose and fix a chart near each $P_i$, and define a linear map by $f \in \mathcal{L}(D) \mapsto$ the sequence of coefficients of the principal parts of the Laurent expansion for $f$ at $P_i$, an element of $\mathbb{C}^N$ where $N = \sum_i n_i$. Elements of the kernel of this map are meromorphic functions on a compact Riemann surface with no poles, i.e. constant functions, so the kernel is $\cong \mathbb{C}$, and by rank-nullity $l(D) \leq N + 1$.

Properties of $l(D)$: $l(0) = 1$, since the only meromorphic functions without poles are constant. For a canonical divisor $K_S = (\omega)$, $L(K_s) = \{f : (fK_S) \geq 0\} \cup \{0\}$,the space of all holomorphic differentials on $S$. If $\deg D < 0$, $\deg(D + (f)) < 0 \forall f$ so $l(D) = 0$. If $D \sim D'$ (i.e. $D = D' + (h)$) then $f \in \mathcal{L}(D)$ iff $fh \in \mathcal{L}(D')$, so $l(D) = l(D')$.

## Riemann-Roch Theorem

Suppose $S$ is a compact connected Riemann surface (recall this means it has non-constant meromorphic functions), then [for any divisor $D$] $l(D) = 1 - g(S) + \deg D + l(K_S - D)$ where $g(S)$ is the genus of $S$ and $K_S$ is any canonical divisor on $S$. This is the second of the two most important results in this course (though we shall still be seing the third-most-important result). The proof is non-examinable and there is not time to cover it in this course; a good, well-written proof can be found in Kurwan's "Complex Algebraic Curves".

Immediate consequences of this result: the $D = 0$ case gives $1 = 1 - g(S) + l(K_S)$, i.e. $l(K_S) = g(S)$ (a remark for interest only; $l(K_S)$ is the geometric genus $p_g$ of $S$; there are many concepts referred to as genus in mathematics; the simplest is topological genus $g$). $D = K_S$ gives $g(S) = 1 - g(S) + \deg K_S + 1 \therefore \deg K_S = 2g(S) - 2 = -\chi(S)$. If $\deg D > 2g(S) - 2$ then $\deg(K_S - D) < 0$, so $l(K_S - D) = 0 \therefore l(D) = 1 - g(s) + \deg D$.

Applications: If $g(S) = 0$ then $S$ is biholomorphic to $\mathbb{P}^1$ $(= S^2)$ (this is related to question 11 on the third example sheet): take $D = P$, a single point in $S$; $\deg P = 1 > 2g(S) - 2$, so $l(P) = 2$ by the above, so $\exists$ a meromorphic function $f$ on $S$ such that the only pole of $f$ is a simple pole at $P$ (since the space of meromorphic functions without poles is only 1-dimensional), so we have [a holomorphic] $f : S \to \mathbb{P}^1$ of degree 1, i.e. a bijection, so $f^{-1}$ is holomorphic [and $f$ is a conformal equivalence].

Theorem: Every compact Riemann surface of genus 2 is hyperelliptic, i.e. has a meromorphic function of degree 2: we have $g(S) = 2$, so by Riemann-Roch 1) $l(K_S) = g(S) = 2 > 0 \therefore \exists$ a non-trivial holomorphic differential $\omega \not\equiv 0$ on $S$, 2) For $P, Q \in S$, possibly coincident, let $D = P + Q$, then $l(K_S - P - Q) = l(P + Q) - 1$.

$\deg(\omega) = -\chi(S) = 2 = \#$ zeroes of $\omega$, since $\omega$ has no poles. Let $P, Q$ be the zeroes of $\omega$, then $l(K_S - P - Q) = l((\omega) - P - Q) = l(0) = 1 \therefore l(P + Q) = 1 + 1 = 2$ [which is $> 1$], so we have a meromorphic (non-constant) function $f$ on $S$ with $\deg f \leq 2$, i.e. $\deg f$ must be 1 or 2. But we cannot have $\deg f = 1$since $g(S) \neq 0$ so $S$ is not the Riemann sphere; thus we must have $\deg f = 2$.

Note that a "general" compact Riemann surface of genus $\geq 3$ is not hyperelliptic, e.g. $\{X^4 + Y^4 = Z^4\} \subset \mathbb{P}^2$ is a surface of genus 3 which is not hyperelliptic.

Suppose the genus is 1 and relabel $S = E$. Let $\mathrm{Cl}^0(E)$ denote the set of linear equivalence classes of divizors of degree 0 on $E$.

We want to give $E$ a group structure. Map the quotient group $\frac{\mathrm{Div}E}{\sim} \to \mathbb{Z}$ by deg; this is a homomorphism with kernel $\mathrm{Cl}^0(E)$. So $\mathrm{Cl}^0(E)$ is the kernel of a homomorphism, so an (additive) group. Choose and fix $P_0 \in E$ (which will become our zero element). Then $\deg D = 0 \Rightarrow \deg(D + P_0) = 1$, so $\deg(K_E - (D + P_0)) = -1$ (since $\deg() = \chi(E) = 0$). So $l(K_E - (D + P_0)) = 0$. Applying Riemann-Roch to $D + P_0$, $l(D + P_0) = 1 - 1 + 1 + 0 = 1$, so there is a unique effective divisor of degree 1, linearly equivalent to $D + P_0$, i.e. there is a unique $P \in E$ such that $D \sim P - P_0$. So we have a bijection $\mathrm{Cl}^0(E) \to E$, so we can consider $E$ as a group.

If $P, Q \in E$, then their sum under the group operation is found or defined to be $R$ such that $(P - P_0) + (Q - P_0) \sim R - P_0$, i.e. $P + Q \sim P + P_0(\star)$. So $P_0$ is the zero element, and $Q$ is the inverse of $P$ iff $R = P_0$, i.e. $P + Q = 2P_0$ (considering $P, Q, R$ as divisors throughout this).

An example: $\Lambda \subset \mathbb{C}$ a lattice, $E = \frac{\mathbb{C}}{\Lambda}$; the points of $E$ are cosets $z + \Lambda$ for $z \in \mathbb{C}$. Set $P_0 = 0 + \Lambda$, then $R$ is the sum of $P$ and $Q$ iff $P + Q - R - P_0 \sim 0$, i.e. there is an elliptic function $f$ such that $f(a) = f(b) = 0, f(c) = f(0) = \infty$ [where $P = a + \Lambda, Q = b + \Lambda, R = c + \Lambda$], with these zeroes and poles being of order 1. By the constraints on elliptic functions, such an $f$ is possible iff $a + b - c - 0 \in \Lambda$, i.e. $a + b \equiv c \mod \Lambda$.

Furthermore (and this relates to queston 11 on the second example sheet

and question 5 on the third example sheet), for $g(E) = 1$, suppose we have meromorphic functions $f$ with $(f) = P_1 + P_2 - 2P_0$, i.e. $P_2$ is the inverse of $P_1$ in the group, and $h$ with $(h) = Q_1 + Q_2 + Q_3 = 3P_0$, i.e. $Q_1, Q_2, Q_3$ sum to zero in the group. Then $h^2 - a_1 f^3$, for a suptable choice of $a_1 \neq 0 \in \mathbb{C}$, has a pole of order $\leq 5$ at $P_0$ and no other poles. So (for suitable $a_2$, which $= 0$ if this pole is of order $\leq 4$), $(h^2 - a_1 f^3) - a_2 fh$ has a pole of order $\leq 4$ at $P_0$ and no other poles; continuing similarly, $(h^2 - a_1 f^3) - a2fh - a_3 f^2 - a_4 h - a_5 f$ has a pole of order $\leq 1$ at $P_0$ and no other poles, so it must be a constant function, $= a_6$; we can wlog take $a_1 = 1$, then $h^3 - f^2 - a_2 fh - a_3 f^2 - a_4 h - a_5 f - a_6 \equiv 0$. We can define $P \in E \setminus \{P_0\} \mapsto 1 : f(P) : h(P)$
$inCsubset \mathbb{P}^2$ where $: Y : Z \mid XZ^2 - a_2 XYZ - a_4 X^2 Z = Y^3 + a_3 XY^2 + a_5 X^2 Y + a_6 X^3\}(\star\star)$, a projective cubic curve; it can be shown that he map extends holomorphicly over $P_0$ by $P_0 \mapsto 0 : 0 : 1$, $E$ is biholomorphic to $C$ and $C$ is non-singular so a Riemann surface. $(\star\star)$ is called a cubic curve in Weierstrass normal form; by a change of variables we may take $a_2 = a_3 = a_4 = 0$. These curves are covered in more detail on the fourth example sheet for this course.

# Analytic Continuation

Suppose $f : D \subsetneq \mathbb{C} \to \mathbb{C}$ is holomorphic. Can we extend $f$ holomorphicly to a larger domain?

Definition: i) A <u>function element</u> is a pair $(f, D)$ with $D \subset \mathbb{C}$ open connected and $f : D \to \mathbb{C}$ holomorphic. ii) $(g, E) \approx (f, D)$, $(g, E)$ is <u>direct analytic continuation</u> of $(f, D)$, if $E \cap D \neq \emptyset$ and $f \mid_{D\cap E} = g \mid_{D\cap E}$ iii) $(g, E)$ is <u>analytic continuation</u> of $(f, D)$, $(g, E) \sim (f, D)$, if there is a finite sequence $(f, D) \approx (f_1, D_1) \approx \cdots \approx (f_n, D_n) \approx (g, E)$; this is an equivalence relation (note that $\approx$ is not, since it is not transitive). The equivalence class of $(f, D)$ under $\sim$ is called a <u>complete analytic function</u> (in the sense of Weierstrass).

The choice of $D$ is important. For example, $(\log z, \mathbb{C} \setminus (\mathbb{R}_{>0}))$ has no nontrivial analytic continuation, as the domain we have chosen is too large. If $D \cap E$ is not connected wee maay have $f = g$ on one component of $D \cap E$ but $f \neq g$ on another, which would be problematic.

Lemma: 1) If $f = g$ on some open disc $U$ then $f = g$ on any open connected $V \supset U$: $f - g \equiv 0$ on $U$, so the identity principle on $(f - g)\mid_V$ implies this is $\equiv 0$. 2) if we have $(f, D)$ and $E$ with $E \cap D \neq \emptyset$ then if $\exists (g, E) \approx (f, D)$ then this $g$ is unique: if we had such $g_1, g_2$ then $g_1 - g_2 \equiv 0$ on $D \cap E$, an open subset of the connected $E$, so $g_1 - g_2 \equiv 0$ on $E$.

A good choice for $D$ is an open disc, since for any two such discs $D_1, D_2$ $D_1 \cap D_2$ is connected, and if $f$ is holomorphic on $D$ then it must be given by a power (Taylor) series convergent on $D$. It is easy to find the maximal disc about any $z_0 \in \mathbb{C}$ - we just take the disk on which this series is convergent, using the <u>root test</u>: for $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, consider whether $\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n (z - z_0)^n|} < 1$ where $\overline{\lim}$ is the "upper limit", the maximum limit of any convergent subsequence. Hadamard's formula is that the radius of convergence is $R = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}}$; this $R$ is the distance from $z_0$ to the nearest singularity of $f$.

Let $D = \{|z - z_0| < R\}, \partial D = \{|z - z_0| = R\}$, $f : D \to \mathbb{C}$ holomorphic. Then $c \in \partial D$ is called a <u>regular</u> point (for $(f, D)$) if $(g, \{|z - c| < \epsilon\}) \approx (f, D)$ for some

$\epsilon > 0$ and some $g$; otherwise $c$ is a $\underline{\text{singular}}$ point. If there are no regular points in $\partial D$ then $\partial D$ is the $\underline{\text{natural boundary}}$ of $(f, D)$.

Examples: 0) $f(z) = \sum_{n=0}^{\infty} z^n$ has $R = 1$ by Hadamard, $z = 1$ is a singular point for $(f, \Delta)$ (recall $\Delta$ is the unit disc), and in fact it is the only such, as $f(z) = \frac{1}{1-z}$. 1) $f(z) = \sum_{n=1}^{\infty} z^{n!}$ $(= z + z^2 + z^6 + \dots)$: we find again $R = 1$, and claim $\{|z| = 1\}$ is the natural boundary of this $(f, \Delta)$: if $\frac{\arg z}{2\pi} = \frac{p}{q} \in \mathbb{Q}$ then $z = re^{2\pi i \frac{p}{q}}$ so $z^{n!} = r^{n!}e^{2\pi i \frac{pn!}{q}} = r^{n!}$ for $n \geq q$; for $0 \leq r < 1$ let $\tilde{f}(z) = \sum_{n=q}^{m} z^{n!} = \sum_{n=q}^{m} r^{n!} \geq r^{m!}(m - q)$ and $\lim_{r \to 1}(m - q)r^{m!} = m - q > M$ for any $M > 0$ by taking $m \in \mathbb{Z}$ such that $m - q > M$. So $\lim_{r \to 1} \tilde{f}(r) > M$ for any $M$, so $f(z)$ has infinite limit on a dense subset of $\{: z :]1\}$, so every $z = e^{i\phi}$ for $\phi \in \mathbb{R}$ is singular (since any neighbourhood of $z$ contains a point at which $f$, informally speaking, is infinite). 2) (sketch) $f(z) = \sum_{n=1}^{\infty} \frac{z^{2^n}}{2^n}$ has agin $R = 1$; moreover the series converges uniformly on $\{|z| \leq 1\}$ so $f$ is continuous on $\{|z| = 1\}$. $\forall z$ with $|z| = 1$ we can find $z$ is singular, as otherwise $\Im(f(e^{it}))$ for some $t \in \mathbb{R}$ is $\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(2^n t)$ but this is not a $C^\infty$ function by the theory of fourier series.

A nontrivial analytic continuation may or may not exist; if it does, we may perhaps obtain a "multivalued function", i.e. have $(f, D) \sim (g, D)$ but $f \neq g$. Riemann suggested that such functions are "defined on the wrong domain", and should be considered as functions on some surface which covers $\mathbb{C}$.

Definition: let $p : X \to Y$ be continuous between topological surfaces, then a continuous $q : V \to X$ is a $\underline{\text{local section}}$ of $p$ over $V \subset Y$ if $p \circ q = \mathrm{id}\,|_V$, e.g. charts on $\frac{\mathbb{C}}{\Lambda}$ are local sections of $p : z \mapsto z + \Lambda$ over $V = p(D)$ where $D \subset \mathbb{C}$ is a small disc.

Theorem: every complete analytic function $\mathcal{F}$ determines a Riemann surface $S(\mathcal{F})$, the Riemann surface of a complete analytic function $\mathcal{F}$, with holomorphic maps $\pi : S(\mathcal{F}) \to \mathbb{C}, u : S(\mathcal{F}) \to \mathbb{C}$ such that if $(f, D) \in \mathcal{F}$ then $f(z) = u \circ q(z)$ for some holomorphic local section $q$ of $\pi$.

Examples: 0) an entire function $f : \mathbb{C} \to \mathbb{C}$ has $S(\mathcal{F}) = \mathbb{C}, \pi = u = \mathrm{id}\,|_{\mathbb{C}}$. A rational function $\frac{p(z)}{q(z)} : \mathbb{C} \setminus \{\text{some finite set}\} \to \mathbb{C}$ has $S(\mathcal{F}) = \mathbb{C} \setminus \{\text{that finite set}\}, \pi = u = \mathrm{id}$. There is a natural $\underline{\text{compactification}}$ $S(\mathcal{F}) = \mathbb{C} \cup \{\infty\} = S^2$. 1) $\sum_{n=1}^{\infty} z^{n!}$ has $S(\mathcal{F}) = \Delta \subset \mathbb{C}, \pi = u = \mathrm{id}_\Delta$; note that $\Delta$ is not biholomorphic to $\mathbb{C}$. 2) Log, the complete analytic function containing $(\sum_{n=1}^{\infty}(-1)^{n+1} \frac{(z-1)^n}{n}, \{|z - 1| < 1\})$. We can show (exercise, using the identity principle) that $\forall (h, D) \in \mathrm{Log}$, $e^{h(z)} = z \forall z \in D$. We can also show that $\forall (s_0, t_0)$ such that $e^{t_0} = s_0 \exists! \tilde{h} : D(s_0, \epsilon) \to \mathbb{C}$ for some $\epsilon > 0$ such that $[\tilde{h}(s_0) = t_0$ and$]$ $(\tilde{h}, D(s_0, \epsilon)) \in \mathrm{Log}$.

We can identify $S(\mathrm{Log}) = \{(s, t) \in \mathbb{C}^2 : e^t - s = 0\}$, a (non-algebraic) curve in $\mathbb{C}^2$. Take charts by first projection $(s, t) \mapsto s$ on suitable open sets covering $S(\mathrm{Log})$. Then we have $\pi : (s, tR \in S(\mathrm{Log}) \mapsto s, u : (s, t) \mapsto t$; [for $D \subset \mathbb{C} \setminus \{0\}$ set $\phi = \pi\,|_D$, then $\phi^{-1}$ is a locas lection of $\pi$ over $D$. Then $(h, D) \in \mathrm{Log}$ is given by $h = u \circ \phi^{-1}$, a "holomorphic branch of Log over $D$".

$u : S(\mathrm{Log}) \to \mathbb{C}$ is a holomorphic bijection, so $S(\mathrm{Log})$ is biholomorphic to $\mathbb{C}$; $u^{-1}$ is uniquely determined [and is $t \mapsto \exp(t)$]. Note that under $\pi : \mathbb{C} \cong S(\mathrm{Log}) \to \mathbb{C}$, $S(\mathrm{Log})$ "spirals" over $\mathbb{C} \setminus \{0\}$ infinitely many times.

3) $\sqrt{z}$, the complete analytic function containing $(\sum_{n=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!} \times (z - 1)^n, D(1, 1))$. $\forall (f, D) \in \sqrt{z}$ we have $f^2(z) = z$; also if $t_0^2 = s_0 \neq 0$ then $\exists!$ (for $D$ fixed) $(f(s), D(s_0, \epsilon)) \in \sqrt{z}$ for some $\epsilon > 0$ with $f(s_0) = t_0$ ($s_0 = 0$ cannot be

included, as then $f^2(z) = z, t \geq 0, f(t) \pm \sqrt{t}$ in real variable is not $C^\infty$ at $t = 0$)

Thus $S(\sqrt{z}) = \{(s,t) \in \mathbb{C}^2 : t^2 - s = 0\} \setminus \{(0,0)\}$, a punctured algebraic curve, $= \{X : Y : Z \mid Z^2 - XY = 0\} \setminus \{1 : 0 : 0, 0 : 1 : 0\}$, i.e. there is a projective curve which is a natural compactification of $S(\sqrt{z})$.

4) "algebraic functions": $f(s)$ defined (locally, near some $s_0 \in \mathbb{C}$)) by $P(s, f(s)) = 0$ for some polynomial $P(s,t)$ with $P(s_0, t_0) = 0, \frac{\partial p}{\partial t}(s_0, t_0) \neq 0$, we get $(f, D(s_0, \epsilon))$ defining a complete analytic function. E.g. $P(s,t) = t^2 - (s^2-1)\dots(s^2-n)$ for $n > 0$, e.g. $n = 4$ - see the handout, and the earlier proof that there exist Riemann surfaces of any genus.

5) Somewhat of a nonexample: beware, the "obvious" algebraic curve might not be the "right" $S(\mathcal{F})$, even for $\mathcal{F}$ algebraic, as the algebraic curve might be singular. E.g. $\mathcal{F}(z) = z\sqrt{1-z}$ (the complete analytic function [presumably defined by some series]). We have projection $\pi : S(\mathcal{F}) \to \mathbb{C}$, and want to use $C = \{(s,t) \in \mathbb{C}^2 : t^2 - s^2(1-s) = 0\}$, but this is singular at $(0,0)$; $p : (s,t) \in C \to s \in \mathbb{C}^2$ has $p^{-1}(0) = (0,0)$, a single point, but $\pi^{-1}(0)$ is two points, corresponding to $(f_1, D(0,\epsilon)), (f_2, D(0,\epsilon))$ where $f_1(z) = z(1 - \frac{1}{2}z - \frac{1}{8}z^2 + \dots), f_2(z) = -f_1(z)$ (note $f_1 \neq f_2$).

Now, the proof of our above theorem, that $\forall (f,D) \in \mathcal{F} \exists$ a local section $q$ of $\pi$ such that $f = u \circ q$. For $z \in \mathbb{C}$, define $(f_1, D_1) \overset{z}{\sim} (f_0, D_0)$ if $z \in D_1 \cap D_0$ and $f_1 = f_0$ near $z$. $\overset{z}{\sim}$ is an equivalence relation; the equivalence class of $(f,D)$ under $\overset{z}{\sim}$ is called the germ of $(f,D)$ at $z$ (note we must have $z \in D$); the notation is $[f,z]$.

Proposition: every germ $[f,z]$ uniquely determines 1) $z \in \mathbb{C}$: $z = \bigcap_{(f,V) \in [f,z]} V$, 2) $f(z) \in \mathbb{C}$: $\forall (f_1, V_1), (f_2, V_2) \in [f,z]$, the definition implies $f_1(z) = f_2(z)$, 3) A complete analytic function $\mathcal{F}$ such that $[f,z] \subset \mathcal{F}$: $(f_1, V_1) \overset{z}{\sim} (f_2, V_2) \Rightarrow (f_1, V_1) \sim (f_2, D_2)$. Put $S(\mathcal{F})$ to be the set of germs in $\mathcal{F}$, $\pi : [f,z] \in S(\mathcal{F}) \mapsto z \in \mathbb{C}, u : [f,z] \in S(\mathcal{F}) \mapsto f(z) \in \mathbb{C}$, and $\forall (f,d) \in \mathcal{F}, q : z \in D \mapsto [f,z] \in S(\mathcal{F})$ [I am unsure about this section]. Now we need to show that $S(\mathcal{F})$ is Hausdorff and has complex structure, and $\pi, u$ are holomorphic with respect to this.

Definition: If $(f,D) \in \mathcal{F}$ then $[f,D]$ defined as the set of all germs $[f,z]$ containing $(f,D)$ is a basic neighbourhood; we have $[f,D] \subset S(\mathcal{F})$, and the map $[f,z] \in [f,D] \mapsto z \in D$ is a bijection. [Then we define] $U \subset S(\mathcal{F})$ is open iff $\forall p \in U \exists [f,D]$ such that $p \in [f,D] \subset U$, i.e. open sets in $S)\mathcal{F})$ are unions of basic neighbourhoods; we clearly have unions of open sets being open and $S(\mathcal{F}), \emptyset$ are open. Suppose $U_1, U_2 \subset S(\mathcal{F})$ open; wlog take $U_j = [f_j, D_j]$. If $U_1 \cap U_2 \neq \emptyset$ then $\exists [f,z] \in U_1 \cap U_2$, then $z \in D_1 \cap D_2$ so $D_1 \cap D_2 \neq \emptyset$ Then $f_1(z) = f_2(z)$, so consider $f_1 - f_2$; we have $f_1 - f_2 \equiv 0$ on some $D(z,\epsilon)$, so $[f_1, D(z,\epsilon)] \subset U_1 \cap U_2$. This holds for any $[f,z] \in U_1 \cap U_2$, so $U_1 \cap U_2$ contains a basic neighbourhood about each of its points, so is open, so we have a valid topology.

To see $S(\mathcal{F})$ is Hausdorff, suppose we have $p_1 \neq p_2 \in S(\mathcal{F})$, and let $p_j = [f_2, z_j]$. If $z_1 = z_2 = z, \exists (f_1, D_1) \in [f_1, z_1]$ and $(f_2, D_2) \in [f_2, z_2]$; then $f_1 - f_2$ is never zero on $D^\star(z,\epsilon)$ for some $\epsilon > 0$ [by identity principle?], so $[f_1, D(z,\epsilon)] \cap [f_2, D(z\epsilon)] = \emptyset$, as $\forall [w \in D^\star(z,\epsilon)], f_1(w) \neq f_2(w)$ so $[f_1, w] \neq [f_2, w]$.

If $z_1 \neq z_2$ then we can take $(f_1, D_1) \in [f_1, z_1], (f_2, D_2) \in [f_2, z_2]$ with $D_1 \cap D_2 = \emptyset$ by Hausdorffness of $\mathbb{C}$, by restricting the $f_i$ to smaller $D_i$ as necessary, so $[f_1, D_1] \cap [f_2, D_2] = \emptyset$.

Now we define the charts on $S(\mathcal{F})$: for $[f,D]$ with $D$ an open disc, $\phi : [f,D] \to D$ given by $\phi = \pi \mid_{[f,D]}$ is our chart (recall $\pi([f,z]) = z$); this is a homeomorphism since for any open $V \subset D$, $[f,V]$ is well defined and open with

$\phi([f,V]) = V$. Given a second chart $\phi_2 : [f_2, D_2] \to D_2$ with $[f,D] \cap [f_2, D_2] \neq \emptyset$ we have $[f, D \cap D_2]$ [open], and $\phi^{-1}(z) = [f,z]$ so $\phi_2(\phi^{-1}(z)) = z$ and we have a complex structure.

Finally, $\pi$ and $u$ are holomorphic maps: for any $[f,z] \in S(\mathcal{F}$, $\pi$ is locally some chart $\phi$ so holomorphic. Recall $u : [f,z] \in S(\mathcal{F}) \mapsto f(z) \in \mathbb{C}$; we have a local expression: for $[f,D] \subset S(\mathcal{F}), z \in \pi([f,D]) = \phi([f,D]), z \in D \overset{\phi^{-1}}{\mapsto} [f,z] \in [f,D] \overset{u}{\mapsto} f(z) \in \mathbb{C}$, where $(f,D) \in \mathcal{F}$ so $f$ is holomorphic [so $u$ is locally holomorphic so holomorphic].

## Digression: covering surfaces

Charts $\phi$ on $S(\mathcal{F})$ are restrictions on $\pi : S(\mathcal{F}) \to \mathbb{C}$, so $\pi$ has ramification points.

Definition: let $\tilde{R}, R$ be topological surfaces. Then a continuous surjective $p : \tilde{R} \to R$ is "covering" (in a preliminary, weak, non-standard sense) if $\forall x \in \tilde{R} \exists$ a neighbourhood $\tilde{U} \subset \tilde{R}$ with $\tilde{U} \ni x$ such that $p\mid_{\tilde{U}}$ is a homeomorphism. (The "correct" definition of covering implies this, but is not equivalent to it). Thus $\pi : S(\mathcal{F}) \to \mathbb{C}$ is a "covering" of its image $\pi(S(\mathcal{F})) \subset \mathbb{C}$.

Proposition: Let $R$ be a Riemann surface, $p : \tilde{R} \to R$ a "covering" [$\tilde{R}$ a topological surface]. Then there is a complex structure on $\tilde{R}$ such that $p$ is holomorphic: for $x \in \tilde{R} \exists \tilde{U}$ with $x \in \tilde{U} \subset \tilde{R}$ such that $p : \tilde{U} \to U$ is a homeomorphism, where $U := p(\tilde{U}) \subset R$. Then there is a chart $\phi : W \to \mathbb{C}$ on $W \subset R$ with $p(x) \in W$. Set $\tilde{\phi} := \phi \circ p : p^{-1}(U \cap W) \subset \tilde{R} \to \mathbb{C}$; this is a composition of homeomorphisms, so a homeomorphism. Then for transition functions, if we have two such $\tilde{\phi}_\alpha, \tilde{\phi}_\beta$ then $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1} = (\phi_\beta \circ p\mid_{W_\beta}) \circ (\phi_\alpha \circ p\mid_{W_\alpha})^{-1} = \phi_\beta \circ p\mid_{W_\beta} \circ (p\mid_{W_\alpha})^{-1} \circ \phi_\alpha^{-1} = \phi_\beta \circ \phi_\alpha^{-1}$, which is holomorphic since $R$ is a Riemann Surface. So these $\tilde{\phi}$ form a complex structure as required.

Definition: let $p : \tilde{R} \to R$ be a "covering" and suppose $\gamma : [0,1] \to R$ is a continuous path. Suppose $w_0 \in \tilde{R}$ is over $\gamma(0)$, i.e. $p(w_0) = \gamma(0)$. Then a lift of $\gamma(t)$ from $w_0$ is a continuous path $\Gamma(t)$ such that $\Gamma(0) = w_0$ and $p \circ \Gamma \equiv \gamma$. A fact from topology is that if a lift $\Gamma$ exists then it is unique.

(If we had the "correct" definition of a covering, then lifts would always exist).

Definition: a "covering" is regular if for any continuous $\gamma : [0,1] \to R$, we have a lift from every point $w_0$ such that $p(w_0) = \gamma(0)$.

$\pi : S(\mathcal{F}) \to \mathbb{C}$ "covers" its image. For $(f,D) \in \mathcal{F}$, $\gamma : [0,1] \to \mathbb{C}$ continuous, $z_0 = \gamma(0) \in D$, $(f,D) \in [f,z_0]$, $\pi([f,z_0]) = z_0$, if $\gamma(t) \in D$ then $\Gamma : [0,1] \to S(\mathcal{F})$ $t \mapsto [f,\gamma(t)]$ is the lift of $\gamma$ from $[f,z_0]$. Generally, if $\gamma : [0,1] \to \pi(S(\mathcal{F})), z_0 = \gamma(0), [f,z_0] \ni (f,D)$ then the lift of $\gamma$ from $[f,z_0]$ (if it exists) is the analytic continuation of $(f,D) \in [f,z_0]$ along $\gamma$; the resulting object is a germ of $\mathcal{F}$ at $z_1 = \gamma(1)$.

[Define] $\pi : S(\mathcal{F}) \to \mathbb{C}$ is regular if there is an analytic continuation along all paths $\gamma$ in $\pi(S(\mathcal{F}))$ from all $[f,z_0]$ with $z_0 = \gamma(0)$.

Facts: 1) A regular "cover" $p : \tilde{R} \to R$ has $\forall x \in R \exists$ a neighbourhood $U \ni x$, $U \subset R$ such that $p^{-1}(U)$ is a disjoint union of connected neighbourhoods in $\tilde{R}$, each homeomorphic to $U$; this property is the topological definition of covering (some sources insist $\tilde{R}$ be connected) (this definition clearly implies our weaker definition). 2) To test for a regular cover: $p : \tilde{R} \to R$ is regular if $\forall z \in R \exists$ a neighbourhood $K \ni z$ such that every connected component of $p^{-1}(\overline{K})$

is compact ($\overline{K}$ being the closure of $K$).

Examples: 1) $z \in \mathbb{C} \mapsto z + \Lambda \in \frac{\mathbb{C}}{\Lambda}$ is a regular cover, 2) $z \in \mathbb{C}^\star \mapsto z^n \in \mathbb{C}^\star$ for some $n \in \mathbb{C}$ is a regular cover - these first two examples pass the aforementioned test, and are covers. 3) The "covering" $\pi : S(\sqrt{1+\sqrt{z}}) \to \mathbb{C}$ is not regular. See question 4 on the fourth example sheet - there is a problem near $z = 1$, $\gamma(t) = 1 - \frac{\epsilon}{2} + \epsilon t$ for $0 \leq t \leq 1$ cannot always be lifted.

Definition: Suppose $\gamma, \sigma : [a, b] \to Y$ are continuous paths in some surface $Y$ with $\gamma(a) = \sigma(a), \gamma(b) = \sigma(b)$. Then a homotopy between $\gamma$ and $\sigma$ is $H(s, t) : [0, 1] \times [a, b] \to Y$ such that $H(s, a) = \gamma(a)\ \overline{(= \sigma(a), \forall s)}$, $H(s, b) = \gamma(b)$, $H(0, t) = \gamma(t)$, $H(1, t) = \sigma(t)$, and $H$ is continuous. $Y$ is simply connected if all $\gamma, \sigma$ with the same endpoints are homotopic (i.e. there is a homotopy $H$ for them).(Equivalently, $Y$ is path-connected and if $\gamma(a) = \gamma(b)$ then $\gamma$ is homotopic to $\sigma(t) \equiv \gamma(a)$.

Fact, which is a theorem in topology: every path-connected surface $R$ admits aregular covering $p : \tilde{R} \to R$ with $\tilde{R}$ simply connected. This $\tilde{R}$ is called the universal regular cover of $R$.

## The Monodromy Theorem

If $p : \tilde{R} \to R$ is a regular covering and $\gamma(t), \sigma(t)$ are homotopic paths in $R$, and $\omega_0 \in \tilde{R}$ is such that $p(\omega_0) = \gamma(a)$ $(= \sigma(a))$, then the lifts $\tilde{\gamma}, \tilde{\sigma}$ of $\gamma, \sigma$ from $\omega_0$ are homotopic; in particular $\tilde{\gamma}(b) = \tilde{\sigma}(b)$. We shall assume this without proof.

Theorem: suppose $U \subset \mathbb{C}$ is open connected, $D = D(z, r) \subset U$, $f : D \to \mathbb{C}$ holomorphic and the germ $[f, z] \ni (f, D)$ has analytic continuation along any path in $U$. Then for any $\gamma, \sigma : [a, b] \to U$ homotopic, $z = \gamma(a) = \sigma(a)$, we obtain the same germ of $f$ at $w = \gamma(b) = \sigma(b)$: $(f, D)$ is $\in$ some complete analytic function $\mathcal{F}$; we have $S(\mathcal{F})$ and $\pi : S(\mathcal{F}) \to \mathbb{C}$ with $U \subset \pi(S(\mathcal{F}))$. Consider $S_U$ defined as the connected component of $[f, z]$ in $S(\mathcal{F})$; the hypothesis about having analytic continuation along any path implies $\pi : S_U \to \mathbb{C}$ is a regular covering, so we have the Monodromy Theorem and hence the result. This can be applied to e.g. $\mathcal{F} = \text{Log}$ or $\mathcal{F} = \sqrt[n]{z}$, which both pass the test of a regular covering.

Corollary: under the hypotheses of the above theorem, if $U$ is simpli-connected, then all analytic continuations of $(f, D)$ in $U$ together give a single-valued holomorphic function $f : U \to \mathbb{C}$: Any $\gamma, \sigma$ with $\gamma(a) = \sigma(a), \gamma(b) = \sigma(b)$ are homotopic, so the germ of $f$ at $w = \gamma(b)$ is unique by the theorem. So e.g. Log, $\sqrt[n]{z}$ define single-valued holomorphic functions over any simply connected $U \subset \mathbb{C} \setminus \{0\}$.

For $a \neq 0 \in \mathbb{C}$, $a\mathbb{Z}$ is an additive subgroup of $\mathbb{C}$. $\frac{\mathbb{C}}{a\mathbb{Z}}$ is, as we know, a Riemann surface (a topological cylinder). For $b \neq 0 \in \mathbb{C}$, consider $\frac{\mathbb{C}}{b\mathbb{Z}}$: $z + a\mathbb{Z} \in \frac{\mathbb{C}}{a\mathbb{Z}} \mapsto \frac{b}{a}z + b\mathbb{Z} \in \frac{\mathbb{C}}{b\mathbb{Z}}$ is a holomorphic bijection (inverse $z + b\mathbb{Z} \mapsto \frac{a}{b}z + a\mathbb{Z}$), so biholomorphic by the inverse mapping theorem.

Now let $\Lambda \subset \mathbb{C}$ be a lattice, $\Lambda' = \alpha\Lambda$ for some $\alpha \neq 0 \in \mathbb{C}$. Then we similarly have $\frac{\mathbb{C}}{\Lambda}$ biholomorphic to $\frac{\mathbb{C}}{\Lambda'}$ by $z + \Lambda \mapsto \alpha z + \Lambda'$; in fact, we have:

Theorem: $\frac{\mathbb{C}}{\Lambda_1}, \frac{\mathbb{C}}{\Lambda_2}$ are conformally equivalent iff $\Lambda' = \alpha\Lambda$ for some $\alpha \in \mathbb{C} \setminus \{0\}$: we have done the reverse implication, for the forward let $f : \frac{\mathbb{C}}{\Lambda_1} \to \frac{\mathbb{C}}{\Lambda_2}$ be a conformal equivalence. Define $\pi_i : \mathbb{C} \to \frac{\mathbb{C}}{\Lambda_i}$ be defined by $z \mapsto z + \Lambda_i$, a holomorphic regular covering. Fix $w_0 \in \mathbb{C}$, defined by $f(\pi_1(0)) = \pi_2(w_0)$.

Suppose $z \in \mathbb{C}$. Choose $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = 0, \gamma(1) = z$. Then $\pi_1(\gamma(t))$ is a path from $\pi_1(0)$ to $\pi_1(z)$¡ so $f(\pi_1(\gamma(t)))$ is a path from $f(\pi_1(0))$ to $f(\pi_1(z))$. Now lift this path from $w_0$, optaining a path $\Gamma(t)$ with $\pi_2(\Gamma(t)) = f(\pi_1(\gamma(t)))$.

If $\tilde{\gamma}$ is homotopic to $\gamma$, we obtain $f(\pi_1(\tilde{\gamma}(t)))$ homotopic to $f(\pi_1(\gamma(t)))$, then by the Monodromy Theorem the lifts $\tilde{\Gamma}(t), \Gamma(t)$ are homotopic; in particular $\tilde{\Gamma}(1) = \Gamma(1)$. Now define $F(z) = \Gamma(1)$, and this is well defined. $F$ is a bijection: we obtain $F^{-1}$ by switching $\Lambda_1$ and $\Lambda_2$, 0 and $w_0$, and $f$ and $f^{-1}$, and applying the same method. $F$ is holomorphic as a function $\mathbb{C} \to \mathbb{C}$: it suffices to prove it is holomorphic near $z_1$. Let $F(z_1) = w_1$, then we have a local holomorphic inverse of $\pi_2 |_{D(w_1,\epsilon)}$, say $\phi$. Then $F |_{D(z,\delta)} = \phi \circ f \circ \pi_1$ is valid for some $\delta > 0$ (as the preimage of $D(w_1, \epsilon)$ is open), so $F$ is locally a composition of holomorphic functions so holomorphic. Then by the inverse function theorem $F$ is biholomorphic. But from complex analysis we know any biholomorphic function $\mathbb{C} \to \mathbb{C}$ is a linear function: $F(z) = az + b$ for some $a \neq 0, b \in \mathbb{C}$.

$F(0) = b = w_0$ by the construction of $F$. Suppose $\omega \in \Lambda_1$, then $\pi_2(F(\omega)) = f(\pi_1(\omega)) = f(\pi_1(0)) = \pi_2(w_0)$ by the definition of $w_0$, so $F(\omega) - w_0 \in \Lambda_2$, i.e. $F(\omega) - F(0) \in \Lambda_2$, i.e. $a\omega \in \Lambda_2$, so $a\Lambda_1 \subset \Lambda_2$, but by the same proof as $F$ is a bijection, $\alpha\Lambda_2 \subset \Lambda_1$ where $\alpha = \frac{1}{a}$ and we have equality as required.

Corollary: there exist uncountably many elliptic curves not biholomorphic to each other.

Remark, for interest: let $\mathcal{M}$ be the set of conformal equivalence classes of elliptic curves; we hope to make it into a "parameter space". It is the set of latticel modulo rescaling. $\forall \Lambda \exists \alpha \in \mathbb{C}^\star$ such that $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ for some $\tau \in$ the upper half-plane $H$. However this is insufficient, as e.g. $\mathbb{Z} + \tau\mathbb{Z} = \mathbb{Z} + (1+\tau)\mathbb{Z}$, or $\mathbb{Z} + i\mathbb{Z} = \mathbb{Z} + (1+i)\mathbb{Z} \sim \mathbb{Z} + (\frac{i-1}{2})$ (exercise: find a suitable value of $\alpha$ to show that this holds). So we need to identify which $\tau$ are equivalent. Consider $PSL(2,\mathbb{Z}) = \frac{\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc=1\}}{\pm 1}$; this acts on $H$ by $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. The generators are $\tau \mapsto \tau + 1, \tau \mapsto -\frac{1}{\tau}$, with corresponding matricies $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ respectively. We can show that $\mathcal{M}$ bijects with the orbits of this action of $PSL(2,\mathbb{Z})$ on $H$. So we need to find the fundamental region, which turns out to be $D = \{\tau \in H : |\Re\tau| \leq \frac{1}{2}, \Re\tau \neq -\frac{1}{2}, |\tau| \geq 1, |\tau| > 1 \text{ if } \Re\tau < 0\}$. Note that the boundaries of this region are identified with their reflections in the imaginary axis, so the resulting surface is a "cigar" shape, with a tube going upwards to infinity but closing up to a line along the bottom.

# The Uniformization Theorem

Definition: a subgroup $G$ of homeomorphisms of a surface $X$ acts properly discontinuously on $X$ if $\forall x \in X \exists$ a neighbourhood $U \ni x$ such that $g_1(U) \cap g_2(U) = \emptyset \forall g_1 \neq g_2 \in G$.

Recall that for any Riemann surface $S$ there is a universal regular cover $\tilde{S}, p : \tilde{S} \to S$ such that $\tilde{S}$ is a simply connected Riemann surface and $p$ is holomorphic.

The uniformization theorem: 1. any simply connected Riemann surface is biholomorphic to either $S^2$ or $\mathbb{C}$ or $\Delta$ (the hard part), 2. (the easy part) every connected Riemann surface is conformally equivalent to a quotient of a simply connected Riemann surface by a subgroup of biholomorphisms acting properly discontinuously.

Corollary: every compact simply connected Riemann surface is biholomorphic to $S^2$ (as we could also deduce from Riemann-Roch.

So for every Riemann surface $S$ there is an $\tilde{S}$ as described with $S$ biholomorphic to $\frac{\tilde{S}}{G}$; we say $\tilde{S}$ uniformizes $S$.

The proof of the first part of the Uniformization Theorem uses some hard analysis applied to PDEs, beyond the scope of this course. There is one book at this level giving a proof, listed in the schedules; otherwise it is necessary to go to actual relearch papers. A part of the proof is the Riemann Mapping Theorem: any simply connected $U \subsetneq \mathbb{C}$ is biholomorphic to $\Delta$. Another part is the theorem on boundary behaviour, which is useful for question 9 on the fourth example sheet: for $D \subsetneq \mathbb{C}$ simply connected, suppose the boundary $\partial D \supset$ some interval $I = \{x \in \mathbb{R} : a < x < b\}$, and $\forall x \in I \exists D(x, \epsilon)$ such that a) $\partial D \cap D(x, \epsilon) = I \cap D(x, \epsilon)$ and b) only one [component of $D(x, \epsilon) \setminus I$] is contained in $D$. Then any biholomorphic map $D \to \Delta$ extends continuously to a homeomorphism $D \cup I \to \Delta \cup$ some arc of the unit circle.

For more on the uniformization theorem, see Ahlfors, 3rd edition, pages 230-234 and 168-173.

In the second part of the uniformization theorem, we have that any $S$ is $\frac{\tilde{S}}{\Gamma}$ for $\Gamma \subset \mathrm{Aut}(\tilde{S})$ acting properly discontinuously. $\mathrm{Aut}S^2 = \frac{SL(2,\mathbb{C})}{\pm 1}$, the set of Möbius maps, $\mathrm{Aut}\mathbb{C} = \{z \mapsto az + b : a \neq 0\}$ and as seen in question 3 on the fourth example sheet, $\mathrm{Aut}\Delta = \frac{SU(1,1)}{\pm 1}$; all these are subgroups of Möbius transformations.

Every Möbius transformation has a fixed point (in $S^2$); recall that we want $\forall x \in S^2$ a neighbourhood $U \ni x$, $U \subset S^2$ such that $g_1(U) \cap U = \emptyset \forall g_1 \in \Gamma$, but if $x$ is fixed, $g_1(x) = x$, then this is impossible. So $S^2$ only uniformizes itself.

In $\mathbb{C}$, if have no fixed points then we must have $a = 1$, so $\Gamma$ is a set of translations $z \mapsto z + b$; it can be shown that if $\Gamma$ is nontrivial then it must $= \omega\mathbb{Z}$ or a lattice. So the surfaces uniformized by $\mathbb{C}$ are $\mathbb{C}$, $\mathbb{C}^\star$ (exercise: cylinders are equivalent to $\mathbb{C}^\star$ - the two infinitely far away ends of the cylinder correspond to 0 and $\infty$), and $\frac{\mathbb{C}}{\Lambda}$. So all other Riemann surfaces are $\frac{\Delta}{\Lambda}$ for some $\Lambda$. Unsurprisingly, the "discrete" $\Gamma \subset \frac{SU(1,1)}{\pm 1}$ give a nontrivial study area; they are called Fuchsian groups.

It is important to note that $\mathbb{C}$ is not biholomorphic to $\Delta$; for example, $\mathrm{Aut}\mathbb{C}$ has a free abelian subgroup of rank 2, $\Lambda$, but it can be checked that the only discrete abelian $T \subset \mathrm{Aut}\Delta$ are cyclic.

The $g = 0$ Riemann surface is $S^2$; the $g = 1$ Riemann surface is uniformized by $\mathbb{C}$. The Riemann surfaces of $g \geq 2$ must be uniformized by $\Delta$; finding the relevant holomorphic universal regular cover can be quite complicated. See question 9 on the fourth example sheet.

A non-examinable sketch of the proof of the second part of the uniformization theorem: we consider cover transformations. These are, for $p : \tilde{S} \to S$, homeomorphisms $q : \tilde{S} \to \tilde{S}$ such that $p \circ q = p$. These are also sometimes called deck transformations - the analogy is that the multiple preimages of redions of $S$ correspond to the decks of a ship.

Step 1: for any cover transformation $q$ we have $q \in \mathrm{Aut}\tilde{S}$, $q$ biholomorphic.

Step 2: cover transformations form a subgroup $\Gamma$ of $\mathrm{Aut}\tilde{S}$ acting properly discontinuously.

Step 3 (transitivity): if $\omega_1, \omega_2 \in \tilde{S}$ are such that $p(\omega_1) = p(\omega_2)$, then there is a unique cover transformation $q$ such that $q(\omega_1) = \omega_2$. Corollary: $S \cong \frac{\tilde{S}}{\Gamma}$ by $z \in S \mapsto p^{-1}(z) \in \frac{\tilde{S}}{\Gamma}$; this is a homeomorphism where $\frac{\tilde{S}}{\Gamma}$ has the quotient topology: $V \subset \frac{\tilde{S}}{\Gamma}$ is open iff $\pi^{-1}(U)$ is open in $\tilde{S}$, where $\pi : x \in \tilde{S} \mapsto \Gamma(x) \in \frac{\tilde{S}}{\Gamma}$. Also, $\frac{\tilde{S}}{\Gamma}$ is Hausdorff.

Step 4: $\frac{\tilde{S}}{\Gamma}$ is a Riemann surface such that $\pi$ is a holomorphic map; furthermore $S$ is biholomorphic to $\frac{\tilde{S}}{\Gamma}$.

A remark for interest: there are three classes of compact Riemann surfaces $S$: we could consider these as $g = 0, g = 1, g > 1$, but it is more natural (and more justifiable, since "0,1,¿1" is not a natural set of classes) to consider these in terms of $\chi(S)$; then our cases are $\chi > 0, \chi = 0, \chi < 0$. Respectively these are uniformized by $S^2, \mathbb{C}, \Delta$; we have respectively a unique example, a 1D connected parameter space (in fact homeomorphic to $\mathbb{C}$) of possibilities, and many possibilities non-homeomorphic to each other. Now consider holomorphic maps to $S^2$: we have respectively any degree occurs, degree 2 always occurs (by e.g. the $\wp$-function), there is only a holomorphic map of degree 2 onto $S^2$ in "exceptional" cases. The constraints on zeroes and poles for these are respectively that the numbers of each are equal, more complicated but tractable constraints, and intricate constraints requiring Riemann-Roch. The curvature is respectively positive, flat and negative (recall that we have a metric under which the disc is a model of the hyperbolic plane).

This classification extends to higher dimensional manifolds; in the first two cases we obtain respectively Fano, Calabi-Yau spaces. In the last case, there are enough possibilities to fill another course.

This is the end of this course. Papers from the last three years are entirely suitable for revision; before that, the course was given in a shorter format, so papers from earlier years may miss some topics.