Representation Theory

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1 Lecture

The example sheets for this course will be on the lecturer's web page.

Some recommended books are J. Alperim and R. Bell's Groups and Representations, I. Issacs' Character Theory of Finite Groups, G. James and M. Liebeal's Representations and Characters of Groups, the classic J-P Serre's Linear Representation of Finite Groups, and M. Artin's Algebra. There is a good set of notes for this course, by C. Toloman, on the DPMMS teaching page.

We view a group as a way to describe the "symmetry" of an object, e.g. the symmetric group S_n of permutations of the set $\{1, \ldots, n\}$ (for some $n \geq 1$; a permutation of a set X is a bijective $f : X \to X$), or the alternating group $A_n \subset S_n$, the set of products of an even number of transpositions $(ij) \in S_n$ (equivalently this is the set of $\sigma \in S_n$ such that the number of pars $i < j$ with $\sigma(i) > \sigma(j)$ is even).

The cyclic group $C_n \cong \frac{\mathbb{Z}_n}{n}$ is the set of rotations preserving a regular n-gon; the dihedral group D_{2n} (which we take as of order $2n$) is the group of rotations and reflections of the n-gon (note that in general this is a proper subgroup of S_n)

The permutation group of a (possibly infinite) set X is the set $\mathrm{Perm}(X)$ of all permutations of X. For example, we can see the gruop $SO(2)$ of all rotations of the plane about the origin as the symmetry group of a circle S^1 in Euclidean geometry; note that it = the circle group S^1.

Example: the group of all isometries of the plane is $\{f : \mathbb{R}^2 \to \mathbb{R}^2 : d(f(x), f(y)) = d(x, y) \forall x, y \in \mathbb{R}^2\}$; it is generated by rotations, reflections and translations. Note that it is equal to the "semidirect product" $O(2) \ltimes \mathbb{R}^2$.

One can define a permutation group to mean $G \subset \mathrm{Perm}(X)$ (for some set X) which is closed, contains the identity and contains inverses of all its elements; it is a fact that composition of permutations is always associative. Another fact is that every group can be viewed as a permutation group.

Definition: let G be a group, X a set; an action of G on X is a function $G \times X \to X$, written gx [or $g(x)$] such that $1x = x \forall x \in x, (gh)(x) = g(h(x)) \forall g, h \in G, x \in X$.

1.1 Proposition

An action of a group G on a set X is equivalent to a homomorphism $G \to \mathrm{Perm}(X)$: just notice that for each $g \in G$, $f_g : X \to X$ defined by $f_g(x) = gx$ is a bijection since it has an inverse function $f_{g^{-1}}$.

Examples: Perm(X) acts on X in the obvious way. $SO(3)$ the group of rotations of $\mathbb{R}^3$ about the origin has finite subgroups [which act on $\mathbb{R}^3$] given by the groups of symmetries of the platonic solids: A_4 for the tetrahedron, S_4 for the cube or octahedron, and A_5 for the icosahedron or dodecahedron.

Definition: let F be a field, V a vector space thereover. Then a (linear) representation of a group G on V is an action of G on V such that $\forall g \in G$ the function $g : V \to V$ is linear.

Examples: we've described linear representations of C_n, D_{2n} on $\mathbb{R}^2$ and of $SO(2)$ on $\mathbb{R}^2, SO(3)$ on $\mathbb{R}^3$. We've also seen representations of S_4, A_4, A_5 on $\mathbb{R}^3$.

For any vector space V over F the set End(V) of linear maps $V \to V$ is a ring under pointwise addition $(f + g)(x) = f(x) + g(x)$ and composition as multiplication $fg(x) = f(g(x))$. The set of invertible elements therof is a group under multiplication, called the general linear group $GL(V)$ $(= \mathrm{Aut}(V))$ (So $GL(V)$ is the group of bijective linear maps $V \to V$ under composition)

Remark: For any group G and vector space V, linear representations of G on V are equivalent to homomorphisms $G \to GL(V)$.

The nicest case is $F = \mathbb{C}$; we will be able to describe all possible $\mathbb{C}$-representations of lots of interesting groups; often this is even very easy. This is one of the most useful theories in all of algebra in terms of its applications in e.g. quantum mechanics.

Definition: For V_1, V_2 represetations of a group G (over a field F), an isomorphism (of representations) $f : V_1 \to V_2$ is a linear isomorphism such that $f g_{V_1} f^{-1} = g_{V_2} \forall g \in G$, i.e. G acts "the same way" on V_1, V_2 if they are identified under f.

Example: let $V = F^n = \{(z_1, \ldots, z_n) : z_i \in F\}$ for some $n \geq 1$; then $GL(F^n)$ is called the group $GL(n, F)$. We have $\mathrm{End}(F^n) = M_n(F)$ the set of $n \times n$ matricies with coefficients in F.

2 Lecture

2.1 Proposition

For any group G, the set of isomorphism classes of n-dimensional complex representations of G can be identified with $\frac{\mathrm{Hom}(G, GL(n,\mathbb{C}))}{GL(n,\mathbb{C})}$, where $GL(n, \mathbb{C})$ acts on the set of homomorphisms $G \to GL(n, \mathbb{C})$ by conjugation $a(\rho)(g) = a\rho(g)a^{-1}$ (the quotient of a set by a group acting on it is the set of orbits): given any representation on an n-dimensional vector space V, choose a basis for V, giving an isomorphism $V \to \mathbb{C}^n$; thus we have a homomorphism $G \to \mathrm{Aut}(\mathbb{C}^n) = GL(n, \mathbb{C})$; the choice of basis for V changes the homomorphism by conjugation by an element of $GL(n, \mathbb{C})$.

Representations of the group $\mathbb{Z}$

A homomorphism $\rho : \mathbb{Z} \to GL(n, \mathbb{C})$ is uniquely determined by $\rho(1) \in GL(n, \mathbb{C})$, and $\rho(1)$ may be any $A \in GL(n, \mathbb{C})$; thus classifying n-dimensional representations of $\mathbb{Z}$ (over $\mathbb{C}$) is equivalent to classifying invertible $n \times n$ matricies up to conjugation.

2.2 Theorem: Jordan Canonical Form

Any $A \in M_n(\mathbb{C})$ is conjugate to a matrix in JCF: let a Jordan block mean an $r \times r$ matrix of the form $\begin{pmatrix} a & 1 & & \\ & a & \dots & \\ & & \dots & 1 \\ & & & a \end{pmatrix}$ for some $a \in \mathbb{C}, r \geq 1$. Then a matrix is in JCF if it has the form $\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \dots & \\ & & & J_n \end{pmatrix}$ where the J_i are Jordan blocks. The JCF of a matrix is unique up to reordering of the blocks.

Note that the determinant of a matrix in JCF is simply the product of its diagonal entries; thus such a matrix is invertible iff all of the a are nonzero; the isomorphism classes of representations of $\mathbb{Z}$ over $\mathbb{C}$ biject with matricies in JCF with nonzero diagonal entries, where we only distinguish up to reordering of blocks.

Examples: A 1D representation of $\mathbb{Z}$ is specified by $a \in \mathbb{C}^\star$: $\rho(1) = a \in \mathbb{C}^\star = GL(1, \mathbb{C})$. A 2D representation of $\mathbb{Z}$ is isomorphic either to some representation $\rho(1) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ for some $a, b \in \mathbb{C}^\star$ or one of the form $\rho(1)] = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ for some $a \in \mathbb{C}^\star$; two different representations of these forms are isomorphic only by $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \sim \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$.

Definition: for any grop G, the trivial representation is the homomorphism $\rho : G \to GL(1, \mathbb{C})$ given by $\rho(g) = 1 \forall g \in G$.

Definition: A representation of a group G is faithful if the corresponding homomorphism $G \to GL(V)$ is injective.

Example: the representation of $\mathbb{Z}$ given by $\rho(1) = a$ for $a \in \mathbb{C}$ is faithful iff a is not a root of unity.

Representations of Finite Cyclic Groups

Let $G = C_n = \frac{\mathbb{Z}}{n} = \langle g : g^n = 1 \rangle = \{1, g, \dots, g^{n-1}\}$. A homomorphism $G \to GL(r, \mathbb{C})$ is determined by $\rho(g) \in GL(r, \mathbb{C})$, which can be any matrix A with $A^n = 1$; thus we have to classify such matricies up to conjugation. We can conjugate A into JCF, then we have $A^n = \begin{pmatrix} J_1 & \\ & \dots \end{pmatrix}^n = \begin{pmatrix} J_1^n & & \\ & J_2^n & \\ & & \dots \end{pmatrix}$; write $J = aI + B$ where $a \in \mathbb{C}^\star, B = \begin{pmatrix} 0 & 1 & & \\ & 0 & \dots & \\ & & \dots & 1 \\ & & & 0 \end{pmatrix}$. Then since I commutes with anything, including B, we can use the binomial theorem to write $J^n = (aI + B)^n = a^n I + na^{n-1}B + \binom{n}{2} a^{n-2} B^2 + \dots$; if this is to $= I$ then we must have $a^n = 1$ and also $na^{n-1}B = 0$, but this implies $B = 0$ and J is a 1×1 block. So any representation of a cyclic group is isomorphic to one of the form

$$\rho(g) = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \cdots & \\ & & & a_n \end{pmatrix}$$ where each $a_i \in \mathbb{C}$ is an nth root of unity (conversely, clearly any such is a valid representation). Two such representations are isomorphic iff the numbers a_i are the same up to reordering.

Definition: For any representations V, W of G, the direct sum $V \oplus W = V \times W = \{(v, w) : v \in V, w \in W\}$ is also a representation of G, by $g(v, w) = (gv, gw)$ [check].

Definition: A subrepresentation W of a representation V of G is a linear subspace $W \subset V$ such that $\forall g \in G \forall w \in W g(W) \in W$, i.e. an invariant subspace of V.

Example: $V \oplus V$ contains $V \oplus 0 \simeq V$ and $0 \oplus W \simeq W$ as subrepresentations.

We have shown above that every finite dimensional representation of a cyclic group C_n over $\mathbb{C}$ is $\simeq$ a direct sum of 1-dimensional representations $L_1 \oplus \cdots \oplus L_r$. In general, G acts on $V \oplus W$ by $g \mapsto \begin{pmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{pmatrix}$ in terms of bases for V and W.

Definition: A representation V of a group G is irreducible if $V \neq 0$ and any G-invariant subspace of V is either 0 or V [i.e. there are no nontrivial subrepresentations?]

Definition: A representation of G is completely reducible if it is isomorphic to a direct sum of irreducible representations; we have shown that all representations of a finite cyclic group are completely reducible, and the irreducible representations of C_n are all 1-dimentional, each given by an nth root of unity.

Finite abelian groups

Every finite abelian group G is (isomorphic to) a product of cyclic groups; more, G can be written uniquely (up to order of factors) as a product of $\frac{\mathbb{Z}}{p^r}$ for distinct primes p.

A representation of $G = \frac{\mathbb{Z}}{a_1} \times \cdots \times \frac{\mathbb{Z}}{a_r}$ is given by matricies $A_1, \ldots, A_r \in GL(n, \mathbb{C})$ such that $A_i^{a_i} = 1 \forall i$ and the A_i all commute; to classify the representations of G up to isomorphism means classifying the sets of such A_i up to $(A_1, \ldots, A_r) \mapsto (gA_1g^{-1}, \ldots, gA_rg^{-1})$ for some $g \in GL(n, \mathbb{C})$.

2.3 Proposition

Any family of commuting matricies each diagonalizable can be simultaneously diagonalized. The A_r are complex representations of finite cyclic groups so diagonalizable, so we may take the A_r all diagonal, and then we automaticly have that they commute. Say each A_i has entries $u_{i1}, \ldots, u_{in}$ along the diagonal; then we need $(u_{ij})^{a_i} = 1 \forall j \forall i$.

2.4 Proposition

Every complex representation of a finite abelian group is completely reducible, and its irreducible representations are 1-dimensional; if $G = \frac{\mathbb{Z}}{a_1} \times \cdots \times \frac{\mathbb{Z}}{a_r}$ then a 1-dimensional representation of G is specified by an a_ith root of unity for some i.

3 Lecture

3.1 Theorem

Every (finite-dimensional complex) representation of a finite group is completely reducible: for a representation V of a group G, V is reducible iff the homomorphism $G \to GL(V)$ is conjugate to a representation $G \to \{\begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix}\} \subset GL(n, F)$; the matricies in this form are precisely those that map $\{\begin{pmatrix} 0 \\ \dots \\ 0 \\ \star \\ \dots \\ \star \end{pmatrix}\} \subset V$ into itself.

To prove that a representation V of a group G is completely reducible, we have to show that it is conjugate to a representation $G \to \{\begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix}\} \subset GL(n, F)$ [presumably this should have been phrased more clearly - I think this needs to be the case whenever V is reducible]

Example: the representation $\mathbb{Z} \to GL(2, \mathbb{C})$ given by $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ is not completely reducible: thinking geometricly, we have a representation of G on a space V with an invariant line $W \subset V$ but such that there is no G-invariant linear subspace $T \subset V$ with $V = W \oplus T$.

Examples of representations

Let G be a finite group. The regular representation of G is the vector space $\mathbb{C}G = \{\sum_{i \in G} a_g g : a_g \in \mathbb{C}\}$ (i.e. the space with basis G), which G acts on by left multiplication. The regular representation is always faithful; it will turn out that for finite G every irreducible representation of G is a subrepresentation of the regular representation.

Let G be a finite group acting on a finite set X; the permutation representation of G is the vector space $\mathbb{C}X = \{\sum_{x \in X} a_x x : a_x \in \mathbb{C}\}$, with G acting on it in the obvious way.

Recall that a hermitian form (or inner product) on a complex vector space V is a function $\langle , \rangle : V \times V \to \mathbb{C}$ which is linear in its first argument, has $\langle x, b_1 y_1 + b_2 y_2 \rangle = \bar{b}_1 \langle x, y_1 \rangle + \bar{b}_2 \langle x, y_2 \rangle$, has $\langle x, y \rangle = \overline{\langle y, x \rangle}$ and has $\forall x \neq 0 \in V, \langle x, x \rangle > 0$.

Definition: Let V be a complex vector space with an inner product. A representation of a group G on V is unitary if $\forall g \in G \forall x, y \in V, \langle gx, gy \rangle = \langle x, y \rangle$, i.e. the matrix representation of g is unitary. A complex representation V of G is- unitarizable if G preserves some inner product on V.

Example: Any permutation representation of a group G is unitarizable, by defining an inner product on $\mathbb{C}X$ such that the basis set X is orthonormal.

3.2 Proposition

Any finite dimensional unitary representation V of any group G is completely reducible: suppose not, $V \neq 0$ and V not irreducible and $W \subset V$ a G-invariant subspace (i.e. $gw \in W \forall w \in W \forall g \in G$), $0 \neq W \neq V$. But then $W^\perp = \{x \in V : \langle x, w \rangle = 0 \forall w \in W\}$ is a G-invariant complement to W in V (because $\langle , \rangle$ is positive definite we must have $W \cap W^\perp = 0$ and then by dimensions $V = W \oplus W^\perp$), so V is the direct sum of two lower-dimensional representations, and by induction on $\dim V$ we have the result.

3.3 Lol, the lecturer forgot his numbering

3.4 Proposition (Weyl's unitary trick

All finite dimensional complex representations of a finite group G are unitarizable: we "average": pick an inner product on V. Then define a new inner product by $\langle x, y \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle gx, gy \rangle$; this is a hermitian form, positive definite and so on. But it is also G-invariant [proof is really obvious, and I'm tired].

3.5 Proposition

Let V be a finite dimensional representation of a finite group G over a field F; suppose either $\text{char} F = 0$ or $\text{char} F = p$ with $p \nmid |G|$. Then V is completely reducible: as a sketch proof, we try to average as before; suppose we have V and W as in 3.2. Let p be a projection map $V \to W$; this determines a (not necessarily G-invariant) complement to W by $\ker p$. Then define a new projection $V \to W$ by $\sigma(x) = \frac{1}{|G|} \sum_{g \in G} gpg^{-1}x$; the reader may verify this is a projection $V \to W$ and $\ker \sigma$ is G-invariant.

4 Schur's Lemma

[Thanks to a substitute lecturer, we have an actual definition: a linear representation of a group G over a field F is a pair (ρ, V) where V is a vector space over F and $\rho : G \to GL(V)$ a homomorphism $g \to \rho(g)$ (we usually just write g for $\rho(g)$) where $\rho(g) : V \to V$ is a linear isomorphism]

We want to consider G-homomorphisms between representations (ρ_i, V_i). A linear $\phi : V_1 \to V_2$ is a G-homomorphism if $\phi(g(r)) = g\phi(r) \forall r \in V \forall g \in G$, i.e. ϕ commutes with the action of G. Write $\text{Hom}^G(V_1, V_2)$ for the F-vector space of G-homomorphisms or "intertwiners" $V_1 \to V_2$.

4.1 Lemma

For V, W irreducible G-spaces over a field F, any G-homomorphism $V \to W$ is either 0 or a G-isomorphism. If V is an irreducible complex (or, in fact, if F is any algebraicly closed field) G-space then any G-endomorphism on V is a scalar endomorphism: Let $\phi : V \to W$ be a G-homomorphism, then $\ker \phi$ is a G-subspace of V, so either $= V$ or $= \{0\}$; in the first case $\phi = 0$, in the second case $\text{Im}\phi$ is a G-subspace of W, so must $= W$, so ϕ is an isomorphism. If ϕ is a G-endomorphism on a complex irreducible G-space V let λ be an eigenvalue of

ϕ (we must have one by algebraic closure), then the eigenspace E_λ is a nonzero G-subspace of V, so must $= V$ and $\phi = \lambda$.

4.2

If V, W are irreducible complex G-representations then $\dim_{\mathbb{C}} \mathrm{Hom}^G(V, W) = 1$ if V, W G-isomorphic, 0 otherwise: if $\phi : V \to W$ is a nonzero G-homomorphism then ϕ is injective ($\ker \phi$ is a non-V G-subspace of V) and surjective ($\mathrm{Im}\phi$ is a nonzero G-subspace of W) so ϕ is a G-isomorphism; if ψ is another such then $\phi^{-1}\psi$ is a G-automorphism on V so $= \lambda\iota$ for some $\lambda \in \mathbb{C}$ and $\psi = \lambda\phi$.

Define $\mathrm{End}^G(V) = \mathrm{Hom}^G(V, V)$. We have seen this is a vector space over F; it is a ring under composition, so forms an algebra over F:

Definition: an (associative) ring (with a unity) containing a distinguished copy of the field F commuting with every element of the ring and with 1 in this copy of F being the unity of the ring, is an algebra over F; it is a division algebra if it is also a division ring (i.e. every nonzero element has a multiplicative inverse).

4.3

If V is an irreducible G-representation over F then $\mathrm{End}^G(V)$ is a division algebra: the reader can check it is an algebra over F with the distinguished copy of F being $\{\lambda\iota : \lambda \in F\}$; it is a division ring since if ϕ is a nonzero element of $\mathrm{End}^G(V)$ then ϕ is an isomorphism so has an inverse.

Schur's lemma over $\mathbb{C}$ [which the lecturer doesn't seem to bother to state, anywhere], now follows from:

4.4

The only finite dimensional division algebra over $\mathbb{C}$ is $\mathbb{C}$ itself: let A be a finite dimensional division algebra over $\mathbb{C}$. For any $\alpha \in A$, the elements $\iota, \alpha, \alpha^2, \ldots$ must be linearly dependent, so there is some nonzero complex polynomial p with $p(\alpha) = 0$; since $\mathbb{C}$ is algebraicly closed we can factorise this as $p(x) = (x - \alpha_1) \ldots (x - \alpha_n)$ for some $\alpha_i \in \mathbb{C}$, then $p(\alpha) = (\alpha - \alpha_1) \ldots (\alpha - \alpha_n) = 0$ and since A is a division algebra we have $\alpha = \alpha_j$ for some j, so $\alpha \in \mathbb{C}$.

Remark: There are three finite dimensional division algebras over $\mathbb{R}$, namely $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

Applications

4.5

Any irreducible complex representation of a finite abelian group G is 1D (another proof of this): let V be such a representation. For $g \in G$ the corresponding map $V \to V$ is a g-endomorphism on V (as G is abelian [it must have an inverse]) so $= \lambda_g\iota$ for some $\lambda_g \in \mathbb{C}$; thus each 1D subspace of V is G-invariant. Since V is irreducible it follows that V is 1D. Note that this fails over $\mathbb{R}$, e.g. on the example sheet we show that C_3 has a 2-dimensional irreducible representation.

4.6 Lecturer can't count, I can't latex

4.7 Exercise

If the finite group G has a faithful irreducible representation ρ over $\mathbb{C}$ then the centre $Z(G)$ is cyclic; in fact $\rho(Z(G)) = \langle e^{\frac{2\pi i}{n}} \iota \rangle$ where $n = |Z(G)|$.

5 Lecture

Recall by Schur's lemma the irreducible representations of a finite abelian group G over $\mathbb{C}$ are 1-dimensional.

Example: $G = \frac{\mathbb{Z}}{4} = \langle g : g^4 = 1 \rangle$; there are 4 irreducible representations given by $g \mapsto$ any 4th root of 1; those given by $\pm i$ are faithful, the other two are not.

Example: $G = \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2} = \langle g, h : g^2 = 1 = h^2 \rangle$; there are 4 irreducible representations of G given by $g \mapsto \pm 1, h \mapsto \pm 1$, none of which are faithful.

Remark: for any finite abelian group, the number of irreducible representation of $\mathbb{G}$ (over $\mathbb{C}$, and up to isomorphism) is equal to $|G|$. However, beware: there is no "natural" 1:1 correspondence between the elements of G and the irreducible representations of G, even given that G is finite abelian. If we choose an isomorphism $G \simeq \frac{\mathbb{Z}}{a_1} \times \cdots \times \frac{\mathbb{Z}}{a_r}$ then we can identify them, but this identification is dependent on our choice of isomorphism.

This statement has two interesting generalizations to a general finite group G: the number of isomorphism classes of irreducible representations of G is equal to the number of conjugacy classes of G. Also, if $V_1, \ldots, V_r$ are the irreducible representations of a finite group G, then $|G| = \sum_{i=1}^{n} \dim(V_i)^2$.

Isotypical Decomposition

Let V be a representation (over $\mathbb{C}$) of a finite cyclic group $G = \frac{\mathbb{Z}}{n}$. Then the generator $g \in G$ acts by a linear map $V \to V$, which we know is diagonalizable.

Let $\xi = e^{\frac{2\pi i}{n}}$; the eigenvalues of G on V are then $\in \{1, \xi, \ldots, \xi^{n-1}\}$. V has a unique decomposition into eigenspaces for G as $V = \bigoplus_{i=0}^{n-1} V(i)$ where $V(i)$ is the ξ^i-eigenspace for $g \in GL(V)$ (i.e. $\{x \in V : gx = \xi^i x\}$). We can think of $V(i) \subset V$ as the sum of all the copies of a certain irreducible representation of G that sit inside V.

Let G be any finite group, and V any complex representation of G. We know that V is completely reducible, i.e. is a direct sum of irreducible representations. So we can write $V = \bigoplus_k W_k^{\oplus m_k}$ [$W_k^{\oplus m_k} = W_k \oplus \cdots \oplus W_k$ m_k times], where $W_1, W_2, \ldots$ are non-isomorphic irreducible representations of V and $m_k \geq 0$. The individual "pieces" $W_k \subset V$ are not at all unique, but the subspace $W_k^{\oplus m_k}$ is uniquely determined; the resulting decomposition of V is called the isotypical decomposition of V.

5.1 Lemma

Let V, V' be complex representations of a finite group G and $f : V \to V'$ a G-linear map (i.e. a linear map commuting with all elements of G). Write $V = \bigoplus_k W_k^{\oplus m_k}, V' = \bigoplus_k W_k^{\oplus m_k}$ (possibly with some of the m_k and/or m'_k being 0), where the W_i are irreducible representations of G. Then f maps the subspace $W_k^{\oplus} m_k$ into $W_k^{\oplus m'_k}$: consider, for each irreducible summand $W_k \subset V$ and $W_l \subset V'$, the composition $W_k \subset V \xrightarrow{f} V' \twoheadrightarrow W_l$ (where $\twoheadrightarrow$ denotes projection); this is a composition of G-linear maps so G-linear. By Schur's lemma it is 0 if $k \neq l$. The linear map $V \to V'$ is given by a block matrix $\begin{pmatrix} W_1 \to W_1 & W_1 \to W_2 & \dots \\ W_2 \to W_1 & W_2 \to W_2 & \dots \\ \dots & \dots & \dots \end{pmatrix}$, so this must be block-diagonal.

5.2 Theorem

Let V be a representation of a finite group. Let $V = \bigoplus V_k$ be a decomposition as we have described. Then 1) this decomposition is unique, independent of how we first decomposed V into irreducibles 2) Every subrepresentation of V which is isomorphic to W_k is contained in V_k 3) The endomorphism algebra $\mathrm{End}^G(V_k)$ is isomorphic to the matrix algebra $M_{m_k}(\mathbb{C})$, coming from our choice of decomposition $V_k = W_k^{\oplus m_k}$ 4) $\mathrm{End}^G(V) = \prod_k M_{m_k}(\mathbb{C})$ (as an algebra over $\mathbb{C}$):

First we proove 2): suppose we have a subrepresentation of V isomorphic to W_k, and consider this as a G-linear map $W_k \hookrightarrow V$; by Lemma 5.1 this must map W_k into $V_k \subset V$. Now 1): notice $V_k \subset V$ is the (non-direct) sum of all subrepresentations of V isomorphic to the irreducible representation W_k; thus it is independent of our original choice of decomposition $V_k = W_k^{\oplus m_k}$. For 3) we need to describe all G-linear maps $V_k \to V_k$ where we can write $V_k = W_k^{\oplus m_k}$ with W_k irreducible; by Schur's lemma any G-linear map $W_k \to W_k$ is a scalar. We have $\mathrm{End}(V_k) = \{\begin{pmatrix} W_k \to W_k & W_k \to W_k & \dots \\ W_k \to W_k & W_k \to W_k & \dots \\ \dots & \dots & \dots \end{pmatrix}\}$; to say that a linear map $V_k \to V_k$ is G-linear means every block (of f considered as an element of $\mathrm{End}(W_k)$) is G-linear, so $\mathrm{End}^G(V)$ is just the set of those block-matricies with each entry a $\dim(W_k) \times \dim(W_k)$ diagonal matrix with all the diagonal entries the same scalar value, so isomorphic to $M_{m_k}(\mathbb{C})$.

For 4), we know that any G-linear map $f : V \to V$ must (by Lemma 5.1) map each V_k into itself. Therefore f is just a block matrix with its blocks being G-linear maps $V_1 \to V_1$, $V_2 \to V_2$ and so on respectively; thus it is isomorphic as described.

The Dual Representation

Let V be a representation $/\mathbb{C}$ of a group G; recall the dual space is $V^\star = \mathrm{Hom}(V, \mathbb{C})$; this is also a representation of G by $(gf)(x) = \overline{f(g^{-1}x)} \forall x \in V, f \in V^\star, g \in G$. The inverse is needed so that G acts on $V^\star$, which we can now see is the case (but would not otherwise be so): $(g_1g_2)(f)(x) = f((g_1g_2)^{-1}(x)) = f(g_2^{-1}g_1^{-1}x) = g_2f(g_1^{-1}x) = g_1(g_2f)(x)$.

5.3 Proposition

For a representation V of a group G (taking everything finite as usual) V is an irreducible representation of G iff $V^\star$ is: will show V reducible $\Rightarrow V^\star$ reducible (which suffices as $(V^\star)^\star = V$). We say we have a G-invariant S with $0 \subsetneqq S \subsetneqq V$;

set $T \subset V^\star$ be the set of linear $f : V \to \mathbb{C}$ that are zero on S, then $0 \subsetneq T \subsetneq V^\star$ and T is G-invariant (note $\dim T = \dim V - \dim S$).

More generally, for any representations V, W of G, $\mathrm{Hom}(V, W)$ (note its dimension is $\dim V \times \dim W$) is also a representation of G: for $f : V \to W$ linear, define $(gf)(x) = gf(g^{-1}x)$; then G acts linearly on $\mathrm{Hom}(V, W)$.

Notice that the subspace of $\mathrm{Hom}(V, w)$ where G acts trivially is precisely $\mathrm{Hom}^G(V, W)$: if $gf = f \forall g \in G$ then $f(x) = gf(g^{-1}x) \forall x \in V$, i.e. $g^{-1}f(x) = f(g^{-1}x)$ so f is G-linear.

6 Tensor Products

Given vector spaces V, W (fin dim $/\mathbb{C}$) define $V \otimes W$ (or $V \otimes_{\mathbb{C}} W$) to be the complex vector space with basis consisting of all pairs $v \in V, w \in W$, written $v \otimes w$, modulo the linear subspace spanned by the relations $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, (aV) \otimes w = v \otimes (aw) = a(v \otimes w) \forall a \in \mathbb{C}$.

6.1 Lemma

Let $e_1, \ldots, e_a$ be a basis for V, $f_1, \ldots, f_b$ a basis for W, then $V \otimes W$ has basis $\{e_i \otimes f_j\}$ (so it has dimension $\dim V \times \dim W$): by definition every element of $V \otimes W$ can be written as a finite sum $\sum_{i=1}^r a_i(v_i \otimes w_i)$; we can write the v_i as linear combinations of $e_1, \ldots, e_a$ and the w_i as linear combinations of $f_1, \ldots, f_b$. Using the relations in $V \otimes W$ we can therefore write our element of $V \otimes W$ as a linear combination $\sum_{i,j} c_{ij}(e_i \otimes f_j)$, so $\{e_i \otimes f_j\}$ span $V \otimes W$.

To see that this set is linearly independent it suffices to construct a linear map $F : V \otimes W \to \mathbb{C}$ with $F(e_{i'} \otimes f_{j'}) = \delta_{ii'}\delta_{jj'}$ for each i, j; to do this we use the linear maps $A : V \to \mathbb{C}, B : W \to \mathbb{C}$ given by $A(e_{i'}) = \delta_{ii'}$, similarly B, which exist; then $F : V \otimes W \to \mathbb{C}$ given by $F(v \otimes w) = A(v)B(w)$ satisfies all the relations so is well defined, and does what we want.

Note that a general element of $V \otimes W$ cannot be written as $v \otimes w$ for $v \in V, w \in W$; rather it is a linear combination of such things.

6.2 Proposition

There are natural isomorphisms for vector spaces U, V, W $U \otimes V \simeq V \otimes U, (U \otimes V) \otimes W = U \otimes (V \otimes W)$, and $(U \oplus V) \otimes W = U \otimes W \oplus V \otimes W$ [lecturer wrote $\times$ rather than $\otimes$ in this last]; we shall not bother to prove these.

If V, W are representations of a group G then $V \otimes W$ is a representation of G by $g(v \otimes w) = gv \otimes gw$, extended by linearity.

6.3 Lemma

If V, W are finite dimensional complex vector spaces then there is a natural isomorphism $\mathrm{Hom}(V, W) \simeq V^\star \otimes W$ (so $\mathrm{Hom}(V^\star, W) \simeq V \otimes W$): we'll define a linear map $V^\star \otimes W \to \mathrm{Hom}(V, W)$ by $f \otimes w \mapsto$ the sinear map $\phi(t) = f(t)w \in W$; we need to show this is well defined on $V^\star \otimes W$, but this is easy (the linear map associated with $(f_1 + f_2) \otimes w$ is the same as that associated with $f_1 \otimes w + f_2 \otimes w$ and so on). Then to see that it's an isomorphism pick bases $e_1, \ldots, e_a$ for V and $f_1, \ldots, f_b$ for W; the "dual basis" $e_1^\star, \ldots, e_a^\star$ for $V^\star$ is given by $e_i^\star(e_j) = \delta_{ij}$.

Then $V^\star \otimes W$ has a basis of the elements $e_i^\star \otimes f_j$; using our bases for V and W, $\mathrm{Hom}(V, W)$ is just the space of $x \times a$ matricies over $\mathbb{C}$; the elements $e_i^\star \otimes f_j$ map to the "elementary" matricies which have one entry 1 and the rest zeroes, which form a basis as required.

Example: Let V be any representation of $G = \frac{\mathbb{Z}}{2} = \{1, g\}$; how can we write down the isotypic decomposition of V? We need to find the $+1$- and -1-eigenspaces of $g : V \to V$, and then $V = (V^{+1} = \{x \in V : gx = x\}) \oplus (V^{-1} = \{x \in V : gx = -x\})$. We can describe the corresponding projections by explicit formulae: $x \mapsto \frac{x+gx}{2}, x \mapsto \frac{x-gx}{2}$ respectively.

Note that these formulae are valid for any representation of $G = \frac{\mathbb{Z}}{2}$ over $\mathbb{C}$, even infinite dimensional ones:

Example: Let V be the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$; let $G = \frac{\mathbb{Z}}{2}$ act on V by $(gf)(x) = f(-x)$. Then V^{+1} is the subspace of even functions and V^{-1} that of odd functions, so the above gives that we can write any continuous function as the sum of an even and an odd function.

Remark: Let V be a representation of G, then $V^\star$ is also; pick a basis for V, which is an isomorphism $V \simeq \mathbb{C}^n$. Then the representation of G on V is given by a homomorphism $\rho_1 : G \to GL(n, \mathbb{C})$. Then the "dual" homomorphism $\rho_2 : G \to GL(n, \mathbb{C})$ (which then $\simeq V^\star$ by the dual basis) is given by $\rho_2(g) = (g^T)^{-1}$ (For $A : V \to V$ the dual map $A^\star : V^\star \to V^\star$ has matrix given by the transpose A^T; $(AB)^T = B^T A^T$ so we need to take the inverse to make this an action, as before).

Lemma: Let V, W be vector spaces over a field F. Then linear maps $V \otimes W \to$ some vector space X biject with <u>bilinear</u> maps $F : V \times W \to X$ (i.e. such F which are linear in each argument): this is trivial from the definition of $V \otimes W$.

So e.g. to construct the natural map $V^\star \otimes W \to \mathrm{Hom}(V, W)$ it suffices to write down a bilinear map $F : V^\star \times W \to \mathrm{Hom}(V, W)$; as before, we do this by $F(f, w)(t) = f(t)w$, and it is easy to check this is bilinear.

Let V be a finite dimensional vector space over a field F. Then a nondegenerate bilinear form on V determines an isomorphism $V \simeq V^\star$ by $x \mapsto (t \mapsto \langle t, x\rangle)$. Beware: an "inner product" for a complex vector space, however, is not a <u>bilinear</u> form, so does not give an isomorphism $V \to V^\star$. This matters: for V a complex representation of a finite group G we have shown this preserves a (Hermitian) inner product, but this is insufficient to relate V and $V^\star$, and in fact V need not be isomorphic to $V^\star$ as representations of G, e.g. if we take the 1-dimensional representation of $\frac{\mathbb{Z}}{n} = \langle g : g^n = 1\rangle$ by $g \mapsto e^{\frac{2\pi i}{n}}$ then the dual representation $V^\star$ is given by $g \mapsto e^{-\frac{2\pi i}{n}}$, and these are not isomorphic for $n \geq 3$.

By contrast, any representation of a finite group G over $\mathbb{R}$ does preserve an inner product, by the same "averaging" proof. This is a non-degenerate bilinear form, so gives an isomorphism $V \to V^\star$, which is G-linear. So every real representation of a finite group is <u>self-dual</u>, for example the $n = 2$ case in the previous example. Any real representation of a group G of dimension n determines an n-dimensional complex representation by $G \to GL(\mathbb{R}, n) \subset GL(\mathbb{C}, n)$.

Lemma: Let V, W be finite dimensional complex vector spaces, $A : V \to V, B : W \to W$ linear maps. Define the linear map $A \otimes B : V \otimes W \to V \otimes W$ by $(A \otimes B)(v \otimes w) = A(v) \otimes B(w)$. Then 1) $\mathrm{tr}(A \otimes B) = tr(A)tr(b)$ 2) $\det(A \otimes B) = \det A^{\dim W} \det B^{\dim V}$: first suppose A, B are diagonalizable. Take bases $e_1, \dots, e_a$ for V and $f_1, \dots, f_b$ for W such that $A(e_i) = c_i e_i, B(f_j) = d_j f_j$ for some $c_i, d_j \in \mathbb{C}$. Then $V \otimes W$ has basis $\{e_i \otimes f_j\}$ and $A \otimes B$ acts by $(A \otimes B)(e_i \otimes f_j) = c_i e_i \otimes d_j f_j = c_i d_j (e_i \otimes f_j)$, so $A \otimes B$ is a diagonal matrix. Then $\mathrm{tr}(A \otimes B) = \sum_{i,j} c_i d_j = \sum_i c_i \sum_j d_j = \mathrm{tr}(A)\mathrm{tr}(B)$; similarly $\det(A \otimes B) = \prod_{i,j} c_i d_j = (\prod_i c_i)^{\dim W} (\prod_j d_j)^{\dim V} = \det A^{\dim W} \det B^{\dim V}$. Then since everything is continuous and the set of diagonalizable matricies is dense in $M_a(\mathbb{C})$, we have the result $\forall A, B$.

Symmetric and Exterior Powers

Let V be a vector space over $\mathbb{C}$ (though most of this section holds for a general field). Then the symmetric group S_n acts on $V \otimes \dots \otimes V$ n times, called $V^{\otimes n}$, by $\sigma(v_1 \otimes \dots \otimes v_n) = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}$; this makes $V^{\otimes n}$ a linear representation of S_n.

Definition: the nth symmetric power $S^n V$ is the subspace of $V^{\otimes n}$ where S_n acts trivially. The sign representation sgn of S_n is the homomorphism $S_n \to GL(1, \mathbb{C})$ by $\sigma \mapsto 1$ if $\sigma \in A_n$, -1 otherwise. The nth exterior power of V, $\Lambda^n V$, is the isotypic subspace for the sign representation of S_n inside $V^{\otimes n}$ (i.e. $\{u \in V^{\otimes n} : \sigma(u) = \mathrm{sgn}(\sigma) u \forall \sigma \in S_n\}$). We can calculate bases for these two spaces in terms of a basis for V:

Example: Let V have basis $\{e_i : 1 \le i \le r\}$. Then $V^{\otimes 2}$ has basis $\{e_i \otimes e_j : 1 \le i, j \le r\}$; S_2 acts on $V^{\otimes 2}$ by $e_i \otimes e_j \mapsto e_j \otimes e_i$ so we can read off that $S^2 V$ has basis $\{e_i : 1 \le i \le r\} \cup \{\frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i) : 1 \le i < j \le r\}$, $\Lambda^2 V$ has basis $\{\frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i) : 1 \le i < j \le n\}$. if V is a representation of G then both G and S_n act on $V^{\otimes n}$, and the two actions commute $(g(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots) = \dots$ [Yes, the lecturer really did write literally this on the blackboard]). So G preserves the S_n-isotypic decomposition. Thus $S^n V$ and $\Lambda^n V$ are representations of G.

7 Perhaps nothing needed numbering in the previous lecture

8 Lecture

Definition: Let V be a finite dimensional representation of a group G over $\mathbb{C}$. The character of V is the function $\chi_V : G \to \mathbb{C} \chi_V(g) = \mathrm{tr}(g) \in \mathbb{C}$ (recall the trace of a linear map $f : V \to V$ exists independently of choice of basis, since $\mathrm{tr}(BAB^{-1}) = \mathrm{tr}(A)$ for any invertible B, or we can define it as the sum of the eigenvalues of f (counted with multiplicity), or equivalently the sum of roots of the characteristic polynomial). Note that this is well defined: if V, W are isomorphic representations of G then they have the same character.

8.1 Theorem

For any representation V of a group G: 1) the character is a class function $(\chi_V(hgh^{-1}) = \chi_V(g) \forall g, h \in G$ 2) $\chi_V(1) = \dim V \in \mathbb{C}$, 3) $\chi_V(g^{-1} = \overline{\chi_V(g)} \forall g \in G$ 4) $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g), \chi_{V \otimes W}(g) = \chi_V(g) \chi_W(g)$ 5) $\chi_{V^*}(g) = \overline{\chi_V(g)}$: for 1), $\chi_V(hgh^{-1}) = \mathrm{tr}(ghg^{-1} \mid_V) = \mathrm{tr}(g \mid_V) = \chi_V(g)$ by properties of trace, for 2) $\chi_V(1) = \mathrm{tr}(I_V) = \dim V$, for 3) let $g \in G$, then $\langle g \rangle$ is a finite cyclic subgroup of G, $\cong \frac{\mathbb{Z}}{n}$, and V is a $\mathbb{C}$-representation of it, so $g\mid_V$ (by which we mean g considered as a linear map $V \to V$) is diagonalizable and its eigenvalues are nth roots of unity. So in some basis for V, $g\mid_V$ is a diagonal matrix with diagonal entries a_i with $(a_i)^n = 1$. Then $g^{-1}\mid_V$ is a diagonal matrix with entries $\frac{1}{a_i}$, so $\chi_V(g^{-1}) = \sum_{i=1}^r \frac{1}{a_i} = \sum_{i=1}^r \overline{a_i}$ (since we know $|a_i| = 1$) which is $\overline{\sum_{i=1}^r a_i} = \overline{\chi_V(g)}$. For 4), $\chi_{V\oplus W}(g) = \operatorname{tr}\begin{pmatrix} g\mid_V & 0 \\ 0 & g\mid_W \end{pmatrix} = \operatorname{tr}(g\mid_V) + \operatorname{tr}(g\mid_W) = \chi_V(g) + \chi_W(g)$; $\chi_{V\otimes W}(g) = \operatorname{tr}(g\mid_{V\otimes W}) = \operatorname{tr}(g\mid_V) \times \operatorname{tr}(g\mid_W)$ as above. Finally, for 5), if a representation V is described by some homomorphism $\rho : G \to GL(r,\mathbb{C})$ then $V^\star$ is described by $G \to GL(r,\mathbb{C})$ by $g \mapsto (\rho(g)^T)^{-1}$. Then for $g \in G$, after a change of basis $\rho(g)$ is diagonal with diagonal entries a_i, $a_i^n = 1 \forall i$. Then $\chi_{V^\star}(g) = \operatorname{tr}(g\mid_{V^\star}) = \operatorname{tr}((g\mid_V^T)^{-1}) = \sum \frac{1}{a_i} = \sum \overline{a_i} = \overline{\sum a_i} = \overline{\chi_V(g)}$.

8.2 Lemma

For any finite dimensional representation V of a finite group G, $\dim_{\mathbb{C}} V^G = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$ (where V^G is the G-invariant subspace), i.e. it is the number of times the trivial representation occurs in a decomposition of V as a direct sum of irreducible representations of G: for any projection $\pi : V \to W$ onto a subspace $W \subset V$, $\operatorname{tr}\pi = \dim W$ (since in a suitable basis π is a diagonal matrix with diagonal entries $\dim W$ 1s and the remainder 0s. We get a projection $V \to V^G$ by $\pi(x) = \frac{1}{|G|}\sum_{g\in G} gx \in V$, so $\dim_{\mathbb{C}} V^G = \operatorname{tr}\pi = \operatorname{tr}(\frac{1}{|G|}\sum_{g\in G} g\mid_V) = \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(g\mid_V) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$.

8.3 Lemma

Let V, W be representations of a finite group G, then define an inner product with $\langle\chi_V,\chi_W\rangle = \dim_{\mathbb{C}}\operatorname{Hom}^G(V,W)$ by for two complex-valued functions α,β on G, $\langle\alpha,\beta\rangle = \frac{1}{|G|}\sum_{g\in G}\alpha(g)\overline{\beta(g)}$. This is a Hermitian inner product on the complex vector space of functions $G \to \mathbb{C}$ (we usually only consider it as applied to the subspace of class functions): $\operatorname{Hom}^G(V,W)$ is the subspace of G-invariant elements of $\operatorname{Hom}(V,W)$, which is a representation of G, so $\dim_{\mathbb{C}}\operatorname{Hom}^G(V,W) = \frac{1}{|G|}\sum_{g\in G}\chi_{\operatorname{Hom}(V,W)}(g) = \frac{1}{|G|}\sum_{g\in G}\chi_{V^\star\otimes W}(g) = \frac{1}{|G|}\sum_{g\in G}\chi_{V^\star}(g)\chi_W(g) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_W(g)$ which $= \langle\chi_W,\chi_V\rangle \in \mathbb{C}$. So $\langle\chi_V,\chi_W\rangle = \overline{\langle\chi_W,\chi_v\rangle} = \overline{\dim_{\mathbb{C}}\operatorname{Hom}^G(V,W)} = \dim_{\mathbb{C}}\operatorname{Hom}^G(V,W)$.

Theorem: 1) For any irreducible complex representation V of a finite group, $\|\chi_V\|^2 = 1$, 2) For any non-isomorphic irreducible representations V, W of G, $\langle\chi_V,\chi_W\rangle = 0$: for 1) $\|\chi_V\|^2 = \langle\chi_V,\chi_V\rangle = \dim_{\mathbb{C}}\operatorname{Hom}^G(V,V) = 1$ by Schur's lemma; the proof of 2) is identical.

8.4 Lol numbering

8.5 Corollary

The number of times an irreducible representation V occurs in a decomposition of a representation W into irreducibles is $\langle\chi_V,\chi_W\rangle$: if $W \simeq \bigoplus_i W_i^{\oplus n_i}$ then $\langle\chi_V,\chi_W\rangle = \sum_i n_i\langle\chi_{W_i},\chi_V\rangle = n_i$ where $V \simeq W_i$ (0 if none of the W_i are $\simeq V$).

8.6 Lol numbering

8.7 Corollary

Two complex representations of a finite group are isomorphic iff they have the same character: By Maschke's theorem, complex representations of G are completely reducible, so by corollary 8.2 the number of times each irreducible representation occurs in V is determined by χ_V.

8.8 Corollary

A representation V of a finite group G is irreducible iff $\|\chi_V\|^2 = 1$: write $V = \bigoplus_i W_i^{\oplus n_i}$ where the W_i are irreducible and not isomorphic to each other and $n_i \geq 0$. Then $\langle \chi_V, \chi_V \rangle = \langle \sum_i n_i \chi_{W_i}, \sum_i n_i \chi_{W_i} \rangle = \sum_i n_i^2 \langle \chi_{W_i}, \chi_{W_i} \rangle = \sum_i n_i^2$.

9 Lecture

Example: consider the standard representation of $\frac{\mathbb{Z}}{n} = \{1, g, \ldots g^{n-1}\}$ on $\mathbb{R}^2$ as the rotations preserving a regular n-gon $g \mapsto \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} \in GL(2, \mathbb{R})$; more generally $g^a \mapsto \begin{pmatrix} \cos \frac{2\pi a}{n} & -\sin \frac{2\pi a}{n} \\ \sin \frac{2\pi a}{n} & \cos \frac{2\pi a}{n} \end{pmatrix}$. If $n \geq 3$ then this 2D representation of $\frac{\mathbb{Z}}{n}$ is irreducible (as a real representation), since there is no 1D subspace of $\mathbb{R}^2$ (i.e. a line through the origin) preserved under G. For $n = 2$ this representation is far from irreducible; we have a continuous family of G-invariant subspaces (all the lines through 0); this occurs because the representation is $W^{\oplus 2}$ where W is the representation $g \mapsto (-1)$.

Over $\mathbb{C}$, our representation of $\frac{\mathbb{Z}}{n}$ (acting on $\mathbb{C}$ by the same matricies) must be reducible, since we know it must be a sum of two 1D representations. Let A be the matrix as above with $g \mapsto A$, then $\det A = \cos^2 \frac{2\pi}{n} + \sin^2 \frac{2\pi}{n} = 1, \mathrm{tr} A = 2 \cos \frac{2\pi}{n}$, so the eigenvalues $a, b \in \mathbb{C}$ of A are nth roots of unity with $ab = 1, a + b = 2 \cos \frac{2\pi}{n} \therefore a = e^{\frac{2\pi i}{n}}, b = e^{-\frac{2\pi i}{n}}$. So this representation is, over $\mathbb{C}$, the direct sum of the 1D representations $g \mapsto (e^{\frac{2\pi i}{n}}), g \mapsto (e^{-\frac{2\pi i}{n}})$.

Consider the standard representation of the dihedral group $D_{2n} = \langle g, r : g^n = 1, r^2 = 1, rgr^{-1} = g^{-1} \rangle$ on $\mathbb{R}^2$ as the symmetries preserving a regular n-gon; the image of $D_{2n} \to GL(2, \mathbb{R})$ is given by $g \mapsto$ the matrix A as before, $r \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; this representation is irreducible (for $n \geq 3$) even over $\mathbb{C}$, because the only subspaces of $\mathbb{C}^2$ that are invariant under $\frac{\mathbb{Z}}{n}$ are two complex lines (i.e. copies of $\mathbb{C}$), and r switches these two lines (the relation $rgr^{-1} = g^{-1}$ implies that r maps any linear subspace of V in which g acts as multiplication by $c \in \mathbb{C}^\star$ to one in which g acts as multiplication by $\frac{1}{c}$).

Example: $S_3 = D_6$ is the smallest nonabelian group; it has an obvious 3D permutation representation $\sigma(x_1, x_2, x_3) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$. [In this representation] S_3 acts as the identity on $\mathbb{C} \cdot (1, 1, 1)$; since S_3 preserves the standard inner product on $\mathbb{C}^3$ this means it preserves $W := (\mathbb{C} \cdot (1, 1, 1))^\perp \subset V = \mathbb{C}^3$, so W is a 2D representation of S^3; we can write $W = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 == 0\} \subset \mathbb{C}^3$.

We claim that this 2D representation is isomorphic to the representation of $S_3 \simeq D_6$ as symmetries of a triangle: it is sufficient to prove they have the same character. The three conjugacy classes in S_3 are those of 1,(123),(12); it is therefore sufficient to consider only the characters of these.

	1	(123)	(12)
Triangle representation	2	-1	0
Trivial representation	1	1	1
3D permutation representation (reducible)	3	0	1

Now since the character of a direct sum is the sum of the characters, we can calculate the line for our 2D representation by W:

	1	(123)	(12)
Triangle representation	2	-1	0
Trivial representation	1	1	1
3D permutation representation (reducible)	3	0	1
W	2	-1	0

In fact, we know all the irreducible representations of S_3: we have

	1	(123)	(12)
Trivial representation	1	1	1
Sign representation	1	1	-1
2D irreducible representation W	2	-1	0

and since the characters of irreducible representations always form an orthonormal set in the vector space of class functions, which is 3D in this case, these three form an orthonormal basis and there are no other irreducible representations of S_3.

9.1 Proposition

The multiplicity of any irreducible representation V in the regular representation $\mathbb{C}G$, for any finite group G, is $= \dim V$: we know this multiplicity is $\langle \chi_V, \chi_{\mathbb{C}G*} \rangle$, and have $\chi_{\mathbb{C}G}(g) = |G|$ for $g = 1$, 0 otherwise. So $\langle \chi_V, \chi_{\mathbb{C}G} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_{\mathbb{C}G}(g)} = \chi_V(1) = \dim V$.

9.2 Corollary

For any finite group G with irreducible complex representations $W_1, \ldots, W_r$, $|G| = \sum (\dim_{\mathbb{C}} W_i)^2$, because as a representation of G, $\mathbb{C}G = \bigoplus_{i=1}^r W_i^{\oplus \dim W_i}$.

9.3 Theorem (Completeness of characters)

For a finite group G, the irreducible characters (i.e. the characters of irreducible representations) form an ON basis for the space of class functions on G.

$\mathbb{C}G$ is a representation of G; it is equal to the space of functions $G \to \mathbb{C}$, which is a commutative ring under $(\alpha\beta)(g) = \alpha(g)\beta(g)$, or forms a noncommutative ring, the group ring of G, under the multiplication of G $((\sum_{g \in G} a_g g)(\sum_{h \in g} b_h h) = \sum_{g,h \in G} a_g b_h gh$.

9.4 Lemma

For any representation V of a group G we have a ring homomorphism $\mathbb{C}G \to \mathrm{End}(V)$ ($\simeq M_n\mathbb{C}$ where $n = \dim_{\mathbb{C}} V$): map each $g \in G$ to $g_V \in \mathrm{End}(V)$ and extend by linearity; since V is a representation of G this gives a ring homomorphism.

9.5 Corollary

(This is a corollary to 9.3) The number of (complex) irreducible representations of G (up to isomorphism) is the number of conjugacy classes in G.

Notice that the center of the group ring $\mathbb{C}G$, $Z(\mathbb{C}G) = \{f \in G : fe = ef \forall e \in \mathbb{C}G\}$, is the vector space of <u>class functions</u> on G: $\sum_{g \in G} a_g g \in Z(\mathbb{C}G) \Leftrightarrow a_g = a_{hgh^{-1}} \forall g, h \in G$. So for any representation V of G and any class function $\phi : G \to \mathbb{C}$, ϕ determines a linear map $\phi\,|_V \colon V \to V$ which commutes with all elements of G, i.e. is G-linear.

Consider the case of an <u>irreducible</u> representation V of G and any class function ϕ: $\phi\,|_V \colon V \to V$ must be a scalar times the identity, by Schur's lemma. We have $\mathrm{tr}(\phi\,|_V) = \mathrm{tr}(\sum_{g \in G} \phi(g) g\,|_V) = \sum_{g \in G} \phi(g) \mathrm{tr}(g\,|_V) = \sum_{g \in G} \phi(g) \chi_V(g) = |G| \langle \phi, \chi_{V^*} \rangle$, so $\phi\,|_V = \frac{|G|}{\dim V} \langle \phi, \chi_{V^*} \rangle$. So if ϕ is a class function that is orthogonal to all irreducible characters, then $\phi\,|_V = 0$ for every irreducible representation V of G, so by complete reducibility $\phi\,|_{\mathbb{C}G} = 0$. But $\phi_{\mathbb{C}G}(1) = \sum_{g \in G} \phi(g) g \Rightarrow \phi = 0$.

10 Lecture

We have now proven "completeness of characters": the characters of complex irreducible representations of a finite group form an orthonormal basis for the complex vector space of class functions on G. The proof relies on taking several points of view on the regular representation $\mathbb{C}G$; in particular, $\mathbb{C}G =$ the let of functions $G \to \mathbb{C}$ by $\sum_{g \in G} \alpha(g) g \leftrightarrow \alpha : G \to \mathbb{C}$. Under this identification, $Z(\mathbb{C}G)$ is the set of class functions $G \to \mathbb{C}$.

A remark: representations of G are equivalent to modules over the group ring $\mathbb{C}G$: if V is a $\mathbb{C}G$-module then it becomes a representation of V by using $G|subset\mathbb{C}G$; conversely if V is a representation of G we can make it into a $\mathbb{C}G$-module by setting $(\sum_{g \in G} a_g g) \cdot x = \sum_{g \in G} a_g (gx) \forall x \in V$. In other words, if $\phi = \sum_{g \in G} a_g g$ then ϕ determines a linear map $V \to V$ by $\phi\,|_V = \sum_{g \in G} a_g (g\,|_V)$.

To see $Z(\mathbb{C}G)$ is the set of class functions $G \to \mathbb{C}$¡ we claim that $\phi = \sum_{g \in G} a_g G$ commutes with all elements of $\mathbb{C}G$ precisely when ϕ commutes with every element of $G \subset \mathbb{C}G$, because we can write $(\sum_{g \in G} a_g g) h = \sum_{g \in G} a_g (gh) = \sum_{k \in G} a_{kh^{-1}} k$, $h(\sum_{g \in G} a_g g) = \sum_{g \in G} a_h hg = \sum_{k \in G} a_{h^{-1}k}$, so $\phi = \sum a_g g$ lies in $Z(\mathbb{C}G)$ iff $\forall h, k \in G$ $a_{kh^{-1}} = a_{h^{-1}k}$: writing $\phi(g) = a_g$ this is the case iff $\phi(kh^{-1}) = \phi(h^{-1}k) \forall h, k \in G$, i.e. ϕ is a class function.

The <u>character table</u> of a finite group G is the (we will later prove, square) matxir showing the characters of all the irreducible representations of G on all conjugacy classes, e.g. for S_3 it is

	1	(123)	(12)
Trivial representation	1	1	1
Sign representation	1	1	-1
2D irreducible representation W	2	-1	0

.

Orthogonality of characters means that the rows of this matrix are orthonormal when suitably interpreted: recall $\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \overline{\chi_2(g)} =$

$\frac{1}{|G|}\sum_{C\subset G \text{ conjugacy classes}} |C|\chi_1(C)\overline{\chi_2(C)}$, so we rewrite the table as

$\|C\|$	1	
C	1	(
Trivial representation	1	
Sign representation	1	
2D irreducible representation W	2	

Then e.g. $\langle 1, \text{sgn}\rangle = \frac{1}{6}(1\times 1\times 1 + 2\times 1\times 1 + 3\times 1\times -1) = 0$, and $\langle W, W\rangle = \frac{1}{6}(1\times 2^2 + 2\times(-1)^2 + 3\times 0^2) = 1$, so the columns really are orthonormal.

Example: If we know all but one of the irreducible representations of a group G, the last is determined by orthogonality. Suppose for S_3 we knew the trivial and sign representations; W is our unknown representation. We know $6 = |G| = 1^2 + 1^2 + (\dim W)^2$, so $\dim W = 2$ and we have a 2 in the first column of W ($\text{tr} I\mid_W = \dim W$). Then we could just use orthogonality of the rows, but there is an easier method: we can interpret the orthonormality of the rows of the characteristic table by saying that if we multiply each column C by $\sqrt{\frac{|C|}{|G|}}$, then the rows become orthonormal in the standard inner product on $\mathbb{C}^n$. This is equivalent to saying this modified matrix is unitary, which is equivalent to saying that the columns are orthonormal. Thus we have:

10.1 Proposition

For any conjugacy classes C, C', $\sum_{\text{irreducible characters } \chi \text{ of } G} \chi(C)\overline{\chi(C')} = \frac{|G|}{|C|}$ if $C' = C$, 0 otherwise.

Therefore, any two different columns of the original character table are orthogonal in the usual inner product on $\mathbb{C}^n$; thus we can easily finish the character table of G.

Example: Let G be a finite cyclic group $\frac{\mathbb{Z}}{n} = \langle g : g^n = 1\rangle$; the n irreducible representations of G are 1D, call them ρ_j, where $\rho_j(g) = (e^{\frac{2\pi i j}{n}})$, so the character table of G is, if we let $\xi = e^{\frac{2\pi i}{n}}$,

$\|C\|$	1	1		1		1
C	1	g	$\dots$	g^k	$\dots$	g^{n-1}
$\rho_0 = 1$	1	1		1		1
ρ_1	1	ξ		ξ^k		ξ^{n-1}
$\dots$						
ρ_j	1	ξ^j		ξ^{jk}		$\xi^{j(n-1)}$
$\dots$						

.

Example: G = the dihedral group D_{2n} of order $2n$, $\langle g, r : g^n = 1, r^2 = 1, rgr^{-1} = g^{-1}\rangle$; take $n \geq 3$ odd, $n = 2m+1$. Then all reflections rg^j are conjugate to r; there are $m+2$ conjugacy classes, so we want $m+2$ irreducible complex representations. For any finite group G, any 1D representation factors through the abelianization $G^{ab} = \frac{G}{[G,G]}$ where the quotient is the commutator of G with itself, i.e. the normal subgroup generated by expressions $aba^{-1}b^{-1}$ for $a, b \in G$, because any homomorphism $G \to GL(1,\mathbb{C}) = \mathbb{C}^\star$ must factor through G^{ab} because $\mathbb{C}^\star$ is abelian. Thus the number of (isomorphism classes of) 1D irreducible representations of G is $|G^{ab}$.

11 Lecture

For n odd, $(D_{2n})^{ab} = \frac{\mathbb{Z}\{g,r\}}{\langle ng=0, 2r=0, r+g-r=-g\rangle}$ (by the numerator we mean $\mathbb{Z}\oplus\mathbb{Z}$ with generators called g, r); this last relation in an abelian group implies $2g = 0$

so $g = 0$ and the group is $\frac{\mathbb{Z}\{r\}}{\langle 2r=0 \rangle} = \frac{\mathbb{Z}}{2}$. So our 1D representations are the two obvious 1D representations given by

	1	$\{g, g^{-1}\}$	$\{g^2, g^{-2}\}$	$\ldots$	$\{g^m, g^{-m}\}$	r
1	1	1	1		1	1
sgn	1	1	1		1	-1

.

Because $rgr^{-1} = g^{-1}$, any representation of D_{2n} containing a subspace W on which g acts by $\xi \in \mathbb{C}^\star$ must also contain rW on which g acts by ξ^{-1}. This suggests the following 2D representations of G: for $1 \le j \le m$, define W_j by $g \mapsto \begin{pmatrix} \xi^j & 0 \\ 0 & \xi^{-j} \end{pmatrix}, r \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where $\xi = e^{\frac{2\pi i}{n}}$. This is a valid representation; it is irreducible "by hand": as a representation of $\frac{\mathbb{Z}}{n} \subset D_{2n}$ it is the direct sum of two non-isomorphic irreducible representations , so the only (non-trivial) $\frac{\mathbb{Z}}{n}$-invariant subspaces of $\mathbb{C}^2$ are these two $\mathbb{C}$-lines, but r switches these two lines, so there are no nontrivial G-invariant subspaces and W_j is irreducible. So our table becomes

	1	$\{g, g^{-1}\}$	$\{g^2, g^{-2}\}$	$\ldots$	$\{g^m, g^{-m}\}$	r
1	1	1	1		1	1
sgn	1	1	1		1	-1
W_j for $1 \le j \le m$	2	$2\cos\frac{2\pi j}{n}$	$2\cos\frac{2\pi 2j}{n}$		$2\cos\frac{2\pi mj}{n}$	0

.

Now consider D_{2n} for n even, $n = 2m$ ($m \ge 2$). Here $(D_{2n})^{ab} = \frac{\mathbb{Z}\{g,r\}}{(ng=0,2r=0,2g=0)} = \frac{\mathbb{Z}\{g,r\}}{(2r=0,2g=0)} \simeq \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$. Then we find the table is

	1	$\{g, g^{-1}\}$	$\ldots$	$\{g^k, g^{-$
1	1	1		1
α_1	1	1		1
α_2	1	-1		(-1)
α_3	1	-1		(-1)
W_j as before, for $1 \le j \le m-1$	2	$2\cos\frac{2\pi j}{n}$		$2\cos 2$

11.1 Proposition

Let G, H be finite groups, then for any representations V of G, W of H, $V \otimes_{\mathbb{C}} W$ is a representation of $G \times H$, and every irreducible representation of $G \times H$ arises uniquely as a product of irreducible representations for V, W in this way:

For any linear map $A : V \to V$ we have a linear map $A \otimes 1 : V \otimes W \to V \otimes W$ by $(A \otimes 1)(v \otimes w) = Av \otimes w$ (this is a bilinear expression in v, w, so this works); likewise for $B : W \to W$ linear we have $! \otimes B : V \otimes W \to V \otimes W$. These $A \otimes 1, 1 \otimes B$ commute [for any A, B], so we have a representation of $GL(V) \times GL(W)$ by $GL(V \otimes W)$ (for any product group $G \times H$ we have $G = G \times 1 \subset G \times H, H = 1 \times H \subset G \times H$ and elements of one of these commute with those of the other). So given homomorphisms $G \to GL(V), H \to GL(W)$ we have a homomorphism $G \times H \to GL(V \otimes W)$.

Now let V be an irreducible representation of G, W an irreducible representation of H; we need to shouw $V \otimes W$ is an irreducible representation of $G \times H$, which we will do by computing $\|\chi_{V \otimes W}\|^2$ (which will $= 1$ iff $V \otimes W$ is irreducible). $\|\chi_{V \otimes W}\|^2 = \frac{1}{|G \times H|} \sum_{g \in G, h \in H} |\chi_{V \otimes W}(g, h)|^2 = \frac{1}{|G||H|} \sum_{g \in G, h \in H} |\chi_V(g)\chi_W(h)|^2 = \frac{1}{|G||H|} \sum_{g \in G} |\chi_V(g)|^2 \sum_{h \in H} |\chi_W(h)|^2 = \|\chi_V\|^2 \|\chi_W\|^2 = 1$ since V, W are irreducible representations of G, H.

Now different (non-isomorphic) choices of irreducible representations of V, W give non-isomorphic representations of $G \times H$: $(V \otimes W) |_{G \subset G \times H} = V^{\oplus \dim W}$ so we can recover V from $V \otimes W$; likewise $V \otimes W$ uniquely determines W.

So, this gives us (# irreducible representations of $G \times$ # irreducible representations of H) distinct irreducible representations of $G \times H$. But $(a,b)(g,h)(a,b)^{-1} = (aga^{-1}, bhb^{-1})$ so the conjugacy classes in $G \times H$ are just those of the form (a conjugacy class in G)$\times$(a conjugacy class in H). So the number of irreducible representations of $G \times H$ is the number of conjugacy classes in G times the number of conjugacy classes in H, and we have constructed all the irreducible representations of $G \otimes H$.

Example: The character table of A_4, of order 12: we notice that $N = \{1, (12)(34), (13)(24), (14)(23)\}$ is a subgroup of A_4, $\simeq \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$; in fact it is normal, with quotient group $\frac{A_4}{N} \simeq \frac{\mathbb{Z}}{3}$ (since this is the only group of order 3). Therefore we have 3 1D representations of A_4, all trivial on N (given by the 3 1D representations of $\frac{\mathbb{Z}}{3}$). Then we can find the last irreducible character a by orthonormality; it must have dimension 3 so that $|A_4| = 12 = 1^2 + 1^2 + 1^2 + (\dim a)^2$, then we can find the rest by orthogonality and construct the character table: setting $\xi = e^{\frac{2\pi i}{3}}$, it is

$\lvert C\rvert$	1	3	4	4
	1	(12)(34)	(123)	(132)
1	1	1	1	1
α_1	1	1	ξ	ξ^2
α_2	1	1	ξ^2	ξ
a	3	-1	0	0

. In fact a is "nicely" realizable as the permutation representation minus the trivial representation.

12 Character table of A_5

Calculating this is at the limits of our ability at this stage (it will be much easier with later results), but it is important to see it early on, because A_5, of order $\frac{5!}{2} = 60$, is the smallest nonabelian simple group; thus in a sense all smaller groups "break down" into abelian "pieces" (e.g. S_3 has $\frac{S_3}{\frac{\mathbb{Z}}{3}} \simeq \frac{\mathbb{Z}}{2}$ or $\frac{A_4}{\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}} \simeq \frac{\mathbb{Z}}{3}$. The conjugacy classes in S_5 are $1, (12), (123), (12)(34), (1234), (12)(345), (12345)$; the only ones of these in A_5 are $1, (123), (12)(34), (12345)$, but it might be that some pairs of elements of A_5 are conjugate in S_5 but not in A_5.

For any $g \in$ a group G, let $Z_G(g)$, the centralizer of g, be $\{h \in G : gh = hg\}$ (or equivalently $\{h \in G : hgh^{-1} = g\}$); this is a subgroup of G, and the conjugacy class of $g \in G$ has order $\frac{|G|}{|Z_G(g)|}$ (by orbit-stabilizer on the action of G on itself by conjugation). In particular, $|C| \mid |G|$ for any conjugacy class $C \subset G$.

Let $g \in A_5$; if we know $Z_{S_5}(g)$ then $Z_{A_5}(g) = Z_{S_5}(g) \cap A_5$; this will be either $Z_{S_5}(g)$ or a subgroup of index 2 therof. In the first case, the S_5-conjugacy class of g splits into two A_5-conjugacy classes (since the size $\frac{|G|}{|Z_G(g)|}$ of each is halved); in the second case (i.e. g commutes with some element of $S_5 \setminus A_5$) the S_5-conjugacy class of g is its A_5-conjugacy class.

Looking back at our classes (checking on the way that their sizes sum to $|A_5| = 60$):

C	1	(123)	(12)(34)	(12345)
$\lvert C\rvert$	1	20	15	$4! = 24$
$\lvert Z_{S_5}(g)\rvert$	1230	6	8	5
Element of centralizer in $S_5 \setminus A_5$	(12)	(45)	(12)	No

; the centralizer of the last is just $\langle (12345) \rangle \subset A_5$, so it splits into two conjugacy classes (12345),(12354) (when a class splits, we can find the other class by conjugating by any element of $Z_{S_5}(g) \setminus A_5$). Thus the conjugacy classes in

A_5 are

$\lvert C\rvert$	1	20	15	12	12
C	1	(123)	(12)(34)	(12345)	(12354)

. So there are 5 irreducible representations of A_5; we would normally look for 1D representations, but because A_5 is nonabelian simple its abelianization is trivial and there are no nontrivial 1D representations. We could of course try and reduce the regular representation of dimension 60 into irreducibles, but this would be very impractical, so we look for smaller "inderesting" representations. We have the 5D permutation representation of A_5, with e.g. $(12)(34) \mapsto \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & 1 \end{pmatrix}$; we see that in general the trace of g in the permutation representation is the number of elements of $\{1,2,3,4,5\}$ fixed by g, so the character of this representation, α, is 5 2 1 0 0. We could compute $\|\alpha\|^2 = 2 = \sum n_i^2$ where $\alpha = \bigoplus_i W_i^{\oplus n_i}$ for irreducible non-isomorphic W_i, so α is the direct sum of two non-isomorphic irreducible representations; we can see that $\mathbb{C}^5$ fixes $\mathbb{C}\cdot(1,1,1,1,1)$, or compute $\langle\alpha, 1\rangle = \frac{1}{60}(1\times 5\times 1 + 20\times 2\times 1 + 15\times 1\times 1 + 0 + 0) = 1$, so α contains the trivial representation precisely once. So we get the next irreducible character of A_5, χ_2, and our table is

$\lvert C\rvert$	1	20	15	12	12
C	1	(123)	(12)(34)	(12345)	(12354)
Trivial representation $1 = \chi_1$	1	1	1	1	1
5D permutation representation - trivial representation χ_2	4	1	0	-1	-1

.

Some standard tricks which are futile in this case: if V is irreducible and L a 1D representation of G then $V \otimes L$ is irreducible; this might be $\simeq V$, but often isn't (however we have no nontrivial 1D representations so this is useless). If V is irreducible then $V^\star$ is irreducible, and has $\chi_{V_\star}(g) = \overline{\chi_V(g)}$, so if we ever find complex characters we have another representation by this.

We now look at $\chi_2 \otimes \chi_2$, a 16D representation of A_5; we want to break it up into irreducibles. We use that for any representation V of G, $V \otimes V \simeq S^2V \oplus \Lambda^2 V$; here $\dim \Lambda^2\chi_2 = \binom{4}{2} = 6, \dim S^2\chi_2 = \binom{4+2-1}{2} = 10$.

12.1 Lemma

Let χ_V be a characteristic of a finite group G, then $\chi_{S^2V}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2)), \chi_{\Lambda^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$: for any $g \in G$, we know g acting on V is diagonalizable; let $e_1, \dots, e_n$ be a basis for V in which g acts by $g(e_i) = a_i e_i$ for some $a_i \in \mathbb{C}$, then g acts on S^2V with eigenvalues $a_i a_j \forall 1 \le i \le j \le n$ (since we have a basis by $\{\frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)\}$), and g acts on $\Lambda^2 V$ with eigenvalues $a_i a_j$ for $1 \le i < j \le n$ (similarly), so $\chi_{S^2V}(g) = \sum_{i\le j} a_i a_j = \frac{1}{2}((\sum a_i)^2 + \sum a_i^2)$ and $\chi_{\Lambda^2 V}(g)/\sum_{i<j} a_i a_j = \frac{1}{2}((\sum a_i)^2 - \sum a_i^2)$. So we can compute some characters, for $\chi_{S^2\chi_2}$ 10 1 2 0 0 (by calculation), and then since the two must sum to 4^2 1^2 0^2 $(-1)^2$ $(-1)^2$, for $\chi_{\Lambda^2\chi_2}$ 6 0 -2 1 1.

The reader should compute that $\|\chi_{S^2\chi_2}\|^2 = 3, \|\chi_{\Lambda^2\chi_2}\|^2 = 2$, so $S^2\chi_2$ is a sum of three different irreducible characters and $\Lambda^2\chi_2$ a sum of two different irreducible characters; computing inner products we find $S^2\chi_2 = 1 + \chi_2 + \beta$ for some irreducible β, and $\Lambda^2\chi_4$ does not contain $1, \chi_2$ or this β. So $\Lambda^2\chi_2$ is the sum of the two irreducible characters we don't know; let them have dimensions u, v and then we know $60 = 1 + 16 + 25 + u^2 + v^2$ so $u^2 + v^2 = 18$ and we must have $u = 3, v = 3$; by orthogonality we can relate the remaining characters. We use a trick to finish: notice that for any characters χ, $\chi_V(g^{-1}) = \overline{\chi_V(g)} = \chi_{V^\star}(g)$, so if g is conjugate to g^{-1} then $\chi_V(g)$ is real, and if every element of G is conjugate to its inverse, $V^\star \simeq V$; this is the case here (the only nontrivial check is $(12345)^{-1} = (54321) = ((15)(24))(12345)((15)(24))^{-1}$), so all our remaining values a, b, c, d are real: our working table looks like

$\lvert C\rvert$	1	20
C	1	(123)
Trivial representation $1 = \chi_1$	1	1
5D permutation representation - trivial representation χ_2	4	1
$S^2\chi_2 - 1 - \chi_2 = \chi_3$	5	-1
χ_4	3	a
χ_5	3	-a

We can use e.g. that the second column multiplied by $\sqrt{\frac{|C|}{|G|}}(= \sqrt{\frac{1}{3}})$ must have squared length equal to 1 in the standard inner product on $\mathbb{C}^5$, so $1 = \frac{1}{3}(1^2 + 1^2 + (-1)^2 + a\bar{a} + a\bar{a})$ so $3 = 3 + 2|a|^2 \therefore |a|^2 = 0 \therefore a = 0$. For the fourth column even this isn't enough; we obtain that $3 = |c|^2 + |1-c|^2$, but this only tells us that c lies on an ellipse in $\mathbb{C}$.

Now we use the trick mentioned above: $\chi_4^\star$ is an irreducible representation of dimension 3, so must be either χ_4 or χ_5. If $\chi_4^\star \simeq \chi_5$ then $\chi_5(g) = \chi_4(g)$, or $\chi_4^\star \simeq \chi_4 \therefore \chi_5^\star \simeq \chi_5$, so both χ_4, χ_5 are real-valued. But since every element g of A_5 is conjugate to g^{-1}, any character χ of A_5 must take real values ($\overline{\chi(g)} = \chi(g^{-1}) = \chi(g)$) i.e. χ is self-dual. So we mave $3 = c^2 + (1-c)^2$ as a real number, and we find $c = \frac{1\pm\sqrt{5}}{2}$. So we can wlog take χ_4 to have $c = \frac{1+\sqrt{5}}{2}$, then $1-c$, the corresponding entry of χ_5, is $\frac{1-\sqrt{5}}{2}$. Then we can use orthogonality to compute the other numbers, finding $b = -1 \therefore 2 - b = -1, d = \frac{1-\sqrt{5}}{2} \therefore 1 - d = \frac{1+\sqrt{5}}{2}$.

13 Lecture

(Actually this lecture started somewhere in the middle of the above calculation, but I feel it morally begins here)

Theorem: For every finite group G, the dimension of any complex irreducible representation divides $|G|$. The proof of this requires some algebra:

Definition: Let $R \subset S$ be commutative rings. An element $x \in S$ is integral over R if there is a [nonzero] monic polynomial with coefficients in R satisfied by x, i.e. $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \in S$ for some $a_i \in R$; equivalently $x \in S$ is integral over R iff the subring $R[x] \subset S$ is finitely generated as an R-module: for the forward implication if x is integral over R then x^n is an R-linear combination of smaller powers of x, and inductively so is any higher power of x, so $R[x]$ is generated by $1, x, \ldots, x^{n-1}$; for the reverse, if $R[x]$ is generated as an R-module by $f_1, \ldots, f_r$, then $R[x]$ must be generated by $1, x, \ldots, x^N$ for some N, so write x^{N+1} as a linear combination of these, giving a monic polynomial with coefficients in R satisfied by x.

An algebraic integer is a complex number which is integral over $\mathbb{Z}$ (clearly this implies that x is an algebraic number, i.e. satisfies some polynomial with coefficients in $\mathbb{Q}$).

Remark: A rational number is an algebraic integer iff it is $\in \mathbb{Z}$; for example, if we had $(\frac{1}{2})^n + a_{n-1}(\frac{1}{2})^{n-1} + \cdots + a_n = 0$ for $a_i \in \mathbb{Z}$ we have an immediate contradiction (the factor of 2^n in the denominator of the first term can't be cancelled, or being more rigorous, multiply the equation by 2^n and then the left hand side is an odd integer so cannot $= 0$).

Example: $\sqrt{N}$ is an algebraic integer since it satisfies $x^2 - N = 0$.

Example: Which elements of $\mathbb{Q}(\sqrt{N})$ are algebraic integers? Precisely those whose trace and norm are integers ($\mathrm{tr}(a+b\sqrt{N}) = (a+b\sqrt{N})+(a-b\sqrt{N}) = 2a$, norm is $(a+b\sqrt{N})(a-b\sqrt{N}) = a^2 - Nb^2$).

Example: nth roots of unity are algebraic integers, e.g. $\xi_3 = \frac{-1+\sqrt{-3}}{2}$ (satisfies $\xi^2 + \xi + 1 = 0$).

13.1 Lemma

If $x, y \in S$ are integral over $\mathbb{Z} \subset S$ then $x + y, xy$ are integral over $\mathbb{Z}$: we know that $\mathbb{Z}[x] \subset S$ is a finitely generated $\mathbb{Z}$-module, and y is integral over $\mathbb{Z}$ so certainly over $\mathbb{Z}[x]$, so $\mathbb{Z}[x, y]$ is a finitely generated $\mathbb{Z}[x]$-module and thus a finitely generated $\mathbb{Z}$-module. So $x + y, xy \in S$ are contained in a ring which is a finitely generated $\mathbb{Z}$-module $\subset S$; any subgroup of a finitely generated abelian group is finitely generated, so $\mathbb{Z}[x + y], \mathbb{Z}[xy]$ are finitely generated and we have the result.

13.2 Lemma

The image of an element $x \in S$ integral over $\mathbb{Z}$ under any ring homomorphism is integral over $\mathbb{Z}$; the proof is immediate.

13.3 Theorem

For V an irreducible complex representation of a finite group G, $\dim(V) \mid |G|$; this follows from the following two lemmas:

13.4 Lemma

The values of any character of G are algebraic integers, since they are sums of roots of unity.

13.5 Lemma

For any conjugacy class C and irreducible representation V, $\chi_V(C)\frac{|C|}{\dim V}$ is an algebraic integer.

Proof of Theorem 13.3 given Lemma 13.5: it's sufficient to prove $\frac{|G|}{\dim V}$ is an algebraic integer, since we know it is rational. Since we assumed V irreducible, $\|\chi_V\|^2 = 1$ i.e. $1 = \frac{1}{|G|}\sum_{g\in G} |\chi_V(g)|^2 = \frac{1}{|G|}\sum_{\text{conjugacy classes } C\subset G} |C|\chi_V(g)\chi_V(g^{-1})$, so $\frac{|G|}{\dim V} = \frac{1}{\dim V}\sum_{C\subset G} |C|\chi_V(g)\chi_V(g^{-1}) = \sum_{C\subset G}(\frac{|C|}{\dim V}\chi_V(g^{-1}))\chi_V(g)$; the bracket is algebraic by Lemma 13.5, and $\chi_V(g)$ is algebraic by Lemma 13.4.

Proof of Lemma 13.5: (Recall the proof of completeness of characters) Let ϕ be any class function $G \to \mathbb{C}$; consider it as an element of $Z(\mathbb{C}G)$, then ϕ acts as a G-linear map on any representation V of G. So if V is irreducible then ϕ acts as a scalar on VW; we find ϕ acts by the scalar $\frac{|G|}{\dim V}\langle\phi, \chi_{V^\star}\rangle$ in this case. Let ϕ be the class function $\phi(g) = 1$ if $g \in C$, 0 otherwise. Then $\phi \in Z(\mathbb{Z}G)$ ($\mathbb{Z}G$ is clearly a ring); we have a ring homomorphism $Z(\mathbb{Z}G) \to \mathbb{C}$ given by $\phi \mapsto$ the scalar $\phi\mid_V$ (for V an irreducible representation of G). We see that $\phi \in Z(\mathbb{Z}G)$ is integral over $\mathbb{Z}$, as $Z(\mathbb{Z}G)$ is a finitely generated $\mathbb{Z}$-module. Therefore the image of ϕ under this homomorphism is integral over $\mathbb{Z}$, but this is $\frac{|G|}{\dim V}\langle \phi, \chi_{V^\star}\rangle = \frac{\chi_V(c)|C|}{\dim V}$.

14 Lecture

Induced Representations

Recall the structure of the two irreducible representations of a dihedral group $D_{2n} \supset$ the cyclic group C_n; as representations of the cyclic group they have the form $L_1 \oplus L_2$ (with in this case $L_2 \simeq L_1^\star$), where $r \in D_{2n}$ maps L_1 to L_2. So we could write $V = L_1 \oplus r \cdot L_1$; from this description we can work out how all of G acts on V. Namely, any element of G has the form h or rh for some $h \in C_n$. We can see how such an element acts on L_1, then an element of D_{2n} acts on rL_1 by: for rx for $x \in L_1$, $g(rx) = (gr)x$, then we can see how such an element acts on any element of V. This is valid, since either $gr \in C_n$ or $gr = r\cdot$ some element of G; $(gr)x$ is then $\in L_1$ or $r \cdot L_1$ respectively.

Definition: Let $H \subset G$ be a finite group, and W a complex representation of H. The induced representation $\mathrm{Ind}_H^G(W)$ is the representation of G given bi $\mathrm{Ind}_H^G(W) = g_1 W \oplus \cdots \oplus g_r W$, where $g_1, \ldots, g_r$ are a set of representatives for $\frac{G}{H}$.

We have $\dim_{\mathbb{C}} \mathrm{Ind}_H^G(W) = [G : H] \dim_{\mathbb{C}} W$.

Recall: For any subgroup $H \subset G$, $\frac{G}{H}$ is the set of cosets $\{aH : a \in G\}$ (which is not generally a group); we have $aH = bH \Leftrightarrow b^{-1}a \in H$, so $\frac{G}{H}$ is a collection of disjoint subsets of G, each of size $|H|$, and $[G : H] = |\frac{G}{H}| = \frac{|G|}{|H|}$.

From this definition we can work out how any element of G acts on $\mathrm{Ind}_H^G(W)$; the group G permutes the subspaces $g_i W$ in exactly the way G acts on $\frac{G}{H}$ by $g(aH) = (ga)H \forall g, a \in G$.

Suppose $g \in G$ maps $g_1 H$ to $g_2 H$; how exactly does it do so? By assumption $gg_1 H = g_2 H$ so $gg_1 = g_2 h$ for some $h \in H$, so we should have $g(g_1 x) = (gg_1)x = (g_2 h)x = g_2(hx) \in g_2 W$ for any $x \in g_1 H \subset W$.

It is not completely clear that this defines a representation of G and doesn't depend on our choice of representatives for cosets; for this we need a fancier definition.

Examples: Let W be the trivial (1D) representation of a subgroup H. Then $\mathrm{Ind}_H^G(\mathbb{C})$ is simply the permutation representation $\mathbb{C}[\frac{G}{H}]$ of G acting on $\frac{G}{H}$.

Example: $\mathrm{Ind}_1^G(\mathbb{C})$ =the regular representation $\mathbb{C}G$.

Let $R_1 \subset R_2$ be rings, typically noncommutative (we will apply this to $\mathbb{C}H \subset \mathbb{C}G$); recall that representations of G are the same thing as $\mathbb{C}G$-modules.

Definition: For any R_1-module M, define $R_2 \otimes_{R_1} M$ to be the free abelian group generated by symbols $r_2 \otimes m$ for $r_2 \in R_2, m \in M$, modulo the relations $(r_1 + r_2) \otimes m = r_1 \otimes m + r_2 \otimes m, r \otimes (m_1 + m_2) = r \otimes m_1 + r \otimes m_2, r \otimes (am) = ra \otimes m \forall r \in R_2, a \in R_1, m \in M$.

Firstly, $R_2 \otimes_{R_1} M$ is an R_2-module by the obvious definition $s(r \otimes m) = sr \otimes m$ (note the order of multiplication does matter). For $H \subset G$ a subgroup, $\mathbb{C}G \otimes_{\mathbb{C}H} M$ is $\mathrm{Ind}_H^G(M)$; this is another way to define the induced representation of a $\mathbb{C}H$-module M. We relate this to the direct definition we use that $\mathbb{C}G$ is a free $\mathbb{C}H$-module, with generators any set of representatives $g_1, \ldots, g_r$ for $\frac{G}{H}$.

H acts on G by $g \cdot h = gh$; this is "very nice"; it's a free action, so G splits into r subsets gH, all of the same size as H. So $\mathbb{C}G = \bigoplus_{i=1}^r g_i \mathbb{C}H$. Therefore $\mathbb{C}G \otimes_{\mathbb{C}H} M = \bigoplus_{i=1}^r g_i M$.

14.1 Theorem

For any rings $R_1 \subset R_2$ and any R_1-module M and R_2-module N, we have $\mathrm{Hom}_{R_2}(R_2 \otimes_{R_1} M, N) = \mathrm{Hom}_{R_1}(M, N \mid_{R_1})$ where $N \mid_{R_1}$ is N considered as an R_1-module: say we have an R_2-linear map $f : R_2 \otimes_{R_1} M \to M$. Define $\phi(m) := f(1 \otimes m) \in N$ for $m \in M$; ϕ is R_1-linear since $\forall a \in R_1, m \in M, \phi(am) = f(1 \otimes am)$; $a\phi(m) = af(1 \otimes m)$ and $a \in R_1 \subset R_2$ so this is $f(a(1 \otimes m)) = f(a \otimes m) = f(1 \otimes am)$ by the definition of $R_2 \otimes_{R_1} M$. We claim f is determined by ϕ: $\forall a \in R_2, m \in M, f(a \otimes m) = af(1 \otimes m) \in N$ as f is R_2-linear. Finally, we have to show that every R_1-linear map $\phi : M \to N$ comes from an R_2-linear map $f : R_2 \otimes_{R_1} M \to N$; we want to define $f(a \otimes m) = a\phi(m) \in N$. This is cleary a well-defined function on the free abelian group on the set of symbols $a \otimes m$, and the relations defining $R_2 \otimes_{R_1} M$ are satisfied for f because $(a, m) \mapsto a\phi(m)$ is bilinear in a and m, and $(ab, m) \mapsto ab\phi(m) = a\phi(bm)$ which is the image of (a, bm) as ϕ is R_1-linear, for any $a \in R_2, b \in R_1, m \in M$.

14.2 Corollary (Frobenius reciprocity)

For groups $H \subset G$ and any representations W of H and V of G, $\mathrm{Hom}_G(\mathrm{Ind}_H^G(W, V)) = \mathrm{Hom}_H(W, V \mid_H)$ where $V \mid_H$ is V viewed as a representation of H; we have proved this.

This means $\mathrm{Ind}_H^G W$ is "the universal representation of G that contains W as a representation of H".

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Example: Let V be any representation of G, than (as a representation of $H \subset G$) contains a representation W of H. Then the smallest G-subrepresentation of V containing W is clearly $\sum_{g \in G} gW = \sum_i g_i W$ where $g_1, \ldots, g_r \in G$ are representatives of $\frac{G}{H}$. Frobenius reciprocity tells us that the H-linear map $W \hookrightarrow V$ extends uniquely to a G-linear map $\mathrm{Ind}_H^G(W) \to V$ (the LHS being $\bigoplus_i g_i W$; the above subspace is the image of this map, but it's not necessarily isomorphic to $\mathrm{Ind}_H^G(W)$, because $\sum g_i W$ need not be a direct sum.

Example: Let $G = \frac{\mathbb{Z}}{2}, H = \{1\}, V = \mathbb{C}$ the trivial representation of G. Let $W = V = \mathbb{C}$, a representation of H. So Frobenius reciprocity gives a G-linear map $\mathrm{Ind}_{\{1\}}^{\frac{\mathbb{Z}}{2}}(\mathbb{C}) \to \mathbb{C}$, which is clearly not injective.

Example: For any subgroup H of a finite group G, every irreducible representation of G is a summand of $\mathrm{Ind}_H^G(W)$ for some H-irreducible representation W: look at $\mathrm{Res}_H^G(V)$; let W be any of the irreducible constituents therof (which must exist since $V \neq 0$ as V is irreducible). Then we have a nonzero H-linear map $W \to \mathrm{Res}_H^G(V)$, so we have a nonzero G-linear map $\mathrm{Ind}_H^G(W) \to V$. Since V is irreducible this map is surjective and $\mathrm{Ind}_H^G \simeq V \oplus$some other representations by complete reducibility.

15.1 Lemma (Simple properties of induction)

1) $\mathrm{Ind}_H^G(W_1 \oplus W_2) \simeq \mathrm{Ind}_H^G(W_1) \oplus \mathrm{Ind}_H^G(W_2)$; this is trivial. 2) Let $H \subset K \subset G$ be subgroups, then $\mathrm{Ind}_K^G \mathrm{Ind}_H^K = \mathrm{Ind}_H^G$; this follows from the definition of $\mathrm{Ind}_H^G(W)$ as $\mathbb{C}G \otimes_{\mathbb{C}H} W$, since for any rings $R_1 \subset R_2 \subset R_3$ and R_1-module M, $R_3 \otimes_{R_2} (R_2 \otimes_{R_1} M) = R_3 \otimes_{R_1} M$.

Induced characters: Given a representation W of H, with character $\chi_W : H \to \mathbb{C}$, we want to compute the character of $\mathrm{Ind}_H^G(W)$. We could try extending χ_W by $\phi(g) = \chi_W(g)$ if $g \in H$, 0 otherwise, but this will not generally be a class function, so we need to average this ϕ to make it one:

15.2 Lemma

If ϕ is a class function on H, extended by 0 to all of G, and $x \in G$, then the "conjugated function" $g \in G \mapsto \phi(x^{-1}gx)$ only depends on the coset xH: we have to show that $\forall g \in G \forall h \in H \phi(x^{-1}gx) = \phi((xh)^{-1}g(xh))$; the RHS here is $\phi(h^{-1}x^{-1}gxh) = \phi(x^{-1}gx)$ as ϕ is invariant under H-conjugation.

So we can define, for any class function ϕ on H, a class function on G by $\mathrm{Ind}_H^G(\phi)(g) = \sum_{i=1}^k \phi(g_i^{-1}gg_i)$ where $g_1, \ldots, g_k$ are representatives for $\frac{G}{H}$ (this is $\frac{1}{|H|}\sum_{a\in G}\phi(a^{-1}ga)$ by the lemma).

15.3 Theorem

For finite groups $H \subset G$ and any representation W of H, the character of the representation $\mathrm{Ind}_H^G(W)$ of G is $\mathrm{Ind}_H^G(\chi_W)$: let $g_1, \ldots, g_r$ be representatives for $\frac{G}{H}$, then $\mathrm{Ind}_H^G(W) = g_1 W \oplus \cdots \oplus g_r W$. Pick any $g \in G$; we have to compute $\mathrm{tr}(g) \in \mathbb{C}$ for G acting on $\mathrm{Ind}_H^G(W)$. We only get a contribution from g acting on $g_i W$ if g maps $g_i W$ into itself, i.e. if $gg_iH = g_iH$, i.e. if $g_i^{-1}gg_i \in H$. Assuming this is so, we need the trace of g acting on g_iW: for each $x \in W$, $g(g_ix)$ is g_ihx for some $h \in H$ ($g_i^{-1}gg_i = h$), so $\mathrm{tr}(g\mid_{g_iW}) = \mathrm{tr}(h\mid_W)$, so $\chi_{\mathrm{Ind}_H^G}(g) = \sum_{\{i\in\{1,\ldots,r\}: g_i^{-1}gg_i \in H\}} \mathrm{tr}(g\mid_{g_iW}) = \sum_{1\le i\le r} \chi_W(g_i^{-1}gg_i$ where we extend χ_W by 0 in $G \setminus H$, but this is just $\mathrm{Ind}_H^G(\chi_W)$.

Example: $\frac{\mathbb{Z}}{4} = \langle(1234)\rangle \subset S_4$; we want to compute the character of the induced representation $\mathrm{Ind}_{\frac{\mathbb{Z}}{4}}^{S_4}(\alpha)$ where α is a faithful 1D representation of $\frac{\mathbb{Z}}{4}$, say $\alpha((1234)) = i$. Then che character of α is:

	1	(1234)	(13)(24)	(1432)
α	1	i	-1	-i

The induced representation of S_4 ($|\frac{\mathbb{Z}}{4}| = 4, |S_4| = 6$ so it is of dimension 6) is then:

$\lvert C\rvert$	1	6	8	3	6
C	1	(12)	(123)	(12)(34)	(1234)
$\mathrm{Ind}_{\frac{\mathbb{Z}}{4}}^{S_4}(\alpha)$	6	0	0	-2	0

Explanations: For (12)(34), only one of the 3 elements of S_4 it is conjugate to lies in H (namely (13)(24)), so $\mathrm{Ind}_{\frac{\mathbb{Z}}{4}}^{S_4}(\alpha)((12)(34)) = \frac{1}{|\frac{\mathbb{Z}}{4}|}|S_4|\frac{1}{3} \times (-1) = \frac{24}{4\times 3}(-1) = -2$. The element (1234) is conjugate to 6 elements of S_4 of which two lie in $\frac{\mathbb{Z}}{4}$, (1234) and (1432), so $\mathrm{Ind}_{\frac{\mathbb{Z}}{4}}^{S_4}(\alpha)((1234)) = \frac{1}{|\frac{\mathbb{Z}}{4}|}(\frac{1}{6}|S_4|i + \frac{1}{6}|S_4|(-i))[= 0]$.

Remark: Extending the field of a representation. Let $F \subset E$ be fields, V a vector space over F, then $E \otimes_F V$ is a vector space over E (if V has basis $\{e_i\}$

so is $\{\sum_{i=1}^n a_i e_i : a_i \in F\}$ then $V_E := E \otimes_F V$ is $\{\sum_{i=1}^n b_i e_i : b_i \in E\}$. So given a representation V of G over F, V_E is a representation of G over E; we have used this already for $\mathbb{R} \subset \mathbb{C}$.

Example: The two 3D representations of A_5 over $\mathbb{C}$ are just two ways of identifying $A_5 \subset GL(3, \mathbb{R})$ with the symmetries of an icosahedron. $(12345), (12354)$ correspond to rotations around an axis through some vertex of the icosahedron. So the character of one of these representations on (12345) has the form $\mathrm{tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0_3^a & a_4 & \end{pmatrix}$ where the a_i are the matrix for a rotation by $\frac{2\pi}{5}$ in $\mathbb{R}^2$. Over $\mathbb{C}$ this is conjugate to $\begin{pmatrix} 1 & & \\ & \xi & \\ & & \xi^{-1} \end{pmatrix}$ where $\xi = e^{\frac{2\pi i}{5}}$ (or another primitive 5th root of unity; the other representation will correspond to $\begin{pmatrix} 1 & & \\ & \xi^2 & \\ & & \xi^{-2} \end{pmatrix}$, so we can see whi there are two representations). So the value of one character on one of the conjugacy classes is $u = 1 + \xi_5 + \xi_5^{-1}$. To write this in terms of square roots we use $\xi^5 = 1, 1 + \xi + \xi^2 + \xi^3 + \xi^4 = 0$, so $u - 1 = \xi + \xi^{-1} \therefore (u-1)^2 = (\xi + \xi^{-1})^2 = \xi^2 + 2 + \xi^{-2} + 1 + (1 + \xi^2 + \xi^3) = 1 - \xi - \xi^4 = 1 - (u-1) = 2 - u \therefore u^2 + 2u + 1 = 2 - u$, and $u^2 - u - 1 = 0$ so $u = \frac{1}{2}(1 \pm \sqrt{5})$, and these are indeed the two characters we found.

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16.1 Lemma

(For $H \subset G$ finite groups, W a representation of H, V a representation of G) $\langle \mathrm{Ind}_H^G \chi_W, \chi_V \rangle_G = \langle \chi_W, \mathrm{Res}_H^G \chi_V \rangle_H$. This is Frobenius reciprocity in terms of characters, because, for any representations V_1, V_2 of G over $\mathbb{C}$, $\dim_{\mathbb{C}} \mathrm{Hom}_G(V_1, V_2) = \langle \chi_{V_1}, \chi_{V_2} \rangle$ [I didn't understand this in the lectures, nor afterwards].

We'll give a formula for $\mathrm{Res}_K^G \mathrm{Ind}_H^G W$ for K, H subgroups of G and W a representation of H. We have $\mathrm{Ind}_H^G(W) = g_1 W \oplus \cdots \oplus g_r W$ for $g_1, \ldots, g_r \in G$ representatives for $\frac{G}{H}$. How does $K \subset G$ act on this? It will break into pieces, one for each K-orbit on $\frac{G}{H}$, where K acts by $k(gH) = (kg)H$ for $k \in K, g \in G$.

Definition: The set K
G/H of <u>double cosets</u> (for K, H subgroups of G) is the set of K-orbits on $\frac{G}{H}$. Equivalently, the double cosets are the $K \times H$-orbits in G, where $k \times H$ acts on G by $(k, h)(g) = kgh^{-1}$ (so this is equivalent to the set of H-orbits on K
G the set of right cosets of K in G).

What is the stabilizer in K of an element sH in $\frac{G}{H}$, i.e. which $k \in K$ have $k(sH) = sH$? $k(sH) = sH$ iff $s^{-1}ks \in H$, i.e. $k \in sHs^{-1}$.

Definition: Let $H_s = K \cap sHs^{-1}$ for any $s \in G$; this is exactly the stabilizer in K of a point of $\frac{G}{H}$.

16.2 Theorem (Mackey's restriction formula)

Let $H, K \subset G$ be subgrous, W a representation of H. Then $\mathrm{Res}_K^G \mathrm{Ind}_H^G W \simeq \bigoplus_s \mathrm{Ind}_{H_s}^K (sW)$, where s runs over a set of representatives in G for K

G/H (the group $sHs^{-1} \subset G$ is isomorphic to H, so the representation W of H must correspond to some representation of sHs^{-1}, which we will call sW. Then for $x \in W$, $(shs^{-1})(sx) = s(hx))$. This is $\mathrm{Ind}_{H_s}^K \mathrm{Res}_{H_s}^{sHs^{-1}}(sW)$: We have $\mathrm{Ind}_H^G W = g_1 \oplus \cdots \oplus g_r W$ for g_i representatives of G/H. Each element of K maps each subspace $g_i W$ into another $g_j W$. So $\mathrm{Res}_K^G \mathrm{Ind}_H^G W = \bigoplus_s K \cdot (sW)$ where s runs over a set of representatives for K
G/H. Since H_s is the stabilizer of sH in G/H, the subgroup $H_s \subset K$ maps the vector space sW into itself (where we consider H_s acting in "the obvious way" on sW) so $K \cdot sW = \bigoplus k_i \cdot sW$ where K_i are representatives in K for K/H_s. But this $= \mathrm{Ind}_{H_s}^K(sW)$ as required.

16.3 Theorem (Mackey's irreducibility criterion

Let $H \subset G$ be groups, W a representation of H. Then Ind_H^G is irreducible if and only if 1) W is irreducible and 2) for each $s \in G \setminus H$, the representations sW and $\mathrm{Res}_{H_s}^H(W)$ are disjoint as representations of $H_s := sHs^{-1} \cap H$.

(We say representations V_1, V_2 of a group G are disjoint if no irreducible component of V_1 is (isomorphic to) an irreducible component of V_2; by Schur it is equivalent that $\mathrm{Hom}_G(V_1, V_2) = 0$). Note that $sW = \mathrm{Res}_{H_s}^{sHs^{-1}}(sW)$, which is a very similar form to $\mathrm{Res}_{H_s}^H(sW)$.

Remark: it is equivalent to state 2) for a set of representatives of H
$G/H \setminus \{1\}$; in particular, if $H \triangleleft G$ then H
G/H is the group G/H, and we just need 2) for all $s \in G/H \setminus \{1\}$.

16.4 Corollary

If $H \triangleleft G$ and W is a representation of H then $\mathrm{Ind}_H^G(W)$ is irreducible if and only if W is irreducible and W, sW are non-isomorphic $\forall s \in G/H \setminus \{1\}$.

Proof of theorem 16.3: A representation V of G is irreducible iff $\|\chi_V\|^2 = 1$. Consider $\langle \mathrm{Ind}_H^G \chi_W, \mathrm{Ind}_H^G \chi_W \rangle_G = \langle \chi_W, \mathrm{Res}_H^G \mathrm{Ind}_H^G \chi_W \rangle_H = \sum_{s \in HG/H} \langle \chi_W, \mathrm{Ind}_{H_s}^H (\mathrm{Res}_{H_s}^{sHs^{-1}}(sW)) \rangle =$ $\sum_{s \in HG/H} \langle \mathrm{Res}_{H_s}^H \chi_W, \mathrm{Res}_{H_s}^{sHs^{-1}} sW \rangle_{H_s}$ where $H_s = sHs^{-1} \cap H$. This is a sum of nonnegative integers, and for $s = 1$ we get $\langle \chi_W, \chi_W \rangle \geq 1$. So $\mathrm{Ind}_H^G \chi_W$ is irreducible iff $\|\chi_W\|^2 = 1$ and all the other terms in the sum are 0, i.e. if W is an irreducible representation of H and for all $s \in H$
$G/H \setminus \{1\}$, W and sW are disjoint as representations of $H_s = sHs^{-1} \cap H$.

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Example: Mackey's criterion for $A_4 \subset S_4$. The character table for A_4 is

$\lvert C \rvert$	1	3	4	4
C	1	(12)(34)	(123)	(132)
χ_1	1	1	1	1
χ_2	1	1	ξ	ξ^2
χ_3	1	1	ξ^2	ξ
χ_4 =permutation representation - 1	3	-1	0	0

where $\xi =$ $e^{\frac{2\pi i}{3}}$.

$A_4^{\mathrm{ab}} = \frac{A_4}{\langle (12)(34), (13), (24) \rangle} \simeq \frac{\mathbb{Z}}{3}$.

By Mackey, for an irreducible representation W of A_4, $\mathrm{Ind}_{A_4}^{S_4} W$ is irreducible iff W is not isomorphic to sW, for some $s \in S_4 \setminus A_4$ (if W is given by a homomorphism $\rho : A_4 \to GL(W)$ then sW is given by $\rho_s(g) = \rho(s^{-1}gs) \forall g \in A_4$ [the lecturer was very confused about where we should use $s^{-1}gs$ or sgs^{-1}, so I would not entirely trust this]). In other words, ρ_s is the composition of conjugation by $s \in S_4 \setminus A_4$, considered as an isomorphism $A_4 \to A_4$, with ρ.

Conjugation by s in this case clearly maps [1] and [(12)(34)] to themselves, but $(32)(123)(32) = (132)$. So we have the following characters for S_4:

C	1	(12)	(123)	(12)(34)	(1234)
ϕ_1	1	1	1	1	1
sign representation ϕ_2	1	-1	1	1	-1
permutation representation - 1 ϕ_3	3	1	0	-1	-1
$\phi_4 = \phi_3 \otimes \phi_2$	3	-1	0	-1	1
$\mathrm{Ind}_{A_4}^{S_4}(\chi_2) =: \phi_5$	2	0	$\xi + \xi^2 = -1$	2	0

.

Recall $(\mathrm{Ind}_H^G \chi_W)(g) = \sum_{s \in \frac{G}{H}} \chi_W(sgs^{-1})$ where $\chi_W(sgs^{-1})$ is taken to be 0 if $sgs^{-1} \notin H$.

We know that $\mathrm{Res}_{A_4}^{S_4} \mathrm{Ind}_{A_4}^{S_4} \chi_4 = \chi_4 \oplus s\chi_4$ ($\simeq \chi_4 \oplus \chi_4$), so, since $\mathrm{Ind}_{A_4}^{S_4} \chi_4$ is reducible, χ_4 extends to an irreducible representation of S_4 [I have no idea about this paragraph at all].

Nilpotent groups and p-groups

Definition: A p-group (for a prime p) is a group of order p^N for some $N \geq 0$.

Definition: A group G is nilpotent if there is a chain of subgroups $G \supset G^1 \supset \cdots \supset G^r = \{1\}$ such that $G^{r-1} \in Z(G)$ the centre of G, $G^{r-2} \in Z(\frac{G}{G^{r-1}})$ and so on.

17.1 Theorem

A p-group is nilpotent: it suffices to show that for $G \neq \{1\}$ a p-group, $Z(G)$ is nontrivial, then we will have the result by induction on N. Every conjugacy class in G has size dividing $|G|$ (since the size of the conjugacy class of $g \in G$ is $\frac{|G|}{|Z_G(g)|}$), so a power of p, so every element $g \in G \setminus Z(G)$ has a conjugacy class of size a multiple of p. So were $Z(G) = \{1\}$ then $|G| = 1+$ a sum of multiples of p, so $|G| = 1 \mod p$, a contradiction.

Definition: A group G is solvable if it has a chain of subgrous $G = G^0 \supset G^1 \supset \cdots \supset G^r = \{1\}$ such that $G^{i+1} \triangleleft G_i \forall i$ and $\frac{G^i}{G^{i+1}}$ is abelian $\forall i$ (clearly any abelian group is nilpotent and any nilpotent group is solvable, but not conversely, e.g. S_3 is solvable but not nilpotent (since $Z(S_3) = \{1\}$)).

Example: Let F be a field. Then the group $SUT_n(F)$ of upper-triangular $n \times n$ matricies with entries from F and diagonals all 1s is nilpotent, but for $n = 3$ it is nonabelian.

Example: $UT_n(F)$ the group of matricies a with entries from F with $a_{ii} \in F^\star, a_{ij} = 0 \forall i > j$ is solvable, but usually not nilpotent.

17.2 Theorem

Every complex irreducible representation of a finite p-group is induced from a 1D representation of some subgroup.

The proof of this uses:

17.3 Lemma

This is a clever lemma. Let G be a finite group, $A \triangleleft G$, V an irreducible representation of G. Then V is induced from a representation W of some subgroup H with $A \subset H \subset G$ such that $W\mid_A$ is isotypical (i.e. a sum of copies of the same irreducible representation) (This includes the case $H = G$ when $V\mid_A$ is isotypic).

17.4 Corollary

(A corollary to 17.3)

Under the assumptions of 17.3, if A is also abelian, then V is induced from a representation W of some subgroup H with $A \subset H \subset G$ such that A acts on W by scalars (i.e. matricies $aI : a \in \mathbb{C}$).

Proof of 17.3: Consider the isotypic decomposition of $V\mid_A$; we have $V\mid_A = V_1 \oplus \cdots \oplus V_n$ where [each] V_i is a sum of copies of some irreducible representation of A. We claim that any element of G permutes the subspaces $V_1, \ldots, V_n$: if V_i is a sum of copies of an irreducible representation W of A then gV_i is a sum of coppies of the irreducible representation gW of A (this is valid, since gW is a representation of $gAg^{-1} = A$). Since V is irreducible, G acts transitively on $\{1, \ldots, n\}$ [by these permutations].

Let $H \subset G$ be the stabilizer of $1 \in \{1, \ldots, n\}$ (so H maps V_1 into itself). Then we see that $V = \mathrm{Ind}_H^G(V_1)$, and V_1 is a representation of H such that $V_1\mid_A$ is isotypic.

17.5 Lemma

(A little lemma)

Let G be a nonabelian p-group. Then G contains a normal, nonabelian, non-central subgroup: We know G is nilpotent; in particular, $\frac{G}{Z(G)}$ (which $\neq 1$ since G is nonabelian) has a nontrivial center. Let g be an element of $G \setminus Z(G)$ that maps into $Z(\frac{G}{Z(G)})$ [under the quotient map]; let $A = \langle Z(G), g\rangle \subset G$. Clearly $A \not\subseteq Z(G)$. A is abelian, since all elements of $Z(G)$ commute with g, so any pair of elements of A commutes. To show A is normal, it suffices to show that any G-conjugate of any element of A lies in A; this is clearly true for elements of $Z(G)$. For a general $x \in G$, $xgx^{-1} = g \times$ (some element of $Z(G)$), because g maps into $Z(\frac{G}{Z(G)})$, so we have the result.

Proof of theorem 17.2: Let V be an irreducible representation of a p-group G of $\mathbb{C}$. If G is abelian, V is 1D and we have nothing to prove. Suppose G is not abelian; by lemma 17.5 G contains an abelian normal subgroup $A \not\subseteq Z(G)$. By corollary 17.4, V is induced from a representation W of a subgroup H with $A \subset H \subset G$ such that A acts by scalars on W. It follows that $\ker(H \xrightarrow{\rho} GL(W)) \neq \{1\}$ (because $\rho(A) \subset Z(\rho(G))$).

18 Lecture

Each value in our character for $\mathrm{Ind}_{A_4}^{S_4}(\alpha)$ above is a sum of two roots of unity. If $g \sim \begin{pmatrix} \xi^i & \\ & \xi^j \end{pmatrix}$, we have $|\mathrm{Re}(\chi(g))| \leq \chi(1)$, with equality iff g acts by 1 on this representation. So our last representation is <u>not</u> faithful; its kernel is $\{1, (12)(34), (13)(24), (14)(23)\}$. So S_4 acts on this 2D representation through its quotient $\frac{S_4}{\text{that group}}$, which is $\simeq S_3$ (the 2D irreducible representation of S_3 has character $2, -1, 0$ on the classes 1,(123),(12) respectively). This is rather special; S_5 cannot map onto S_n for $2 < n < 5$ as A_5 is simple.

Why does S_4 map onto S_3? One can think of S_4 as the group of rotations of $\mathbb{R}^3$ preserving the cube (it permutes the four diagonals of the cube), then the homomorphism $S_4 \twoheadrightarrow S_3$ comes from the way S_4 permutes the three coordinate axes.

Theorem: Every irreducible complex representation of a p-group is induced from a 1D representation of some subgroup: suppose we know this for all p-groups of order $< |G|$. Let V bet an irreducible representation of G, $\rho : G \to GL(V)$. If $\ker(\rho) \neq \{1\}$ then V is a representation of the smaller p-group $\frac{G}{\ker\rho}$; we have $V = \mathrm{Ind}_{\frac{H}{\ker\rho}}^{\frac{G}{\ker\rho}}$(some 1D representation α), where H is the inverse image under $G \to \frac{G}{\ker\rho}$ of some subgroup of $\frac{G}{\ker\rho}$. We can think of α as a 1D representation of H, and then $V = \mathrm{Ind}_H^G(\alpha)$ and we are done. So we may assume V is a <u>faithful</u> irreducible representation of G. If G is abelian we are done (V must be 1D), so we can assume G is not abelian; by the previous lecture G has an abelian normal subgroup $A \lhd G$ with $A \not\subseteq Z(G)$. By another lemma from the previous lecture, there is a subgroup H with $A \subset H \subset G$ and irreducible representation W of H such that $V = \mathrm{Ind}_H^G(W)$ and A acts by scalars on W. But then this representation can't be faithful, because scalars commute with <u>every</u> matrix, so this case can't arise.

18.1 Theorem (Burnside's $p^a q^b$ theorem

Every group of order $p^a q^b$ for p, q prime is solvable; equivalently the order $|G|$ of any nonabelian simple group G has at least 3 distinct prime factors. (This is the best possible result, since A_5 is a nonabelian simple group with $|A_5| = 60 = 2^2 \times 3 \times 5$ having exactly 3 distinct prime factors. This result is a key first step in the classification of all finite simple groups, which was only finished in around 2005: Theorem (roughly): Any finite simple group is isomorphic to one of: 1) $\frac{\mathbb{Z}}{p}$ for p prime 2) A_n for $n \geq 5$ 3) "Groups of Lie p type": roughly, groups of matricies over a finite field, e.g. $PSL_n(\mathbb{F}_q)$ where $q = p^r$ a prime power, $n \geq 2$ or $PSO_n(\mathbb{F}_q)$ or $PSP_{Z_n}(\mathbb{F}_n)$ or a few variants of these. 4) 26 "sporadic" finite simple groups, including the "Monster").

The theorem follows from these two lemmas:

18.2 Lemma

For each character χ of a finite group G and each $g \in G$, $|\chi(g)| = \chi(1) \Leftrightarrow \frac{\chi(g)}{\chi(1)}$ is a nonzero algebraic integer.

18.3 Lemma

If G has order $p^a q^b$ then there is a nontrivial irreducible character χ of G and $g \neq 1 \in G$ such that $\frac{\chi(g)}{\chi(1)}$ is a nonzero algebraic integer.

Given these two lemmas, Burnside follows: STP that a nonabelian simple group G of order $p^a q^b$ does not exist. Suppose G is such a group; by lemma 18.3 there is $g \neq 1 \in G$ and an irreducible character $\chi \neq 1$ such that $\frac{\chi(g)}{\chi(1)}$ is an algebraic integer. By lemma 18.2, $|\chi(g)| = \chi(1) = \dim V$ for the space V with character χ. Say $\chi(1) = \dim V = n$, then $\chi(g)$ is a sum of n roots of unity, since it is the trace of a diagonal $n \times n$ matrix with entries roots of unity. Now $|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$ with equality iff there are $z \in \mathbb{C}, a_i \in \mathbb{R}_{\geq 0}$ such that $z_i = a_i z \forall i$. So $|\chi(g)| = 1 \Rightarrow$ all the eigenvalues of g acting on V are equal, i.e. g acts as a scalar on V, so g commutes with all of V. But $Z(G) = \{1\}$ since G is nonabelian simple. Also, this representation is faithful, because it's nontrivial and G is simple. So $g \in Z(G)$ so $g = 1$, a contradiction.

Proof of lemma 18.2: Let χ be a character of a finite group G, $g \in G$. The forward implication is easy: if $|\chi(g)| = \chi(1)$ then g acts by a scalar c, a root of unity, on V. Then $\chi(g) = nc$ and so $\frac{\chi(g)}{\chi(1)} = \frac{nc}{n} = c$, which is a root of unity so an algebraic integer. For the reverse, suppose $\frac{\chi(g)}{\chi(1)}$ is an algebraic integer. Write $n = \dim V = \chi(1)$ and let $c_1, \ldots, c_n$ be the eigenvalues of g on V; they are roots of unity. Thus $\frac{\chi(g)}{\chi(1)} = \frac{c_1 + \cdots + c_n}{n}$. Clearly $|\frac{\chi(g)}{\chi(1)}| \leq 1$. Appealing to Galois theory, this is a number in $\mathbb{Q}(\zeta)$ where $\zeta = e^{\frac{2\pi i}{N}}$ for $N = |G|$. This is the splitting field of the polynomial $x^N - 1$. Then it is a fact (a theorem in Galois theory) that $\{\alpha \in \mathbb{Q}(\zeta) : g(\alpha) = \alpha) \forall g \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})\} = \mathbb{Q}(\zeta)^{\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} = \mathbb{Q}$.

Consider the "norm" of $\frac{\chi(g)}{\chi(1)}$, by which we mean the product of all "Galois conjugates" of $\frac{\chi(g)}{\chi(1)} \in \mathbb{Q}(\zeta)$. By the fact, this norm lies in $\mathbb{Q}$ (since it's fixed by the Galois group), but it's an algebraic integer because all Galois conjugates of an algebraic integer [are algebraic integers]. So this norm is an "ordinary" integer $\in \mathbb{Z}$. But the norm is a product of expressions $\frac{\text{sum of n nth roots of unity}}{n} \in \mathbb{C}$, all of which have absolute value ≤ 1, so the norm must be ± 1, and we see that our original number $\frac{\chi(g)}{\chi(1)}$ must have absolute value 1.

19 Lecture

All these arguments are due to Burnside in around 1900. To prove the theorem, it remains to prove Lemma 18.2: claim: we can find a conjugacy class $C \subset C, C \neq \{1\}$ of size a power of p, and an irreducible character $\chi \neq 1$ with $\chi(C) \neq 0$ such that $\chi(1)$ is not a multiple of p. Given this claim, $\frac{\chi(C)}{\chi(1)}$ is clearly nonzero, and it is an algebraic integer using the fact that for any irreducible character χ and conjugacy class C, $\frac{|C|}{\dim \chi}\chi(C)$ is an algebraic integer, which comes from: for an irreducible V, we can view V as a $\mathbb{C}G$-module, and $Z(G)$ acts by scalars on V. This gives a ring homomorphism $Z(\mathbb{Z}G) \to \mathbb{C}$; because $Z(\mathbb{Z}G)$ is a finitely generated $\mathbb{Z}$-module and also a ring, the image of this consists of algebraic integers. But we found an explicit formula for this homomorphism; in particular, for each conjugacy class $C \subset G$, $\sum_{g \in C} g \mapsto \frac{|C|}{\dim \chi}\chi(C)$. So given the claim, we know that $|C|\frac{\chi(C)}{\chi(1)}$ is an algebraic integer, and $|C|$ is a power of p and $\chi(1)$ is not a multiple of p. To deduce that $\frac{\chi(C)}{\chi(1)}$ is an algebraic integer, we use that $\chi(1), |C|$ are relatively prime: thus $1 = a\chi(1) + b|C|$ for some $a, b \in \mathbb{Z}$, so $\frac{\chi(C)}{\chi(1)} = a\chi(C) + b\frac{|C|}{\chi(1)}\chi(C)$, and both these terms are algebraic integers (the second by the above, the first because $\chi(C)$ is a sum of roots of unity), so the claim implies the lemma. Proof of the claim: To find a conjugacy class $C \neq \{1\}$ of size a power of p, it's equivalent to find a $g \neq 1$ with $|Z_G(g)| = p^r q^b$ for some $0 \leq r \leq a$ (as then $|C| = \frac{|G|}{|Z_G(g)|}$ is a power of p). Use Sylow's theorem: there is a subgroup $H \subset G$ of order q^b. This H is a q-group, so it has nontrivial centre (for $b \geq 1$; for $b = 1$ the group is nilpotent and we have the result). Let $g \in Z(H) \setminus \{1\}$, then $Z_G(g)$ is a subgroup containing the q-Sylow subgroup, so has order as required. Use the orthogonality of columns in the character table for the columns $\{1\}$ and C; this tells us that $1 + \sum_{\text{nontrivial irreducible characters}} \chi(C)\overline{\chi(1)} = 0$ Since $\chi(1)$ is a dimension this is $1 + \sum_{\chi \neq 1} \chi(C)\chi(1)$; if every irreducible representation $\neq 1$ had dimension a multiple of p, then we would have $1 + (\text{algebraic integer}) \times p = 0$, so $\frac{1}{p}$ would be an algebraic integer, which is false. So there is an irreducible character $\chi \neq 1$ with $\chi(1)$ not a multiple of p; in fact we may insist this is so for $\chi(C) \neq 0$, as we can remove the terms with $\chi(C) = 0$ from the above sum.

Compact Groups

Definition: A topological group G is a group that is also a topological space such that the multiplication map $G \times G \to G : (g, h) \mapsto gh$ and map $g \mapsto g^{-1}$ are continuous.

Example: Any group G can be viewed as a topological group with the discrete topology (since then every function $G \to S$ is continuous).

Example: $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are topological groups when considered as subspaces of $\mathbb{R}^{n^2}, \mathbb{C}^{n^2}$.

A compact group is a topological group which is compact as a topological space.

Some examples of compact groups: finite groups, the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ under multiplication, $O(n) \subset GL(n, \mathbb{R})$ (i.e. $\{A \in GL(n, \mathbb{R}) : AA^T = 1\} = \{A \in GL(n, \mathbb{R}) : \|Ax\| = \|x\| \forall x \in \mathbb{R}^n\} = \{A \in GL(n, \mathbb{R}) : \langle Ax, Ay \rangle = \langle x, y \rangle \forall x, y \in \mathbb{R}^n\}$) ($O(n)$ contains rotations and reflections). Note that if $A \in O(n)$ then $\det A = \pm 1$.

Definition: $SO(n)$, the special orthogonal group, is $\{A \in O(n) : \det A = 1\}$; intuitively this is the group of rotations but not reflections.

Examples: $O(1) \simeq \frac{\mathbb{Z}}{2}, SO(1) = \{1\}, SO(2) \simeq S^1, \frac{O(2)}{S^1} \simeq \frac{\mathbb{Z}}{2}$. $SO(3)$ is the group of rotations about various axes in $\mathbb{R}^3$.

Why is $O(n)$ compact? One can describe it as the set of orthonormal bases in $\mathbb{R}^n$, e.g. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2) \Leftrightarrow \{\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\}$ is an ON basis for $\mathbb{R}^2$. The set of ON bases of $\mathbb{R}^n$ is $\{(v_1, \ldots, v_n) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n : \langle v_i, v_j \rangle = \delta_{ij}\}$ which is a closed subset of $\mathbb{R}^{n^2}$ (it's the inverse image of a single point of $\mathbb{R}$ under a continuous function), and it's clearly bounded since $\|v_i\| = 1 \forall i$, so it's compact.

Similarly, $U(n) = \{A \in GL(n, \mathbb{C}) : A\overline{A^T} = 1\}$ is a compact group. If $A \in U(n)$ then $|\det A| = 1$.

Definition: $SU(n) = \{A \in U(n) : \det A = 1\}$; we have $\frac{U(n)}{SU(n)} \simeq S^1$ with the isomorphism being the usual determinant.

Example: $U(1) \simeq SO(2) \simeq S^1$. $SU(1) = \{1\}$. $SU(2)$ is homeomorphic to S^3: take an element $A \in U(2)$ and ask what $A(e_1), A(e_2) \in \mathbb{C}^2$ are (e_i being the standard basis vectors). Clearly $A(e_1) \in S^3$ considered as $\{z \in \mathbb{C}^2 : \|z\| = 1\}$; if we know $A(e_1)$ then $A(e_2)$ can be any element of length 1 in the $\mathbb{C}$-line $(Ae_1)^\perp \simeq \mathbb{C} \subset \mathbb{C}^2$. If $A \in SU(2)$ then A is uniquely determined by $A(e_1) \in S^3$, and $A(e_1)$ may take any value in S^3, so $SU(2) \simeq S^3$.

In fact S^1 and S^3 are the only spheres that have the structure of a topological group.

20 Lecture

Quaternions are the noncommutative $\mathbb{R}$-algebra of dimension 4 (as a $\mathbb{R}$-vector space), with basis $1, i, j, k$. Multiplication is given by (Hamilton) $i^2 = j^2 = k^2 = -1, ij = k, ji = -k$ and cyclic permutations of these last two. This is a division algebra or "noncommutative field": $\forall x \neq 0 \in \mathbb{H} \exists y \in \mathbb{H} : xy = yx = 1$; write $y = x^{-1}$. We cannot write $\frac{a}{b}$ for general $a, b \in \mathbb{H}$ since ab^{-1} does not generally $= b^{-1}a$. To prove that we do have a division algebra, define conjugation in $\mathbb{H}$ by $\overline{a + bi + cj + dk} = a - bi - cj - dk$; with this definition we find $(a + bi + cj + dk)(\overline{a + bi + cj + dk}) = a^2 + b^2 + c^2 + d^2 \in \mathbb{R}$, so define $|x| = \sqrt{x\bar{x}}$ for $x \in \mathbb{H}$; this is real, ≥ 0, and > 0 if $x \neq 0$. So if $x \neq 0 \in \mathbb{H}$then $\frac{\bar{x}}{|x|^2}$ is an inverse for x as required.

Notice that $\{x \in \mathbb{H} : |x| = 1\} \simeq S^3$, e.g. by considering $\mathbb{H}$ as $\simeq \mathbb{R}^4$. This is a group under multiplication because in fact $|xy| = |x||y| \forall x, y \in \mathbb{H}$. Note the parallel with $S^1 \subset \mathbb{C}$.

This group S^3 is isomorphic to the group $SU(2) = \{\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1, a, b \in \mathbb{C}\{$. Clearly this is homeomorphic to S^3 which we can consider as $\{(a, b) : |a|^2 + |b|^2 = 1, a, b \in \mathbb{C}\}$, but we want a group isomorphism to $S^3 \subset \mathbb{H}$. Multiplication on $SU(2)$ is given by $\begin{pmatrix} a_1 & \dots \\ b_1 & \dots \end{pmatrix} \begin{pmatrix} a_2 & \dots \\ b_2 & \dots \end{pmatrix} = \begin{pmatrix} a_1a_2 - \bar{b}_1b_2 & \dots \\ b_1a_2 + \bar{a}_1b_2 & \dots \end{pmatrix}$, but this is exactly the formula for multiplication of quaternions: $(a_1 + jb_1)(a_2 + jb_2) = (a_1a_2 - \bar{b}_1b_2) + j(b_1a_2 + \bar{a}_1b_2)$ (any quaternion can be writtten $a + jb$ for $a, b \in \mathbb{C}$) using the fact that for $z \in \mathbb{C} \subset \mathbb{H}, zj = j\bar{z}$.

Schur's lemma for $\mathbb{C}$-representations of topological gorups is true by the same argument as for finite groups. Averaging arguments work for compact groups G using "Haar measure" $\int_G$: {continuous functions$G \to \mathbb{C}\} \to \mathbb{C}$. It has the properties $\int_G 1 = 1$ (i.e. the volume of G is 1), left- and right-invariance $\int_G f(g)dg = \int_G f(gh)dg = \int_G f(hg)dg \forall h \in G$. If G is finite we can just define Haar measure by $\int_G f(g)dg = \frac{1}{|G|}\sum_{g \in G} f(g) \in \mathbb{C}$. For $G = S^1$, Haar measure is just $\int_G f(e^{i\theta})dg := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta$. Haar measure exists for all compact groups, but not for e.g. $\mathbb{R}$.

For $SU(2)$ and other groups, we will later give explicit formulae for Haar measure. For $SU(2) \simeq S^3$, Haar measure is the standard form on S^3, scaled to have volume 1.

Definition: A finite dimensional representation of a topological group G on a complex vector space V is a continuous group homomorphism $G \to GL(V) \simeq GL(n, \mathbb{C})$. The character of a representation is given by $\chi_V(g) = \mathrm{tr}(g \mid_V) \in \mathbb{C}$ as before; this is a continuous class function on G.

20.1 Proposition

Every finite dimensional complex representation of a compact group G is unitarizable; therefore representations of G are completely reducible.

20.2 Theorem

Characters of complex irreducible representations of a compact group G have L^2-norm 1 and the characters of two non-isomorphic irreducible representations are orthogonal, where we define $\langle \chi_1, \chi_2 \rangle := \int_G \chi_1(g)\overline{\chi_2(g)}dg \in \mathbb{C}$.

20.3 Corollary

A representation V of a compact group G is irreducible iff $\|\chi_V\|^2 = 1$: write $V = W_1^{\oplus n_1} \oplus \cdots \oplus W_r^{\oplus n_r}$ for the W_i irreducible (by 20.1). Then by 20.2, $\|\chi_V\|^2 = \langle \sum n_i \chi_{W_i}, \sum n_j \chi_{W_j} \rangle = \sum n_i^2$.

Completeness of character means: the Hilbert space of L^2 class functions, by which we mean class functions with finite length, is a "Hilbert space direct sum" of the characters of irreducible representations; it has an orthonormal basis given by the character of irreducible representations: the spcae of L^2 class functions on G is $\simeq \bigoplus_{i=1}^{\infty} \mathbb{C} \cdot \chi_{V_i}$, i.e. any L^2 class function has the form $\sum_i a_i \chi_{V_i}$ for some $a_i \in \mathbb{C}$ with $\sum |a_i|^2 < \infty$.

20.4 Theorem

Every complex irreducible representation of S^1 is isomorphic to the 1D representation $z \mapsto z^n$ for some integer n: by Schur's lemma, every irreducible representation of an abelian group is 1D, so we have to classify continuous homomorphisms $S^1 \to \mathbb{C}^\star = GL(1, \mathbb{C})$. Clearly the image of such a homomorphism must be $\subset S^1$¿ Notice that $\rho : S^1 \to \mathbb{C}^\star$ must send any nth root of 1 to another nth root of 1. Since ρ is continuous, there is a $n > 0$ such that $\rho(\{e^{2\pi i\theta} : \theta \in [-\frac{1}{n}, \frac{1}{n}\}) \subset \{e^{i\theta} : \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$. So in particular $\rho(e^{\frac{2\pi i}{n}}) = e^{\frac{2\pi i a}{n}}$ for some integer $a \in (-\frac{n}{4}, \frac{n}{4})$. We claim $\rho(z) = z^a \forall z \in S^1$; first, we'll show that $\rho(e^{2\pi i \frac{1}{2^r n}}) = e^{2\pi i \frac{a}{2^r n}} \forall$ integers $r \geq 0$. We know this for $r = 0$; if we know this for smaller values of r, then $\rho(e^{2\pi i \frac{1}{2^r n}}) = \pm e^{2\pi i \frac{a}{2^r n}}$ But we know that $\rho(e^{2\pi i \frac{1}{2^r n}}) \in \{e^{i\theta} : \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$, so the $\pm$ must be $+$ and we have induction: $\rho(z) = z^a$ for $z = e^{2\pi i \frac{1}{2^r n}}$ for all $r \geq 0$. But since ρ is a homomorphism, then $\rho(z) = z^a$ for all $(2^r n)$th roots of 1, for all $r \geq 0$, and this forms a dense set of points in S^1, and $\rho(z), z^a$ are continuous, so $\rho(z) = z^a \forall z \in S^1$.

21 Lecture

Fourier series: The Hilbert space $L^2(S^1) = \{f : S^1 \to \mathbb{C} : \int |f|^2 < \infty\}$ has an ON basis given by the characters of the irreducible representations of S^1. In this particular case the characters are simply the functions $f_n(\theta) = e^{in\theta}$ or $f_n(z) = z^n$ for $n \in \mathbb{Z}$, so we can expand any function f in terms of characters as $f(\theta) = \sum_{n\in\mathbb{Z}} \langle f, f_n\rangle f_n(\theta)$ where $\langle f, f_n\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{f_n(\theta)}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta$, the standard formula for the coefficients a_n of the Fourier series of f. So we can see this course as a generalization of Fourier analysis to all finite groups; we can use the representation theory for e.g. $(\frac{\mathbb{Z}}{n})^2$ for problems like encoding videos. The coefficients of a_n for n near 0 are usually the most important.

21.1 Conjugacy classes in $U(n)$ and $SU(n)$

Every unitary matrix is diagonalizable, in fact by a unitary matrix. Moreover, a unitary matrix has diagonal entries in $S^1 \subset \mathbb{C}$. So any $A \in U(n)$ is conjugate to a diagonal matrix with diagonal entries $e^{i\theta_1}, \ldots, e^{i\theta_n}$ for $\theta_1, \ldots, \theta_n \in \mathbb{R}$. So two unitary matricies are conjugate iff they have the same eigenvalues (possibly in different orders), so the set of conjugacy classes in $U(n)$ is $\simeq \frac{(S^1)^n}{S_n}$ where S_n acts on $(S^1)^n$ by permuting the n points. Moreover, the set of conjugacy classes in $SU(n)$ is $\ker((S^1)^n \to s^1 : (z_1, \ldots, z_n) \mapsto z_1 z_2 \ldots z_n)/S_n$. So any matrix in $SU(2)$ is conjugate to a matrix $\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$ and this is unique except that we can replace θ with $-\theta$. So the set of conjugacy classes in $SU(2)$ is $= [0, \pi]$. Under $SU(2) = S^3 \subset \mathbb{H} \simeq \mathbb{R}^4$, the matricies $\{\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}\}$ form a subgroup $\simeq S^1$ which corresponds to $S^1 \subset \mathbb{C} \subset \mathbb{H}$. In fact, the conjugacy classes in S^3 are exactly the subsets $\{x \in S^3 : \mathrm{Re} x = a\}$ for $a \in [-1, 1]$; here $\theta \in [0, \pi]$ and $a \in [-1, 1$ are related by $a = \mathrm{Re}(e^{i\theta}) = \cos\theta$. Finally, notice that for any $A \in SU(2)$, $A \sim \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$, $\mathrm{tr} A = \mathrm{tr}\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$.

21.2 Proposition

Elements of $SU(2)$ are conjugate iff they have the same value of $\frac{1}{2}\mathrm{tr} A \in [-1, 1]$.

Haar measure on $SU(2)$ is the standard volume form on $S^3 \subset \mathbb{R}^4$, scaled to have total volume 1. To check this, one should check that left and right multiplication by unit quaternions $\in S^3$ preserve the Euclidean metric on $\mathbb{R}^4$, and hence volumes on S^3; this follows from the formula $|xu| = |ux| = |x|$ for $|u| = 1, x \in \mathbb{H} \simeq \mathbb{R}^4$ where $|x|$ is the Euclidean length of x (In fact, more generally, $|ab| = |a||b| \forall a, b \in \mathbb{H}$).

21.3 Theorem (Weyl integration formula)

Let f be a continuous class function on $SU(2)$. Then $\int_{SU(2)} f(g)dg = \frac{1}{4\pi}\int_0^{2\pi} f(\theta)|\Delta(\theta)|^2 d\theta = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin^2\theta d\theta$, where Δ is the Weyl denominator $\Delta(\theta) = e^{i\theta} - e^{-i\theta}$, and $f(\theta)$ means $f\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \in \mathbb{C}$ (The two last versions are the same since $\Delta(\theta) = e^{i\theta} - e^{-i\theta} = i\sin\theta$): we have $\int_{S^3} f(g)dg = \int_0^{\pi}\int_{S^2 \times d\theta} f(g)dxd\theta$ where $x \in S^2$ with a given value of θ (the notation is quite silly and confusing, but the reader should be able to understand the result). This is $\int_0^{\pi}(\text{area of } S^2 \text{ for this} \theta) f(\theta)d\theta = \int_0^{\pi} 4\pi \sin^2\theta f(\theta)d\theta$. We have to scale this formula by a constant to make $\int_{S^3} 1dg =$

1: $\int_0^\pi 4\pi \sin^2\theta d\theta = 2\pi^2$ (= the volume of S^3 in the standard Euclidean metric), so using Haar measure, $\int_{SU(2)} f(g)dg = \frac{1}{2\pi^2}\int_0^\pi 4\pi\sin^2\theta f(\theta)d\theta = \frac{2}{\pi}\int_0^\pi \sin^2\theta f(\theta)d\theta = \frac{1}{\pi}\int_0^{2\pi}\sin^2\theta f(\theta)d\theta$.

Symmetric and exterior powers

Recall: for a complex vector space V, S_n acts on the space $V^{\otimes n}$ and we defined $S^nV = \{x \in V^{\otimes n} : \sigma(x) = x \forall \sigma \in S_n\}, \Lambda^n V = \{x \in V^{\otimes n} : \sigma(x) = \text{sgn}(\sigma)x \forall \sigma \in S_n\}$. Because representations of S_n are completely reducible, we can also think of these as quotients of $V^{\otimes n}$: $S^nV = \frac{V^{\otimes n}}{x_1\otimes\cdots\otimes x_n = x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}\forall\sigma\in S_n}, \Lambda^nV = \frac{V^{\otimes n}}{x_1\otimes\cdots\otimes x_n = (\text{sgn}(\sigma))x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}\forall\sigma\in S_n}$. We use the notation $x_1x_2\dots x_n$ for the image of $x_1\otimes\cdots\otimes x_n$ in S^nV and $x_1\wedge\cdots\wedge x_n$ for the image of it in Λ^nV; we have $x_1x_2 = x_2x_1$ and $x_1\wedge x_2 = -x_2\wedge x_1$ and so on. In particular, if we choose a basis $y_1,\dots,y_r$ for V then S^nV becomes the vector space of homogenous polynomials of degree n in $y_1,\dots,y_r$.

22 Irreducible Representations of $SU(2)$

The trivial representation of dimension 1 is irreducible. Also $SU(2)$ has a "standard" $2D$ rep by considering it as a subset of $GL(2,\mathbb{C})$ (to check this is irreducible, consider its restriction to $S^1 = \{\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta}\end{pmatrix} : \theta\in\mathbb{R}\}\subset SU(2)$; it is the sum of the two non-isomorphic 1D representations $z\mapsto z$ and $z\mapsto z^{-1}$, so the only possible (nontrivial) $SU(2)$-invariant subspaces of V are the two lines $\mathbb{C}\cdot(1,0)$ and $\mathbb{C}\cdot(0,1)$, but these are not preserved by, for example, $\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\in SU(2)$, so V is irreducible). For any $n\geq 0$, we shall show that S^nV is an irreducible representation of $SU(2)$, and that every irreducible representation of $SU(2)$ is isomorphic to S^nV for some $n\geq 0$.

If we take e_1, e_2 a basis for V then S^nV has a basis $e_1^n, e_1^{n-1}e_2,\dots,e_2^n$, so $\dim_\mathbb{C} S^nV = n+1$. To describe the character of S^nV, it's enough to describe its restriction to $S^1\subset SU(2)$: for $z\in S^1$, $\begin{pmatrix} z & \\ & z^{-1}\end{pmatrix}\in SU(2)$ maps $e_1^ae_2^{n-a}$ to $(ze_1)^a(z^{-1}e_2)^{n-a} = z^{2a-n}e_1^ae_2^{n-a}$. So $\chi_{S^nV}(z) = \sum_{a=0}^n z^{2a-n} = z^n + z^{n-2} + \cdots + z^{-n}$; the reader will simply need to learn this for the exam.

Example: $\chi_1(z) = 1, \chi_V(z) = z + z^{-1}, \chi_{S^2V} = z^2 + 1 + z^{-2}$.

22.1 Theorem

The representation S^nV of $SU(2)$ is irreducible: STP $\|\chi_{S^nV}\|^2 = 1$. We have $\|\chi_{S^nV}\|^2 = \int_{SU(2)} |\chi_{S^nV}(g)|^2dg = \frac{1}{4\pi}\int_0^{2\pi}|\chi_{S^nV}(z)|\cdot|\Delta(z)|^2d\theta$ (where we consider $z = e^{i\theta}$). This is $\frac{1}{4\pi}\int_0^{2\pi}(z^n + z^{n-2} + \cdots + z^{-n})(z^n + z^{n-2} + \cdots + z^{-n})(z - z^{-1})(z^{-1} - z)d\theta$ since $\overline{z^n} = z^{-n}$ for $z = e^{i\theta}\in S^1$; this is $\frac{1}{4\pi}\int_0^{2\pi}(z^{n+1} - z^{-(n+1)})(-z^{n+1} + z^{-(n+1)})d\theta = \frac{1}{4\pi}\int_0^{2\pi}(-z^{2n+2} + 2 - z^{-(2n+2)})d\theta$. We have $\frac{1}{2\pi}\int_0^{2\pi} z^n d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}d\theta = \{1 \text{ if } n = 0, 0 \text{ otherwise}\}$, so this is $\frac{1}{4\pi}2\times 2\pi = 1$. So we have the result.

Why is this all the irreducible representations of $SU(2)$? Let W be a finite dimensional representation of $SU(2)$. Then the character of W (on $S^1 \subset SU(2)$) is a Laurent polynomial (with integer coefficients) $b_1 z^{a_1} + b_2 z^{a_2} + \cdots + b_r z^{a_r}$, $b_i \in \mathbb{Z}, a_i \in \mathbb{Z}$, because $W|_{S^1}$ is a direct sum of 1D representations $z \mapsto z^a, a \in \mathbb{Z}$. Moreover, the character of any such W is an even Laurent polynomial, $f(z) = f(z^{-1})$, because $\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$ is conjugate to $\begin{pmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{pmatrix}$ in $SU(2)$. So $\chi_W(z) = \chi_W(z^{-1})$. Then just observe that che characters of the irreducible representations of $SU(2)$ [that we have already found] span the abelian group of all even Laurent polynomials. So every irreducible representation of $SU(2)$ is isomorphic to $S^n V$ for some $n \geq 0$.

Tensor products of representations of $SU(2)$

What is the representation $V \otimes V$ (i.e. how does it decompose into irreducibles)? It's a 4D representation of $SU(2)$. We always have $V \otimes V \simeq S^2 V \oplus \Lambda^2 V$ (this is a general result for any group and any representation). In $SU(2)$ $S^2 V$ is an irreducible representation of dimension 3; $\Lambda^2 V$ is then a 1D representation, so must be irreducible and $\simeq \mathbb{C}$ (the trivial representation).

What does it mean that $SU(2)$ acts trivially on $\Lambda^2 V$? For any group acting on an n-dimensional vector space V, G acts on $\Lambda^n V$ which has dimension $\binom{n}{n} = 1$ ($\dim_{\mathbb{C}} \Lambda^a(\mathbb{C}^b) = \binom{b}{a}$). So the action of G on $\Lambda^n V$ must be given by a homomorphism $G \to \mathbb{C}^\star$. If G acts on V by $\rho : G \to GL(n, \mathbb{C})$ then it acts on $\Lambda^n V$ by $\det \rho : G \to \mathbb{C}^\star$. And $\det \rho = 1$ for $\rho : SU(2) \to GL(2, \mathbb{C})$.

We can answer similar questions about e.g. $V^{\otimes 3} = V \otimes V \otimes V$ or $V \otimes S^2 V$ "by hand".

Example: Let $W = V \otimes S^2 V$, a representation of $SU(2)$ of dimension $2 \times 3 = 6$. We want to decompose it into irreducibles; its character is $\chi_W(z) = \chi_V(z)\chi_{S^2V}(z) = (z + z^{-1})(z^2 + 1 + z^{-2}) = z^3 + 2z + 2z^{-1} + z^{-3})$. This must decompose (uniquely) as a sum of characters of irreducible representations of $SU(2)$; in this clase it is clearly $(z^3 + z + z^{-1} + z^{-3}) + (z + z^{-1})$, so there is an isomorphism of representations of $SU(2)$ $V \otimes S^2 V \simeq S^3 V \oplus V$ (the dimensions are correct here, because $\dim S^3 V = 4$). The $SU(2)$-linear map $V \otimes S^2 V \to SU^3 V$ is just multiplication of polynomials. Some elements of its kernel are e.g. $e_1 \otimes (e_1 e_2) - e_2 \otimes (e_1^2) \neq 0 \in \ker(V \otimes S^2 V \to S^3 V)$.

22.2 Theorem (Clebsch-Gordon)

For any integers $0 \leq p \leq q$, $S^p V \otimes S^q V \simeq S^{p+q} V \oplus S^{p+q-2} V \oplus \cdots \oplus S^{q-p} V$ as a representation of $SU(2)$: It suffices to compute characters. We have $\chi_{S^pV \otimes S^qV} = (\frac{z^{q+1} - z^{-(q+1)}}{z - z^{-1}}(z^p + z^{p-2} + \cdots + z^{-p})$ (as $\chi_{S^qV} = z^q + z^{q-2} + \cdots + z^{-q} = \frac{z^{q+1} + z^{-(q+1)}}{z - z^{-1}}$. This $= \sum_{a=0}^{p} (\frac{z^{p+q+1-2a} - z^{p-q-1-2a}}{z - z^{-1}}) = \sum_{a=0}^{p} \chi_{S^{p+p-2a}V}(z)$, because this last is $\sum_{a=0}^{p} \frac{z^{p+q-2a+1} - z^{-p-q+2a-1}}{z - z^{-1}}$, and rearranging the sums these are equal.

22.3 Theorem

$SO(3) \simeq \frac{SU(2)}{\{\pm 1\}}$, where $-1 = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in SU(2)$, $SO(4) \simeq \frac{SU(2)\times SU(2)}{\{(1,1),(-1,-1)\}}$.

Proof sketch: think of $SU(2)$ as the group S^3 of quaternions of length 1. Then S^3 acts on the vector space $\mathbb{R}^3$ of "pure quaternions" $\mathbb{R}\cdot i + \mathbb{R}\cdot j + \mathbb{R}\cdot k = \{z \in \mathbb{H} : \mathrm{Re} z = 0\}$ by $a(z) = aza^{-1} \in \mathbb{R}^3$. This gives a homomorphism $SU(2) \to O(3) = SO(3) \cup$ another component; since $SU(2)$ is connected it maps into $SO(3)$.

Also, $U(2) = \frac{SU(2)\times S^1}{\{(1,1),(-1,-1)\}}$.

We map $SU(2) \to SO(3)$ by: $g \in S^3$ acts on $\{x \in \mathbb{H} : \mathrm{Re} x = 0\} \simeq \mathbb{R}^3$ by $x \mapsto gxg^{-1} \in \mathbb{R}^3$; this homomorphism is onto and its kernel is $\{\pm 1\}$. We map $SU(2) \times SU(2) \to SO(4)$ by for $x \in \mathbb{H} \simeq \mathbb{R}^4$, $(g,h)(x) = gxh^{-1}$. This maps into $O(4)$ since multiplication by a unit quaternion preserves length. But the image of $SU(2) \times SU(2)$ must be connected, and $(1,1) \mapsto 1$, so this maps into $SO(4)$. The kernel is $\{(1,1),(-1,-1)\}$ and the homomorphism is surjective, so $SO(4) = \frac{SU(2)\times SU(2)}{\{(1,1),(-1,-1)\}}$.

For $U(2)$, it's easy that any element of $U(2)$ is AB with $A \in SU(2)$ and $B = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$ for some $\theta \in \mathbb{R}$. So $U(2) = \frac{SU(2)\times S^1}{\{(1,1),(-1,-1)\}}$.

For compact Lie groups G, H (as or finite groups) any complex irreducible representation of $G \times H$ has the form $V \otimes_{\mathbb{C}} W$ where V is an irreducible representation of G and W an irreducible representation of H. Moreover, each irreducible representation of $G \times H$ arrises uniquely in this way. So the irreducible representations of $SO(3)$ are exactly those irreducible representations of $SU(2)$ in which $-1 \in SU(2)$ acts by identity. Now the irreducible representations of $SU(2)$ are $S^n V$ for $n \geq 0$ where $V \simeq \mathbb{C}^2$ is the standard representation of $SU(2)$; $-1 \in SU(2)$ acts by $x \mapsto -x \forall x \in V$ so $-1 \in SU(2)$ acts by $x_1 \times \cdots \times x_n \mapsto (-x_1)\ldots(-x_n) = (-1)^n x_1 \ldots x_n$. So the irreducible representations of $SO(3)$ over $\mathbb{C}$ are $\mathbb{C}, S^2V, S^4V, \ldots$; call these $\mathbb{C} = W_0, W_1, W_2, \ldots$ respectively, since V is not a representation of $SO(3)$ so using the S^nV notation would be misleading. Then $\dim_{\mathbb{C}} W_i = 2i+1$.

Here $\mathbb{C} = W_0$ is the trivial representation of $SO(3)$. W_1 is the standard representation of $SO(3)$ on $\mathbb{R}^3$, tensored with $\mathbb{C}$ (so $W_1 \simeq \mathbb{C}^3$ as a vector space). What is S^2W_1? We can analyze this as a representation of $SU(2)$; here W_1 has character $z^{-2}+1+z^2$ so S^2W_1 has character $(z^{-2})^2+1^2+(z^2)^2+z^{-2}\times 1+z^{-2}\times z^2+1\times z^2 = z^{-4}+z^{-2}+2+z^2+z^4$; this is because in general, for any vector spaces A and B, there are natural isomorphism $S^n(A\oplus B) = \bigoplus_{j=0}^n S^jA \otimes_{\mathbb{C}} S^{n-j}B$ and $\Lambda^n(A\oplus B) = \bigoplus_{j=0}^n \Lambda^jA \otimes_{\mathbb{C}} \Lambda^{n-j}B$.

In our case, $W_1 \mid_{S^2} = L_{-2} \oplus L_0 \oplus L_2$ where $L_a \simeq \mathbb{C}$ with S^1 acting by $z \mapsto z^a$, for $S_1 \subset SU(2) \twoheadrightarrow SO(3)$ [No, I have no idea].

We read off that (as representations of $SU(2)$) $S^2(S^2V) \simeq S^4V \oplus \mathbb{C}$. So in terms of $SO(3)$, $S^2W_1 \simeq W_2 \oplus \mathbb{C}$. Geometrically, we can think of W_1 as the vector space of $\mathbb{R}$-linear functions $\mathbb{R}^3 \to \mathbb{C}$, so S^2W_1 is the space of homogenous polynomials $\mathbb{R}^3 \to \mathbb{C}$ of degree 2, $\mathbb{C}\cdot\{x^2, y^2, z^2, xy, xz, yz\}$. This contains the trivial representation of dimension 1 spanned by the function $x^2+y^2+z^2$ (as $SO(3)$ preserves lengths).

The irreducible representations of $U(2)$ have the form: an irreducible representation of $SU(2) \otimes_{\mathbb{C}}$ an irreducible representation of S^1, such that $(-1,-1) \in$

$SU(2) \times S^1$ acts as the identity. These will be $S^nV \otimes L_m$ for some $n \geq 0, m \in \mathbb{Z}$; for such a representation $(-1,-1)$ acts by $(-1)^n(-1)^m = (-1)^{m+n}$, so the irreducible representations of $U(2)$ are indexed by (n,m) for $n \geq 0, m \in \mathbb{Z}, n+m \equiv 0$ mod 2. It seems reasonable to write A for $V \otimes_{\mathbb{C}} L_1$; this representation A is the standard 2D representation of $U(2)$: $\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$ acts on A by the scalar $e^{i\theta}$.

We get some other irreducible representations of $U(2)$ as S^nA for $n \geq 0$. $S^nA = S^nV \otimes L_n$¿ For $U(2)$, Λ^nA is not a trivial representation, though it is 1D; it is L_2 in the above notation, because $\det \neq 1$ on $U(2)$. Notice that for a 1D representation B of any group, $B^{\otimes n}$ is a 1D representation for <u>any</u> $n \in \mathbb{Z}$ (where we consider $B^{\otimes -1} = B^\star$).

Conclusion: the irreducible representations of $U(2)$ are exactly $S^nA \otimes (\Lambda^2 A)^{\otimes m}$ for any $n \geq 0$ and $m \in \mathbb{Z}$ (this $= S^nV \otimes L_{n+2m}$ as a representation of $SU(2) \times S^1$).

In short: all linear algebra constructions applied to a 2D vector space are "built up from" S^n and Λ^2.

We could similarly describe the representations of $SO(4)$ in detail, but we will not.

Representations of $SU(n)$ (especially $SU(3)$)

A character of a representation of $SU(n)$ is determined by its restriction to the <u>maximal torus</u> $T = \{\begin{pmatrix} z_1 & & \\ & \dots & \\ & & z_n \end{pmatrix} : z_i \in S^1, z_1 \dots z_n = 1\}$. The complex irreducible representations of $T \simeq (S^1)^{n-1}$ are 1D since $(S^1)^{n-1}$ is abelian. The irreducible representations of $(S^1)^{n-1}$ are of the form $(z_1, \dots, z_n) \mapsto z_1^{a_1} \dots z_n^{a_n}$ for $a_i \in \mathbb{Z}$, where $z_1^{a_1+1} \dots z_n^{a_n+1}$ gives the same representation since $z_1 \dots z_n = 1$ (so we could describe all representations as $(z_1, \dots, z_n) \mapsto z_1^{a_1} \dots z_{n-1}^{a_{n-1}}$). So the character of any finite dimensional representation of $(S^1)^{n-1}$ is a Laurent polynomial in $z_1, \dots, z_{n-1}$ (or in $z_1, \dots, z_n$ if we remember $z_1 \dots z_n = 1$).

Let W be a representation of $SU(n)$. Its character (restricted to $(S^1)^{n-1}$) is a Laurent polynomial in $z_1, \dots, z_n$; in fact, it must be a <u>symmetric</u> function in $z_1, \dots, z_n$ (but remember that $z_1 \dots z_n = 1$, so functions which might not initially appear symmetric can still be symmetric, e.g. $z_1 + z_2 + z_1^{-1}z_2^{-1}$ below).

Example: the character of the standard 3D representation of $SU(3)$ is $z_1 + z_2 + z_3 = z_1 + z_2 + z_1^{-1}z_2^{-1}$.

For $SU(3)$, we can visualize the representations of $SU(3)$: each "Laurent monomial" $z_1^{a_1}z_2^{a_2}, a_i \in \mathbb{Z}$ can be viewed as a point in a lattice $\simeq \mathbb{Z}^2$: draw a "triangular" or "hexagonal" lattice, in which z_1 is the point 1, z_1z_2 the point $\frac{\pi}{3}$ around a circle, z_2 the point $\frac{2\pi}{3}$ around, then $z_1^{-1}, z_1^{-1}z - 2^{-1}, z_2^{-1}$ respectively. So the character of a representation of $SU(3)$ is determined by its "multiplicities" ($\in \mathbb{Z}, \neq 0$ at each of the "weights" $z_1^{a_1}z_2^{a_2}$, and it will be symmetric under the action of S_3. We can draw representations by dots on the lattice, e.g. the character of the trivial representation is a dot at the origin. The (3D) standard representation has character a triangle of three dots at $z_1, z_2, z_3 = z_1^{-1}z_2^{-1}$; the dual $V^\star$, also of dimension 3, has representation the reflection of this in the y axis, $z_1^{-1} + z_2^{-1} + z_1z_2$. The character of Λ^2V, using that $\Lambda^2(A_1 \oplus \dots \oplus A_n) = \bigoplus_i \Lambda^2 A_i \oplus \bigoplus_{i<j} A_i \otimes A_j$, is

$\chi_{\Lambda^2 V} = z_1z_2 + z_1(z_1^{-1}z_2^{-1}) + z_2(z_1^{-1}z_2^{-1}) = z_1z_2 + z_1^{-1} + z_2^{-1} = \chi_{V^\star}$ and in fact $\Lambda^2 V \simeq V^\star$ ($\dim \Lambda^2 V = \binom{3}{2} = 3$, so dimensions are correct). The isomorphism between them comes from the natural product $\Lambda^2 V \otimes_{\mathbb{C}} V \to \Lambda^3 V \simeq \mathbb{C}$, $(v_1 \wedge v_2) \otimes v_3 \mapsto v_1 \wedge v_2 \wedge v_3$, which is a dual pairing; that is, the associated map $\Lambda^2 V \to V^\star$ is an isomorphism.

Remark: V (and $V^\star$) are irreducible representation of $SU(3)$, because each of their weights forms a single orbit under S_3.

The symmetric power S^2V has character a large triangle, i.e. the three points $z_1^2, z_2^2, z_1^{-2}z_2^{-2}$ and also the three "edge" points $z_1z_2, z_1^{-1}, z_2^{-1}$ (using $S^1(A_1 \oplus \cdots \oplus A_n) = \bigoplus_i S^2 A_i \oplus \bigoplus_{i<j} A_i \otimes A_j$). This is irreducible (one way to see this is that Weyl's integration formula generalizes, with $\Delta(z) = \prod_{1 \le i < j \le 3}(z_i - z_j)$. $S^2(V^\star) \simeq (S^2V)^\star$ of dimension 6 is also irreducible, with character the reflection in the y axis of the above.

Now look at $V \otimes V^\star$, of dimension 0. This is reducible, snice it contains a trivial representation: $(V \otimes V^\star)SU(3) = \mathrm{Hom}_{SU(3)}(V, V) \simeq \mathbb{C}$ (since V is irreducible). But "$V \otimes V^\star$" is an 8D irreducible representation of $SU(3)$; its character is a weight 2 point at the origin and then 6 points (of weight 1) in a hexagon: $z_1^2z_2, z_1z_2^2, z_1^{-1}z_2^2, z_1^{-2}z_2^{-1}, z_1^{-1}z_2^{-2}, z_1z_2^{-1}$.

[It is] easy to check that the characters of the representations $S^aV \otimes S^bV^\star$ span (the abelian group of) all symmetric Laurent polynomials in $z_1, z_2, z_3 = z_1^{-1}z_2^{-1}$, so every irreducible representation of $SU(3)$ occurs inside one of these representations.

Any character of a representation of $SU(3)$ is determined by its part in the "cone of dominant weights" spanned by z_1 and z_1z_2. Each irreducible representation has a unique "highest weight" [by which I *think* the lecturer means the one furthest from the origin] in this cone. The character of $S^aV \otimes S^bV^\star$ contains the monomial $z_1^a(z_1z_2)^b$ with coefficient 1,and all other weights occuring in this representation are "smaller"; they are contained within the hexagon about the origin with vertices $S^3\cdot$ that vertex. So there is a unique irreducible representation $\Gamma_{a,b}$ of $SU(3)$ for any $a, b \ge 0$, contained in $S^aV \otimes S^bV^\star$, whose character contains $z_1^a(z_1z_2)^b$. So $S^aV \otimes S^bV^\star = \Gamma_{a,b} \oplus$ (some $\bigoplus_{(a',b')<(a,b)} \Gamma_{a',b'}^{\oplus ?}$).

Characters of the irreducible representation of $SU(3)$, $\Gamma_{a,b}$ for $a, b \ge 0$, are known (Weyl). In particular (Weyl dimension formula) $\dim \Gamma_{a,b} = \frac{1}{2}(a+1)(b+1)(a+b+2)$.

Examples: $\Gamma_{0,0}$ is the trivial representation of dimension 0, $\Gamma_{1,0} = V, \Gamma_{0,1} = V^\star, \Gamma_{2,0} = S^2V, \Gamma_{0,2} = S_2V^\star, \Gamma_{1,1}/V \otimes V^\star - \mathbb{C}$.

[The lecturer realised at this point that he had to stop talking as it was noon, so I guess this is the end of the course.]