Methods

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This course has four parts: Fourier Series and Fourier Transforms, Green's functions and the dirac delta functions, finite domains and the classical differential equations (the wave equation, the heat equation and laplace's equation) and the Euler-Lagrange equation.

Part I
FS and FTs

1 Fourier Series

1.1 Periodic funcs

$f(t)$ is periodic $f(t+T)=f(t)$; T is the period. Note that such a function need not "join up" continuously.

Consider $\left\{\cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) : n \in \mathbb{N}\right\}$, both periodic on $[0,2L]$. Thes are orthogonal [taking integral of product over the whole period as our vector product]; if we let $S_{mn}=\int_0^{2L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx$ this is $\frac{1}{2}\int_0^{2L}\cos\frac{(n-m)\pi x}{L}dx - \frac{1}{2}\int_0^{2L}\cos\frac{(n+m)\pi x}{L}dx$. This is clearly 0 for $m \neq n$, but otherwise if $m=n$ it is L - it is $L\delta_{mn}$. Similarly, $C_{mn}=\int_0^{2L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx$ is $2L\delta_{mn}$. In fact, this set of functions form an orthogonal basis for the space of all periodic functions on $[0,2L]$ - we can express any "well-behaved" periodic f on $[0,2L]$ in terms of it, i.e. as $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}$. These a_n, b_n are constants, called the fourier coefficients, and they define the function just as well as its "normal" definition. It might be hard to find them, but we can use the orthogonality - $\int_0^{2L}f(x)\sin\frac{m\pi x}{L}dx = \int_0^{2L}\frac{1}{2}a_0\sin\frac{m\pi x}{L}+\sum_{n=1}^{\infty}a_n\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}+b_n\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx$ which if we are assuming the function is well behaved is $\frac{1}{2}a_0\int_0^{2L}\sin\frac{m\pi x}{L}dx+\sum_{n=1}^{\infty}a_n\int_0^{2L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx+b_n\int_0^{2L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx$ which is simply Lb_m, all other terms dropping out, so we can find the b_m by $\frac{1}{L}\int_0^{2L}f(x)\sin\frac{m\pi x}{L}dx$; similarly $a_m=\frac{1}{L}\int_0^{2L}f(x)\cos\frac{m\pi x}{L}dx$, and this is also true for a_0; $\frac{a_0}{2}\times 2L=\int_0^{2L}f(x)\,dx$ so $\frac{a_0}{2}$ is the mean value of f on the domain. Fourier transforms are essentially an infinite version of these calculations.

Note that we are integrating over a single period; any period will give the same result.

We can also express this in terms of the period T; $a_m = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi mt}{T}dt, b_m = \frac{2}{T}\int_0^T f(t)\sin\frac{2\pi mt}{T}dt$. Sometimes it is easier to scale the x values so $L = \pi$ and work on these rather than frequently multiplying by $\frac{\pi}{L}$.

Example

The sawtooth function, $f(x) = x$ on $[-L, L]$, has $a_m = \frac{1}{L}\int_{-L}^{L} x\cos\frac{m\pi x}{L}dx$ which is the integral of an odd (since x is odd and the cos is even) function on a symmetric domain so 0; this is a general result, even functions have pure cos FS, odd functions have pure sin FS. b_m here is $\frac{1}{L}\int_{-L}^{L} x\sin\frac{m\pi x}{L}dx$ which we can find is $\frac{2L}{m\pi}(-1)^{m+1}$, i.e. $f(x) = \frac{2L}{\pi}\left(\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} - \dots\right)$.

1. The finite sums $f_N(x) = \sum_{n=1}^{N} b_n \sin\frac{n\pi x}{L} \to f(x)$ in general almost everywhere, i.e. except for a set of measure 0.

2. There is a persistant overshoot here at $x = L$; this is Gibbs' phenomenon and will be seen in more detail in question 5 of the example sheet.

3. The smoothness of $f(x)$ determines the convergence of the series - if the pth derivitive is the lowest discontinuous derivative, the error in the n-term series will be of order $n^{-(p+1)}$.

As an exercise the reader should find the FS of $\frac{x^2}{2}$ on $-L < x < L$.

1.2 Dirichlet Definition

If $f(x)$ is a bounded periodic function with a finite number of minima, maxima and discontinuities the Fourier series converges to $f(x)$ at all points where f is continuous and $\frac{1}{2}(f(x+) + f(x-))$ ($f(x+)$ means $\lim_{y\downarrow x} f(y)$ and similarly) at points where f is discontinuous.

1.3 Integration and Diff of FS

FS can always be integrated termwise. If $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}$, $F(x) = \int_{-L}^{x} f(u)\,du = \frac{1}{2}a_0(x+L) + \sum_{n=1}^{\infty}\frac{a_n L}{n\pi}\sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} -\frac{b_n L}{n\pi}\left((-1)^n\cos\frac{n\pi x}{L}\right)$. We already know the FS for x, so we can express this F as an FS: $F(x) = \frac{a_0 L}{2} + L\sum_{n=1}^{\infty}\frac{b_n(-1)^n}{n\pi} + L\sum_{n=1}^{\infty}\frac{a_n - (-1)^n a_0}{n\pi}\sin\frac{n\pi x}{L} - L\sum_{n=1}^{\infty}\frac{b_n}{n\pi}\cos\frac{n\pi x}{L}$.

However, differentiation is not guaranteed to work, e.g. $f(x) = 1$ for $0 < x < 1$, -1 for $-1 < x < 0$ has a perfectly good FS but if we differentiate this termwise we obtain a divergent series. Still, for an f which is cnts when extended as a $2L$-periodic function and piecewise differentiable on $[-L, L]$, $\exists$ a FS rer of $f'(x)$; suppose have such an f with $f' = g = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L} + B\sin\frac{n\pi x}{L}$. $A_0 = \frac{1}{L}\int_0^{2L} g(x)\,dx = \frac{1}{L}\int_0^{2L} f'dx = \frac{f(2L) - f(0)}{L} = 0$, $A_n = \frac{1}{L}\int_0^{2L} f'\cos\frac{n\pi x}{L}dx = \frac{1}{L}f\cos\frac{n\pi x}{L}\,|_0^{2L} + \frac{n\pi}{L^2}\int_0^{2L} f\sin\frac{n\pi x}{L} = \frac{n\pi}{L}b_n$. Similarly $B_n = -\frac{n\pi}{L}a_n$. This is very useful for PDEs, as if we have for example to find $\psi(x,y)$ for which $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$ we can write $\psi=\frac{a_0(y)}{2}+\sum_{n=1}^{\infty}a_n(y)\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n(y)\sin\frac{n\pi x}{L}$ reducing it to an ODE.

1.4 Alternative Fourier Representations

$\cos\frac{n\pi x}{L}=\frac{1}{2}\left(e^{i\frac{n\pi x}{L}}+e^{-i\frac{n\pi x}{L}}\right),\sin\frac{n\pi x}{L}=\frac{1}{2i}\left(e^{i\frac{n\pi x}{L}}-e^{-i\frac{n\pi x}{L}}\right)$, so we can write $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}$ as $\sum_{n=-\infty}^{\infty}c_ne^{i\frac{n\pi x}{L}}$ where $c_n=\frac{a_n-ib_n}{2},c_{-n}=\frac{a_n+ib_n}{2},c_0=\frac{a_0}{2}$. This works since cplx exps are also orthogonal, $\int_0^{2L}e^{i\frac{n\pi x}{L}}e^{-\frac{m\pi x}{L}}dx=2L\delta_{mn}$, so we find c_n by $\frac{1}{2L}\int_0^{2L}fe^{-i\frac{m\pi x}{L}}$. Note that there are not more independent coefficients than before, because f being real gives the restriction that $c_{-n}=c_n^\star$. This complex form has a simple derivative expansion, if $f'=\sum_{n=-\infty}^{\infty}c_ne^{i\frac{n\pi x}{L}}$ and $f=\sum_{n=-\infty}^{\infty}\hat{c_n}e^{i\frac{n\pi x}{L}}$ then we find $c_n=\frac{in\pi}{L}\hat{c_n}$.

Half range series

For f defined only on $[0,L]$ we can extend this either by requiring f odd or requiring f even ; normally the latter is far easier to work with for a function with $f(L)=0$ and the former with one where it does not but $f(0)=0$, though both are formally equivalent.

1.5 Parseval's Thm

We're often interested in energy, $E=\int_0^{2L}(f(x))^2dx$. If we define $f=\sum_{n=-\infty}^{\infty}c_ne^{i\frac{n\pi x}{L}}$ and $g=\sum_{n=-\infty}^{\infty}d_ne^{i\frac{n\pi x}{L}}$ we have $\int_0^{2L}fgdx$ (assuming well-behaved, as we shall do without mentioning from now on) is $\sum_{n=-\infty}^{\infty}c_n\sum_{m=-\infty}^{\infty}d_m\int_0^{2L}e^{i\frac{\pi x(m+n)}{L}}dx=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}c_nd_m2L\delta_{n,-m}$ (that is, δ_{no} where $o=-m$) which is $2L\sum_{n=-\infty}^{\infty}c_nd_{-n}=2L\sum_{n=-\infty}^{\infty}c_nd_n^\star$. When $f=g$ this gives $2L\sum_{n=-\infty}^{\infty}|c_n|^2$. In terms of a_n and b_n this is $L\left(\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)\right)$ [?].

Now we shall consider non-finite domains

2 Fourier Transforms

2.1 Relationship to FS

Recall that f periodic on $[-L,L]$ has $f(x)=\sum_{n=-\infty}^{\infty}c_n^{(L)}e^{i\frac{n\pi x}{L}}$ with $c_n^{(L)}=\frac{1}{2L}\int_{-L}^{L}f(x)e^{-i\frac{n\pi x}{L}}dx$; the dirichlet conditions include that $\int_{-L}^{L}|f(x)|dx$ so we require f abs integrable, i.e. $\int_{-\infty}^{\infty}|f(x)|dx$ has some real value M_1, and also f square integrable, i.e. $\int_{-\infty}^{\infty}|f(x)|^2dx=M_2\in\mathbb{R}$. Then $\lim_{L\to\infty}2LC_n^{(L)}=\lim_{L\to\infty}\int_{-L}^{L}f(x)e^{-i\frac{n\pi x}{L}}dx$ (for fixed n) is $\int_{-\infty}^{\infty}f(x)dx$, the modulus of which is $\leq M_1$ so this is real and so since $2L\to\infty$, $C_n^{(L)}\to 0$ for fixed n.

If we consider an interval on the real line, when L is small, points of the form $\frac{n\pi}{L}$ are rare, while as L grows we have more and more points of this form in the interval. Therefore, as $L \to \infty$ we replace this set of points with a cnts variable k.

2.2 Defn of FT

We define $\tilde{f}(k) = \lim_{L\to\infty} 2LC^{(L)}_{\frac{kL}{\pi}}$ (if $k = \frac{n\pi}{L}$, $n = \frac{kL}{\pi}$) which is $\int_{\infty}^{\infty} f(x) e^{-ikx} dx$; this is one of a class of definitions for the FT of f. k is called the wavenumber; a wavelength λ is $\frac{2\pi}{k}$. The general FT is $\tilde{f} = A\int_{\infty}^{\infty} f(x) e^{-ikx} dx$; we will use $A = 1$ but $A = \frac{1}{\sqrt{2\pi}}$ is also frequently used, especially in engineering applications. The FT has many beautiful properties; the reader should verify from the definition:

1. Linearity $\widetilde{(f_1 + f_2)} = \tilde{f}_1 + \tilde{f}_2$.
2. Translation: $g(x) = f(x-a) \Rightarrow \widetilde{g}(k) = e^{-ika}\widetilde{f}(k)$.
3. Freq shift: $g(x) = e^{ik_0x} f(x) \Rightarrow \widetilde{g}(k) = \widetilde{f}(k - k_0)$; the astute reader should notice this is the dual of the previous property.
4. Scaling: $g(x) = f(ax) \Rightarrow \widetilde{g}(k) = \frac{1}{a}\widetilde{f}\left(\frac{k}{a}\right)$.
5. $g(x) = f'(x) \Rightarrow \widetilde{g}(k) = ik\widetilde{f}(k)$
6. $g(x) = xf(x) \Rightarrow \widetilde{g}(k) = i\widetilde{f}'(k)$.

5 is very important; if f is cnts, piecewise cntsly diffable and its FT exists, meaning $\lim_{|x|\to\infty} f(x) = 0$, then $\int_{-L}^{L} f' e^{-ikx} dx = \left[f(x) e^{-ikx}\right]_{-L}^{L} + ik\int_{-L}^{L} f(x) e^{ikx} dx$ which gives the required result as $L \to \infty$.

Concept

Say we have a problem in physical space involving derivs; these are hard, especially when they include PDEs, but if we transform into what is variously called wavenumber, transform, fourier or spectral space it becomes simpler since PDEs become ODEs and ODEs become algebraic; we can then find our solution and transform back. Though not covered in this course, we can use numerical methods very effectively in spectral space, and when discretising we can use the very efficient FFT to transform back and forth in $O(n\log n)$ time.

2.3 Inversion of FTs

Recall that $\frac{f(x_+)+f(x_-)}{2} = \sum_{n=-\infty}^{\infty} e^{i\frac{n\pi x}{L}} C_n^{(L)} = \sum_{n=-\infty}^{\infty} e^{i\frac{n\pi x}{L}} \frac{1}{2L}\int_{-L}^{L} f(y) e^{-i\frac{n\pi y}{L}} dy$. Now for $h = \frac{\pi}{L}$ this becomes $\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} he^{inhx}\int_{-L}^{L} f(y) e^{-inhy} dy$; as $L \to \infty$, $h \to 0$ but $nh \to k$. So $\frac{f(x_+)+f(x_-)}{2}$, which is $f(x)$ assuming f cnts at x, is $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\left[\frac{\widetilde{f}(k)}{A}\right]dk$; this defines an inverse FT; for our case $A=1$ at cnts points we have $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\widetilde{f}(k)\,dk$; compare this with $\widetilde{f}(k)=\int_{-\infty}^{\infty} f(x)\,e^{-ikx}dx$.

Consider $f(x)=1$ for $|x|<a$, 0 otherwise; this problem is important since FTs are often used on digital samples of waves. $\widetilde{f}(k)=\int_{-a}^{a} e^{ikx}dx=\int_{-a}^{a}\cos kx dx=2\frac{\sin ka}{k}$; the inversion is easy with techniqes from the complex methods course, but still possible with elementary methods; $f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin ka}{k}e^{ikx}dk=\frac{2}{\pi}\int_0^{\infty}\frac{\sin ka\cos kx}{k}dk=\frac{1}{\pi}\int_0^{\infty}\frac{\sin k(a+x)}{k}dk+\frac{1}{\pi}\int_0^{\infty}\frac{\sin k(a-x)}{k}dk$; define this to be I_1+I_2; this was Q3 in the DE course last year. Let $I(\lambda,\alpha)=\int_0^{\infty}\frac{\sin\lambda x}{x}e^{-\alpha x}dx$ for $\lambda\in\mathbb{R}$ and $\alpha>0$; $\frac{dI}{d\lambda}=\int_0^{\infty}\cos\lambda x e^{-\alpha x}dx=\Re\int_0^{\infty}e^{-(\alpha+\lambda i)x}dx=\Re\left(\frac{1}{\alpha+\lambda i}\right)=\Re\left(\frac{\alpha}{\alpha^2+\lambda^2}\right)$. Therefore $\int dI=\int\frac{\alpha}{\alpha^2+\lambda^2}d\lambda=\arctan\frac{\lambda}{\alpha}+c$. As $\alpha\to\infty$, $I\to 0$, so $c=0$, and as $\alpha\to 0$, $I(\lambda,\alpha)$ must therefore $\to\frac{\pi}{2}\text{sgn}\lambda$ (where $\text{sgn}\lambda$ denotes the sign of λ), so $I_1=\frac{1}{2}\text{sgn}(\alpha+x), I_2=\frac{1}{2}\text{sgn}(\alpha-x)$, which gives $f(x)=1$ for $|x|<\alpha$, 0 for $|x|>\alpha$, and $\frac{1}{2}$ for $|x|=\alpha$ as we would expect, since returning to earlier in the proof we clearly have $I_1=0$ when $x=a$ and $I_2=0$ when $x=-a$.

2.4 Convolution and Paseval's Thm

We often encounter problems in spectral space involving products $\widetilde{f}\widetilde{g}=\widetilde{h}$ (all functions of k) of FTs; how do we find $h(x)$ for this? Requiring f,g have piecewise cnts first derivs and one of the FTs and the other function (wlog f and $\widetilde{g}$) abs integrable, $h(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{f}(k)\,\widetilde{g}(k)\,e^{ikx}dk=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{g}(k)\int_{-\infty}^{\infty}f(u)\,e^{-iku}du e^{ikx}dk$; since f and $\widetilde{g}$ abs integrable we can exchange the order of integration and this is $\int_{-\infty}^{\infty}f(u)\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{g}(k)\,e^{ik(x-u)}dkdu=\int_{-\infty}^{\infty}f(u)\,g(x-u)$; this is called $f\star g$ the convolution. As an exercise the reader should prove the dual of this, if $h=fg$ then $\widetilde{h}(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{f}(u)\,\widetilde{g}(x-u)\,du$. A corollary of this is Parseval's Relation: $g(x)=f^{\star}(-x)\Rightarrow\widetilde{g}(k)=\widetilde{f}^{\star}(k)$, since $\widetilde{g}(k)=\int_{-\infty}^{\infty}f(-x)\,e^{-ikx}dx$ so $\widetilde{g}^{\star}(k)=\int_{-\infty}^{\infty}f(-x)\,e^{ikx}dx$ and substituting $y=-x$ this is $\int_{-\infty}^{\infty}f(y)\,e^{-iky}dy$.

Then for this f and g, $h(x)=\int_{-\infty}^{\infty}f(u)\,g(x-u)\,du=\int_{-\infty}^{\infty}f(u)\,f^{\star}(u-x)\,du$; this must equal $\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{h}(k)\,e^{ikx}dk$ but also $\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{f}^{\star}(k)\,\widetilde{f}(k)\,e^{ikx}dk=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widetilde{f}(k)\right|^2e^{ikx}dk$; we can consider this as representing energy.

2.5 "One-sided" FTs

For f even, $\widetilde{f}(k)=A\int_{-\infty}^{\infty}f(x)\,e^{-ikx}dx=2A\int_0^{\infty}f(x)\cos(kx)\,dx$; this is equivalent to f being defined only on $(0,\infty)$with particular properties at $x=0$. The inverse is $f(x)=\frac{1}{\pi a}\int_0^{\infty}\widetilde{f}(k)\cos kxdk$. The convention is that we choose $A=\frac{1}{2}$, so we define the FCT (C being short for cos) for even funcs or funcs defined only for $x\geq 0$ by $\widetilde{f_C}(k)=\int_0^{\infty}f(x)\cos kxdx$, inversed by $f(x)=\frac{2}{\pi}\int_0^{\infty}\widetilde{f_C}(k)\cos kxdk$; similarly for odd funcs or funcs defd only on $x\geq 0$ we have

$\widetilde{f_S}(k) = \int_0^\infty f(x) \sin kx dx$, $f(x) = \frac{2}{\pi} \int_0^\infty \widetilde{f_S}(k) \sin kx dx$. Convolution thms exist for these but are more complicated; some are covered on the ex sheet for this course. The usefulness of these becomes apparent when we recall that many eqns from classical physics involve second derivatives e.g. $\frac{\partial^2 x}{\partial t^2}$; we can relatively easily find $\widetilde{g_C}(k)$ and $\widetilde{g_S}(k)$ for $g = f''$. We choose which series to use for a semi-infinite domain by the boundary conds - if we have dirichlet conds (f) at the boundary we use the sin series, if von Neumann conds (f'), the cos series.

Second derivs are very important for many applications of the FCT and FSTs. For f defined on $[0, \infty)$ let $g = f'$ and $h = f''$, assume f is cnts and well behaved and sufficiently smooth that these have FTs. $\widetilde{h_C}(k) = \int_0^\infty \frac{d^2 f}{dx^2} \cos kx dx = \left[\frac{df}{dx} \cos kx\right]_0^\infty + k \int_0^\infty \frac{df}{dx} \sin kx dx = -\frac{df}{dx} |_{x=0+}$ ($\frac{df}{dx}$ evaluated at $x = 0+$, since of course it is not defined for $x = 0-$) $+ [kf(x) \sin kx]_0^\infty - k^2 \int_0^\infty f(x) \cos kx dx = -\frac{df}{dx} |_{x=0+} - k^2 \int_0^\infty f(x) \cos kx dx = -\frac{df}{dx} |_{x=0+} - k^2 \widetilde{f_C}(k)$; similarly from a few steps back $\widetilde{g_C}(k) = -f |_{x=0+} + k\widetilde{f_S}(k)$; the reader should verify $\widetilde{h_S}(k) = kf |_{x=0+} - k^2 \widetilde{f_S}(k), \widetilde{g_S}(k) = -k\widetilde{f_C}(k)$, which is why we pick to use sin or cos series as above.

2.6 Applications of FTs

As stated above, FTs can be used to solve ODEs very efficiently when the inversion is possible; for example, if $y''(x) - \lambda^2 y(x) = -f(x)$ for $x \in (-\infty, \infty)$ for some fixed $\lambda > 0$ and f for which the FT exists, assuming $y \to 0$ and $y' \to 0$ as $|x| \to \infty$; unfortunately we will simply fail to find ys not satisfying these conditions, we take FTs to obtain $-k^2 \widetilde{y} - \lambda^2 \widetilde{y} = -\widetilde{f}$ (all of these as functions of k) so $\widetilde{y} = \frac{\widetilde{f}}{\lambda^2 + k^2}$ (compare with the same result for Laplace's equation) and so we can find y very easily by convolution if we can find the IFT of $\frac{1}{\lambda^2 + k^2}$. As before this is trivial with techniques from the complex methods course, but, at least having already found the result with such means, we can demonstrate that it works by elementary methods; consider $h(x) = \frac{e^{-\mu|x|}}{2\mu}$ for $\mu > 0$ and note this is even, so $\widetilde{h} = \frac{1}{2\mu} 2 \int_0^\infty \exp(-x(\mu + ik)) dx$; since the function is even its FT will be real so this is $\Re\left(-\frac{\exp(-x(\mu+ik))}{\mu(\mu+ik)}\right)_0^\infty = \Re\left(\frac{1}{\mu(\mu+ik)}\right) = \frac{1}{\mu^2+k^2}$, so $\widetilde{g}(k) = \frac{1}{\lambda^2+k^2}$ is the FT of $g(x) = \frac{e^{-\lambda|x|}}{2\lambda}$; therefore by the convolution thm $y(x) = \frac{1}{2\lambda} \int_{-\infty}^\infty f(u) \exp(-\lambda|x-u|) du$; this is a very powerful result since previously we have always had to calculate it individually for each f.

2.7 Application of FTs to linear systems

Suppose we have some linear operator acting on an input to give an output, such as an (analogue, audio) amplifier, which can change the relative amplitudes of different frequencies and also the phase of the parts (because it adds a delay). Say the input or "pulse" is $I(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \widetilde{I}(\omega) e^{i\omega t} dt$; this is called the "synthesis" of the pulse. $\widetilde{I}(\omega)$ is cplx; ω represents the frequency; $-ve$ ω which would initially appear meaningless corresponds to an opportunity to consider $\sin\omega t$ and $\cos\omega t$ independently. $\widetilde{I}(\omega)=\int_{-\infty}^{\infty}I(t)\,e^{-i\omega t}dt$ is the "resolution" of the pulse. We represent the effect of the amplifier or similar by letting the output be $O(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{O}(\omega)\,e^{i\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{R}(\omega)\,\widetilde{I}(\omega)\,e^{i\omega t}d\omega$ where $\widetilde{R}(\omega)$ is the "response" or "transfer" func (of course $R(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{R}(\omega)\,e^{i\omega t}d\omega$); this is a product so we use the conv thm and obtain $O(t)=\int_{-\infty}^{\infty}I(u)\,R(t-u)\,du$. We take $I(t)$ to be 0 for $t<0$ and by causality (there should be no response for a zero input) we also take $R(t)=0$ for $t<0$, so this is $O(t)=\int_0^t I(u)\,R(t-u)\,du$. This makes sense; $R(t)$ is the effect of the response to the signal from t ago.

We know the input and want to find the output; typically we will get an expression of the form $\left(a_0\frac{d^n}{dt^n}+a_1\frac{d^{n-1}}{dt^{n-1}}+\cdots+a_n\right)O(t)=\left(b_0\frac{d^m}{dt^m}+b_1\frac{d^{m-1}}{dt^{m-1}}+\cdots+b_m\right)I(t)$, with $m\le n$ and $a_0,b_0\neq 0$; for simplicity we will take $m=0$ and $b_0=1$, then in spectral space we have $\widetilde{I}(\omega)=\left(a_0i^n\omega^n+a_1i^{n-1}\omega^{n-1}+\cdots+a_n\right)\widetilde{O}(\omega)-\widetilde{R}(\omega)=\frac{1}{a_0i^n\omega^n+a_1i^{n-1}\omega^{n-1}+\cdots+a_n}$; then by partial fractions $\widetilde{R}(\omega)=\sum_{m=1}^{n}\frac{T_m}{i\omega-C_m}$ where there is the possibility of repeated roots.

For the existence and stability of solutions to this we need $\Re(c_m)<0$; then (again using techniques from CM) we can solve straightforwardly by noting that $f(t)=\frac{t^{k-1}e^{ct}}{(k-1)!}$ (for $\Re(c)<0$) has $\widetilde{f}(k)=\frac{1}{(i\omega-c)^k}$ (repeated roots give the higher powers on the denominator). This is the kind of response we would expect from an amplifier - the polynomial term means there will initially be growth, but the exponential term ensures long-term decay.

There are 3 classical equations we shall use as examples: Laplace's eqn $\nabla^2\psi=0$ or $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$, which is elliptical, the Diffusion eqn $\frac{\partial T}{\partial t}-\kappa\frac{\partial^2 T}{\partial x^2}=0$ which is parabolic, and the wave equation $\frac{\partial^2 y}{\partial t^2}-c^2\frac{\partial^2 y}{\partial x^2}=0$ which is hyperbolic. In general say we want to solve $\frac{d^2y}{dx^2}+\cdots=f(x)$; previously we have always had to guess a PI. Now we write this as $\mathcal{L}y=f(x)$ for some linear operator $\mathcal{L}$. We want to invert this to find $y=\mathcal{L}^{-1}f$; to do this we use a Green's function which is a G such that $\mathcal{L}G=\delta(x-\psi)$ where δ is the Dirac delta

Part II
Second order linear ODEs

3 Generalised functions - dirac δ fn, Heaviside step fn

This course will be entirely nonrigorous

3.1 Defn of δ fn

We define this by its properties: $\delta(x-\psi) = 0 \forall x \neq \psi$ and $\int_{-\infty}^{\infty} \delta(x-\psi)\,d\psi = 1$; heuristically this is a zero function with an infinitely tall spike at $x = \psi$. We have the sampling property; for $f(x)$ a func cnts in the nhood of ψ we have $\int_{-\infty}^{\infty} f(x)\,\delta(x-\psi)\,dx = f(\psi)$. We can define δ rigorously in terms of distributions or as the limit of some non-unique sequences; the obvious sequence of functions $\to \delta$ is $p_n(x) = \frac{n}{2}$ for $|x| < \frac{1}{n}$ and 0 otherwise, but as on the ex sheet we can do this with cnts funcs by $p_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$. We do not even need to use positive-valued functions; $p_n(x) = \frac{\sin nx}{\pi x}$ has the relevant properties, i.e. that $p_n(x) \to 0 \forall x \neq 0$ and $\int_{-\infty}^{\infty} p_n(x)\,dx = 1$, so is just as valid a sequence as any of the others.

3.2 Properties

1. If $f(x)$ cnts in some ndood of $x = \psi$ $\int_a^b f(x-\psi)\,dx = f(\psi)$ for $a < \psi < b$ and 0 for $a > \psi$ or $b < \psi$.
2. $\delta(at) = \frac{1}{|a|}\delta(t)$ for $a \neq 0$, where by "=" we mean the two functions behave in the same way wrt the sampling property
3. $g(x)\,\delta(x) = g(0)\,\delta(x)$ for g cnts at 0
4. $\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|f'(x_i)|}$ where x_i are the simple zeroes of f, i.e. $f(x_i = 0)$ but $f'(x_i) \neq 0$; this is covered in more detail on the example sheet
5. The integral of the δ fn $\int_{-\infty}^{x} \delta(\psi)\,d\psi$ is 0 for $x < 0$, 1 for $x > 0$

We define this as the Heaviside step fn; clearly $\frac{dH}{dx} = \delta$. As is often done we will choose $H(0) = \frac{1}{2}$, primarily since this is what we will get when we reconstruct it after an FT. Notice that "derivatives get worse" - H is the derivative of $f(x) = 0$ for $x \leq 0$, x for $x \geq 0$ which is even cnts, H is bounded but δ is not. We would expect the derivatives of δ to be even worse; we define them by integration by parts $\int_{-\infty}^{\infty} \delta'(x-\psi)\,f(x)\,dx = [\delta(x-\psi)\,f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x-\psi)\,f'(x)\,dx = -\frac{df}{dx}|_{x=\psi} = f'(\psi)$ provided f suficiently smooth, and generally (again assuming f sufficiently smooth) $\int_{-\infty}^{\infty} f(x)\,\delta^{(n)}(x)\,dx = (-1)^n f^{(n)}(0)$.

3.3 Fourier properties of δ fn

Consider $f(x) = \delta(x)$ on $-L < 0 < L$; $f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$ where $c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\,e^{-\frac{in\pi x}{L}} = \frac{1}{2L}\int_{-L}^{L} \delta(x)\,e^{-\frac{in\pi x}{L}} = \frac{1}{2L}$ so $f(x) = \frac{1}{2L}\sum_{-\infty}^{\infty} e^{\frac{in\pi x}{L}}$. Since we know this must also equal $\sum_{m=-\infty}^{\infty} \delta(x-2mL)$ we have a value for this divergent FS - this is called Poisson's integral formula.

We can apply this to our previous exampse: $f(x) = 1$ for $0 < x < 1$ and -1 for $-1 < x < 0$ should have $f'(x) = 2\sum_{m=-\infty}^{\infty} \delta(x-2m) - 2\sum_{m=-\infty}^{\infty} \delta(x-1-2m)$ which applying this gives us that $f'(x)$ should be $4(\cos \pi x + \cos 3\pi x + \cos 5\pi x + \dots)$, which is precisely the series we find by taking the FT and then applying the formula for differentiation in spectral space.

FT

The inversion formula is $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\int_{-\infty}^{\infty} f(u)\, e^{-iku} du dk = \int_{-\infty}^{\infty} f(u)\left(\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{ik(x-u)}dk\right)du$; the bracketed term is behaving as $\delta(u-x)$, so we can define the δ fn by this. Notice that the δ function is entirely local, but its transform space equivalent is delocalised.

$\delta(u-x) = \delta(x-u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-u)}du$ so the FT $\widetilde{\delta}(k) = \int_{-\infty}^{\infty}\delta(x)\, e^{-ikx}dx =$ 1 - an absolutely local function in physical space maps to an absolutely nonlocal one in transform space. Likewise $f(x) = 1$ has $\widetilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx}dx = 2\pi\delta(k)$. For $f(x) = \cos\omega x$ $\widetilde{f}(x) = \pi(\delta(k+\omega)+\delta(k-\omega))$; for $f(x) = \frac{1}{2}(\delta(x+a)+\delta(x-a))$ $\widetilde{f}(k) = \cos ka$ and similarly $g(x) = \sin\omega x$ has $\widetilde{g}(x) = \frac{\pi}{i}(\delta(k+\omega)-\delta(k-\omega))$, $g(x) = \frac{1}{2}(\delta(x+a)-\delta(x-a))$ has $\widetilde{g}(k) = \sin ka$ [check, I haven't] - again localised functions become nonlocal and vice versa when transformed, and this is in general true. Gaussians are just local enough that they are transformed into other gaussians.

$H(x) + H(-x) = 0$; note this is cnts at 0 because of how we defned H. So $\widetilde{H}(k) + \widetilde{H}(-k) = 2\pi\delta(k)$ [I am entirely confused as to which δs should be $\widetilde{\delta}$s and vice versa; I'm assuming none since any are silly, but in lecture all had tildes] but we also have $H'(x) = \delta(x)$ so $ik\widetilde{H}(k) = \widetilde{\delta}(k) = 1$; this appears to be a contradiction, but isn't because $y = \frac{1}{x} + C\delta(x)$ has $xy = 1$ for any constant C; we can easily see $\widetilde{H}(k) = \pi\delta(k) + \frac{1}{ik}$.

The reader may verify $f(x) = \text{sgn}x$ with $\frac{1}{2}\text{sgn}0 = 0$ has $\widetilde{f} = \frac{1}{ik}$ so $f(x) = \frac{1}{2}\text{sgn}(x-a)$ has $\widetilde{f} = \frac{e^{-ika}}{ik}$; for $f(x) = \delta^{(n)}(x)$ $\widetilde{f} = (ik)^n$ so $g(x) = x^n$ has $\widetilde{g}(k) = 2\pi(i)^n\delta^{(n)}k$; we can apply this to what's known as d'Alembert's solution or the Cauchy problem for the wave eqn $u_{tt} = c^2u_{xx}$; we are given $u(x,0) = \phi(x)$ and $\frac{\partial}{\partial t}u(x,0) = \psi(x)$ and want to find $u(x,t)$. Taking FTs wrt x $\frac{\partial}{\partial t^2}\widetilde{u}(k,t) = -k^2c^2\widetilde{u}(k,t)$; take wlog $c > 0$ then $\widetilde{u} = A(k)\, e^{ikct} + B(k)\, e^{-ikct}$; $\widetilde{u}(k,0) = \widetilde{\phi}(k) = A(k) + B(k)$ and $\frac{d}{dt}\widetilde{u}(k,0) = ikc(A(k)-B(k)) = \widetilde{\psi}(k)$ so $\widetilde{u}(k,t) = \widetilde{\phi}\cos kct + \frac{\widetilde{\psi}}{2ikc}\left(e^{ickt} - e^{-ikct}\right)$; call this T_1+T_2, then $T_1 = \frac{1}{2}\int_{-\infty}^{\infty}(\delta(u+ct)+\delta(u-ct))\,\phi(x-u)\,du = \frac{1}{2}(\phi(x+ct)+\phi(x-ct))$; the wave if not initially moving splits into two waves moving in opposite directions with speed c.

$T_2 = \frac{1}{4c}\int_{-\infty}^{\infty}(\text{sgn}(x+ct+u) - \text{sgn}(x-ct-u))\,\psi(u)\,du = \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(u)\,du$ - the height at point x is affected by the velocity from any point close enough for the wave to have moved from there to x.

So $u(x,t) = \frac{1}{2}\phi(x-ct) + \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(u)\,du$; this equation works for a situation where information propagates with finite speed c, unlike laplace's eqn which works for e.g. pressure in an incompressible liquid where information is propagated instantly, or the heat eqn which again requires instant information propogation

3.4 Inhomog lin 2nd order ODEs

$\mathcal{L}y = \frac{d^2y}{dx^2} + \beta(x)\frac{dy}{dx} + \gamma(x)y = f(x)$ (we have divided by a coeff $\alpha(x)$ which we want to take always nonzero in the domain we're interested in so that this is a true second order DE. The homog eqn $\mathcal{L}y = 0$ generally has two nontrivial lin ind sols y_1, y_2, then the gen sol to it is $Ay_1 + By_2$ and the gen sol of the full eqn is $y_p + Ay_1 + By_2$ where y_p is some particular integral soln to $Ly_p = f$; A, B must be determined by two distinct boundary conds; typically when solving on $[a, b]$ we will either have the boundary value problem where $y(a), y(b)$ given, the initial value problem $y(a), y'(a)$ given, or possibly some other constraints such as letting $b \to \infty$ and having $y \to 0$ or merely bounded as $x \to \infty$.

Defn of Green's fn

A green's fn $G(x;\zeta)$ is defd as the sol of $\mathcal{L}G = \delta(x-\zeta)$ with $G(a;\zeta) = 0 = G(b,\zeta)$; these apply to much more than 2nd order ODEs but we shall just use this example. Then the soln to $\mathcal{L}y = f$ with $y(a) = 0 = y(b)$ is $y(x) = \int_a^b G(x;\zeta)f(\zeta)\,d\zeta$; G acts as an inverse to $\mathcal{L}$ since $L\int_a^b G(x;\zeta)f(\zeta)\,d\zeta = \int_a^b \mathcal{L}Gf(\zeta)\,d\zeta = \int_a^b \delta(x-\zeta)f(\zeta)\,d\zeta = f(x) = \mathcal{L}y$; this allows us to solve such a DE for general f

3.5 Constructuon of GF

For $\zeta > x$ $\mathcal{L}G = 0$; we have a sol of this in terms of the homg sols $G = Ay_1 + By_2$; similarly $G = Cy_1 + Dy_2$ for $x > \zeta$; note A, B, C, D here depend on ζ. We want to find four constants from our two boundary conds; we can only apply the first cond to the $\zeta > x$ part since it is for a minimal value of x, so $Ay_1(a) + By_2(a) = 0$, similarly $Cy_1(b) + Dy_2(b) = 0$. We need two conds at the $x = \zeta$ join; clearly we have G cnts at $x = \zeta$ as were it discnts we have at least a finite jump in it so $\frac{dG}{dx} \propto \delta(x-\zeta)$ or worse and so $\frac{d^2G}{dx^2} \propto \delta'(x-\zeta)$ or worse, but there is no such term in the DE. So we have $Ay_1(\zeta) + By_2(\zeta) = Cy_1(\zeta) + Dy_2(\zeta)$; we need one final condition. For an eqn of the form we are trying to solve, the jump $\left[\frac{dG}{dx}\right]_{\zeta-}^{\zeta+}$ must $= 1$; this is only true where we have divided out α the coeff of $\frac{d^2y}{dx^2}$, in general it would be $\frac{1}{\alpha}$. We obtain this by integrating the defn of the GF so we have $\int_{\zeta-\epsilon}^{\zeta+\epsilon} \frac{d^2G}{dx^2}dx + \int_{\zeta-\epsilon}^{\zeta+\epsilon} \beta\frac{dG}{dx}dx + \int_{\zeta-\epsilon}^{\zeta+\epsilon} \gamma\frac{dG}{dx}dx = \int_{\zeta-\epsilon}^{\zeta+\epsilon} \delta(x-\zeta)\,dx = 1$. Since $G, \frac{dG}{dx}, \beta, \gamma$ bounded the second and third integrals $\to 0$ and we have the condition stated above by taking the limit as $\epsilon \to 0$, so $Cy_1'(\zeta) + Dy_2'(\zeta) - Ay_1'(\zeta) - By_2'(\zeta) = 1$. We now have four conditions so can find A, B, C, D. Note that when we now use this to express y, $y(x) = \int_a^b G(x;\zeta)f(\zeta)\,d\zeta = \int_a^x (Cy_1(\zeta) + Dy_2(\zeta))f(\zeta)\,d\zeta + \int_x^b (Ay_1(\zeta) + By_2(\zeta))f(\zeta)\,d\zeta$; the "upper" and "lower" parts are reversed since we are now integrating wrt ζ, so $\zeta < x$ when $x > \zeta$ and vice versa. $G(x;\zeta)$ is in fact symmetric in x, ζ.

We do not need the homog BCs: G solves the forced problem with homog BCs, but since the problem is linear we can add a sol to the unforced problem with inhomog BCs to construct the general solution.

Notice how the GF adds up the contribution of the forcing over the length of the domain. We can interpret this by example; a string of length L suspended at both ends at height 0, with mass $\mu(x)$ per unit length at tension $T(x)$ moving only vertically and with small deflections $y << L$.

If the string is at angle θ_1 above the horizontal at some x and $\theta_2$at $x+\delta x$, since we are assuming no motion in the x direction the forces in this direction must be balanced and $T(x)\cos\theta_1 = T(x+\delta)\cos\theta_2$. Since deflections are small the θ_i are small and $\cos\theta_i \approx 1$, so $T(x) = T(x+\delta)$. Now in the y direction, $T(x)\sin\theta_2 - T(x)\sin\theta_1 - \mu\delta x g = \mu\delta x\frac{\partial^2 y}{\partial t^2}$; $\sin\theta_1 \approx \tan\theta_1 = \frac{\partial y}{\partial x}\mid_x; \sin\theta_2 \approx \frac{\partial y}{\partial x}\mid_{x+\delta x} \approx \frac{\partial y}{\partial x}\mid_x + \delta x\frac{\partial^2 y}{\partial x^2}\mid_x$. Substituting this back into the equation, $T\delta x\frac{\partial^2 y}{\partial x^2} - \mu\delta x g = \mu\delta x\frac{\partial^2 y}{\partial t^2}$ so $T\frac{\partial^2 y}{\partial x^2} - \mu g = \mu\frac{\partial^2 y}{\partial t^2}$; this is the forced wave eqn, which reduces to the wave eqn for a light string $\mu \to 0$.

If we take the static problem $\frac{\partial}{\partial t} = 0$ we have $T\frac{d^2y}{dx^2} = \mu g$ or $\frac{d^2y}{dx^2} = \frac{\mu}{T}g = f(x)$. If $f(x)$ is constant (i.e. $\mu(x)$ constant) this gives $y = \frac{\mu g}{2T}x^2 + k_1 x + k_2$; if we require $y = 0$ at $x = 0, L$ we have $y(x) = \frac{\mu g}{2T}x(x-L)$. Consider a light string with a heavy point mass of mass m attached at $x = \zeta$. This forms a triangle with the string to the left of the mass being at angle θ_1 above the horizontal and that to the right at θ_2; there is the same tension T on both sides and we have $T(\sin\theta_1 + \sin\theta_2) = mg$; the θ_i are small so $T\left(-\frac{y(\zeta)}{\zeta} - \frac{y(\zeta)}{L-\zeta}\right) = mg = f(\zeta)$ so $y(\zeta) = \frac{mg(\zeta-L)\zeta}{TL}$ and the soln is $y = f(\zeta)\frac{x(\zeta-L)}{L}$ for $0 \le x \le \zeta$, $f(\zeta)\frac{(z-L)\zeta}{L}$ for $\zeta \le x \le L$. These are linear as there is no force anywhere except the point mass, so $T\frac{d^2y}{dx^2} = 0$. The reader should apply BCs to show $y = f(x)G(x;\zeta)$; the GF is the response from a completely localized force. Now if we imagine N masses attached to the string we have $y(x) = \sum_{i=1}^{N} f_i(\zeta_i)G(x;\zeta_i)$ as the problem is linear [recall the string has natural length L]. Now if we take a continuum limit we have $y(x) = \int_0^L f(\zeta)G(x-\zeta)d\zeta$. As a very important exercise the reader should show that for $f(x) = \mu g$ constant we can recover $y(x) = \frac{\mu g}{2T}x(x-L)$ from this.

2.5 Application of GFs to IVP and rel to response fn

$\mathcal{L}y = f(x)$ for $x \ge a$, $y, \frac{dy}{dx} = 0$ at $x = a$. We construct a Green's fn for $\mathcal{L}G = \delta(x-\zeta)$; for $a \le x < \zeta$ $\mathcal{L}G = 0$ so $G = Ay_1(x) + By_2(x)$; applying the BCs $Ay_1(a) + By_2(a) = 0 = Ay_1'(a) + By_2'(a)$. Since y_1, y_2 lin ind this gives $A = B = 0$ so $G = 0$ for $a < x < \zeta$. Though this might initially seem problematic it is actually good as it means $G = 0$ for $\zeta > x$; for fixed ζ G is nonzero only on a finite interval (see the end of this paragraph). For $x > \zeta$ $G = Cy_1 + Dy_2$ and by continuity, $Cy_1(\zeta) + Dy_2(\zeta) = 0, Cy_1'(\zeta) + Dy_2'(\zeta) =$ the jump in $\frac{dG}{dx}$, which we are assuming to be 1. The basic sol is then$\int_a^\infty G(x;\zeta)f(\zeta)d\zeta = \int_a^x G(x;\zeta)f(\zeta)d\zeta + \int_x^\infty G(x;\zeta)f(\zeta)d\zeta = \int_a^x G(x;\zeta)f(\zeta)d\zeta$ and we have causality - $y(x)$ is determined by the forcing or response between a and x.

As an example, to solve $\frac{d^2y}{dx^2} + y = f(x)$ with $y(0) = y'(0)$ we have $G = 0$ for

$x < \zeta$, $C\sin x + D\cos x$ for $x > \zeta$; by continuity we have $C\sin\zeta + D\cos\zeta = 0$ and from the jump condition $C\cos\zeta - D\sin\zeta = 1$, so $C = -D\frac{\cos\zeta}{\sin\zeta}$ and $G(x;\zeta) = \cos\zeta\sin x - \sin\zeta\cos x = \sin(x-\zeta)$ for $x > \zeta$. So $y(x) = \int_0^x f(\zeta)\sin(x-\zeta)\,d\zeta$; note the resemblance to the convolution of an FT $h(x) = \int_{-\infty}^{\infty} f(u)\,g(x-u)\,du$.

If we instead take the FT of this eqn, converting to time we have $\frac{d^2y}{dt^2} + y = f(t)$ so $-\omega^2\tilde{y} + \tilde{y} = \tilde{f}$ and $\left(1-\omega^2\right)\tilde{y} = \tilde{f}$. Recall that $xy = 1 \Rightarrow y = \frac{1}{x} + c\delta(x)$ so $\tilde{y} = \frac{\tilde{f}}{1-\omega^2} + c_1\delta(1+\omega) + c_2\delta(1-\omega) = \frac{\tilde{f}}{2(\omega+1)} - \frac{\tilde{f}}{2(\omega-1)} + c_1\delta(1+\omega) + c_2\delta(1-\omega)$. Now we have that for $h(t) = \sin t H(t), g(t) = \sin t, f(t) = H(t)$ $\tilde{h} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(u)\,\tilde{g}(\omega-u)\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\pi\delta(u) + \frac{1}{iu}\right)\frac{\pi}{i}\left(\delta(\omega-u-1) - \delta(\omega+u+1)\right) = \frac{1}{2(\omega+1)} - \frac{1}{2(\omega-1)} + \frac{\pi}{i}\delta(\omega-1) - \frac{\pi}{i}\delta(\omega+1)$ so $\tilde{y} = \tilde{f}\tilde{h}$ and by the convolution thm $y = \int_{-\infty}^{\infty} f(u)\sin(t-u)\,H(t-u)\,du = \int_0^t f(u)\sin(t-u)\,du$.

To see why these two methods are equivalent, consider a damped oscilator; we get an eqn of the form $\frac{d^2y}{dt^2} + 2p\frac{dy}{dt} + \left(p^2+q^2\right)y = f(t)$ so taking FTs $\left(-\omega^2 + 2pi\omega + p^2 + q^2\right)\tilde{y} = \tilde{f}$ and $\tilde{y} = \frac{\tilde{f}}{-\omega^2+2pi\omega+p^2+q^2}$ which we can consider as $\tilde{R}\tilde{f}$, then $y = \int_0^t R(t-\tau)\,f(\tau)\,d\tau$ where $R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{-\omega^2+2pi\omega+p^2+q^2}d\omega$. Then $\frac{d^2R}{dt^2} + 2p\frac{dR}{dt} + \left(p^2+q^2\right)R = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-\omega^2+2pi\omega+p^2+q^2}{-\omega^2+2pi\omega+p^2+q^2}e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}d\omega = \delta(t)$; $R(t)$ is the GF. Therefore the GF is sometimes called the impulse response.

3 Sturm-Liouville

A correction; where ζ (and on a few occations, also ψ) was used above, ξ was meant; this should not affect anything but some readers might be confused by the change of variables between previous sections above and this; ξ is the correct notation inasmich as it is that used in the lectures and on the example sheets.

3.1 Motivation and def of self-adjoint operator

If $Ay = \lambda y$ for A a matrix and $\lambda \in \mathbb{C}, y$ a vector then λ is an eval, y an evec; we want to generalize this to diff operators. The equivalent prob is $\frac{d^2y}{dx^2} + \beta(x)\frac{dy}{dx} + \gamma(x)\,y = \mathcal{L}y = -\lambda K(x)\,y$; wlog we take K real and positive, for λ some scalar. This can always be written in canonical SL form $\mathcal{L}y = -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\,y = \lambda w(x)\,y$; w the weight is wlog real and positive as before, by multiplying by $-e^{\int^x\beta(u)du}$: $-\frac{d}{dx}\left(e^{\int^x\beta(u)du}\frac{dy}{dx} - \gamma e^{\int^x\beta(u)du}\right) = \lambda K(x)\,e^{\int^x\beta(u)du}y$ i.e. $p = e^{\int^x\beta(u)du}, q = -\gamma e^{\int^x\beta(u)du}, w = k(x)\,e^{\int^x\beta(u)du}$.

$\mathcal{L}y = -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\,y$ is the SL operator

If a mat A is self-adjoint $A = A^\dagger$ (recall [except we haven't done it yet] $a^\dagger_{ij} = a^\star_{ji}$) we know it has real evals λ and for distinct λ the associated evecs are orthog and form a basis for the underlying vec sp.

For a general operator $\mathcal{L}$ on $[a,b]$ the adjoint of $\mathcal{L}$, $\mathcal{L}^\dagger$ has the property that $\forall$ pairs of functions y_1, y_2 satisying appropriate (see later) BCs, $\int_a^b y_1\mathcal{L}y_2dx =$

$\int_a^b y_2 \mathcal{L}^\dagger y_1 dx$; [unintelligible comparison with matricies].

If $\mathcal{L} = \mathcal{L}^\dagger$ we say $\mathcal{L}$ is self-adjoint; the SL operator is sely-adjoint w/ appropriate BCs as $\int_a^b y_1 \mathcal{L} y_2 dx = \int_a^b y_1 \left(-\frac{d}{dx} \left(p(x) \frac{dy_2}{dx} \right) + q(x) y_2 \right) dx = \left[-y_1 p \frac{dy_2}{dx} \right]_a^b + \int_a^b \frac{dy_1}{dx} p(x) \frac{dy_2}{dx} + y_1 q(x) y_2 dx = \left[-y_1 p \frac{dy_2}{dx} \right]_a^b + \left[y_2 p \frac{dy_1}{dx} \right]_a^b + \int_a^b -y_2 \frac{d}{dx} \left(p(x) \frac{dy_1}{dx} \right) + y_1 q(x) y_2 dx = \left[p \left(y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right) \right]_a^b + \int_a^b y_2 \mathcal{L} y_1 dx$, so $\mathcal{L}$ is self-adjoint provided $T_1 = \left[p \left(y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right) \right]_a^b$ is 0 because of BCs on the y_i, e.g:

1. $y = 0$, $y' = 0$ or even $y + ky' = 0$ for some constant k at $x = a$ [and b].
2. $y(a) = y(b)$ and $y'(a) = y'(b)$
3. $p(a) = p(b) = 0$; this means a, b are singular points of the ODE

3.2 Properties of self-ajoint operator

Evals are real as given $\mathcal{L} y_n = \lambda_n y_n w$ taking the cplx conj we have $\mathcal{L} y_n^\star = \lambda_n^\star y_n^\star w$ so $\int_a^b y_n \mathcal{L} y_n^\star dx = \int_a^b y_n \lambda_n^\star w y_n^\star dx$ but since $\mathcal{L}$ is self-adjoint this $= \int_a^b y_n^\star \mathcal{L} y_n dx = \int_a^b \lambda_n y_n w y_n^\star$ so $(\lambda_n^\star - \lambda_n) \int_a^b w |y_n|^2 dx = 0$; the integrand is always positive so $\lambda_n^\star = \lambda_n$ i.e. λ_n is real.

Efuncs of distinct evals are orthog in a particular sense: say $\mathcal{L} y_n = \lambda_n y_n w, \mathcal{L} y_m = \lambda_m y_m w, \lambda_n \neq \lambda_m$. Then $\int_a^b y_m \mathcal{L} y_n dx = \int_a^b y_n \mathcal{L} y_m dx$ by self-adq, so $\lambda_n \int_a^b y_m w y_n dx = \lambda_m \int_a^b y_m w y_n dx$ and $(\lambda_n - \lambda_m) \int_a^b y_m w y_n dx = 0$ so $\int_a^b y_m w y_n dx = 0$. Thus we can create an ON set of efuncs by $Y_n(x) = \frac{y_n(x)}{\sqrt{\int_a^b w y_n^2 dx}}$ and $\int_a^b Y_m w Y_n dx = \delta_{mn}$. Without proof, this set is in fact a complete basis; any func $f(x)$ on $[a, b]$ w/ the same BCs as the efuncs (e.g. $f(a) = f(b) = 0$ can be expressed as $\sum_{n=1}^\infty a_n y_n(x) = \sum_{n=1}^\infty A_n Y_n(x)$ $(A_n = a_n \sqrt{\int_a^b w y_n^2 dx})$; if we multiply this by $w(x) Y_m(x)$ and integrate over $[a, b]$ then $\int_a^b f(x) w Y_m(x) dx = \sum_{n=1}^\infty A_n \int_a^b w Y_n(x) Y_m(x) = A_m$ by orthogonality; we have properties very much like the FS though note the FS is not generally normalized.

We have a Parseval's Thm: consider $I = \int_a^b w(x) (f(x) - \sum_{n=1}^\infty A_n Y_n(x)) dx = \int_a^b w f^2 dx - 2 \sum_{n=1}^\infty A_n \int_a^b w f Y_n dx + \sum_{n=1}^\infty A_n \sum_{m=1}^\infty A_m \int_a^b w Y_n Y_m dx = \int_a^b w f^2 dx - 2 \sum_{n=1}^\infty A_n^2 + \sum_{n=1}^\infty A_n^2 = \int_a^b w f^2 dx - \sum_{n=1}^\infty A_n^2$, but by the above completeness we know $I = 0$ so $\int_a^b w f^2 dx = \sum_{n=1}^\infty A_n^2$.

Note that if the basis is not complete, e.g. if the operator is not self-adj, we have the above $I > 0$ so only $\int_a^b w f^2 dx \geq \sum_{n=1}^\infty A_n^2$, Bessel's ineq. This makes sense - the series is incomplete, so it does not have as much energy as the full function.

Returning to the case with a self-adj op, $S_N(x) = \sum_{n=1}^N A_n Y_n(x)$ has $\lim_{N \to \infty} S_N(x) = f(x)$ (except at discontinuities as with FS); the mean square error in representing $f(x)$ by $S_N(x)$ is $\epsilon_N = \int_a^b w(f - S_N)^2 \, dx \to 0$ as $N \to \infty$; how does ϵ_N vary with A_i?

$\frac{\partial \epsilon_N}{\partial A_i} = -2\int_a^b w \left(f - \sum_{m=1}^N A_m Y_m\right) Y_i dx = -2\int_a^b w f y_i dx + 2\sum_{m=1}^N A_m \int_a^b Y_m Y_i dx = -2A_i + 2A_i = 0$, so the actual A_i are an extreme point of the (root mean square) error, and we can easily show that it is in fact a minimum; the efunction eqn minimizes the rms error between the finite sums and the actual f.

3.3 Example of SL problem; example of the important technique of separation of vars

Say we have a light unforced finite string of length L; $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ with $y(0,t) = y(l,t) = 0$. The initial conds are $y(x,0) = \phi(x)$, $\frac{\partial y}{\partial t}(x,0) = \psi(x)$ given and the problem is to find $y(x,t)$; we use separation of variables i.e. assume this is $X(x)T(t)$. Ignoring points where either of X, T are zero this gives us $\frac{\ddot{T}}{T} = c^2 \frac{X''}{X}$; the LHS depends only on t and the RHS only on x as last year so both must be $=$ some constant $-\lambda c^2$. Then $X'' + \lambda X = 0, \ddot{T} + \lambda c^2 T = 0$; it is quite common to get SL eqns from separation of vars (we can write e.g. the DE for X as $-\frac{d}{dx}\left(1\frac{dX}{dx}\right) + 0x = 1\lambda x$. For $\lambda < 0$ $X = A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right)$; applying the BCs, $X(0) = 0 \Rightarrow \alpha = 0, X(l) = 0 \Rightarrow \beta = 0$ so there are no nontrivial sols; we take $\lambda > 0$ (meaning $-\lambda c^2 < 0$; c being real is a reasonable assumption) and then $X'' = -\lambda X$ so $X = \alpha\cos\sqrt{\lambda}x + \beta\sin\sqrt{\lambda}x$; $X(0) = 0 \Rightarrow \alpha = 0, X(l) = 0 \Rightarrow \beta\sin\sqrt{\lambda}l = 0$ so we have an infinite no. of sols $\sqrt{\lambda_n} = \frac{n\pi}{L}$ i.e. $\lambda_n = \frac{n^2\pi^2}{L^2}$ for integer n.

The efunctions $Y_n(x)$ are $\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$, [should this constant have some n dependence? Probably not] or $\beta_n \sin\frac{n\pi x}{L}$; these are called the normal modes of the system. Note $n = 1$ gives only a half period of the sin wave in the string; $n = 2$ called an overtone is the first full period.

For the time dependence we have $\ddot{T}_n + \frac{n^2\pi^2c^2}{L^2}T_n = 0$ so $T_n(t) = \gamma_n \cos\frac{n\pi ct}{L} + \delta_n \sin\frac{n\pi ct}{L}$; a specific sol to the problem is then $y = X_n T_n = \sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right)$ for new constants $A_n = \frac{\gamma_n}{\beta_n}, B_n = \frac{\delta_n}{\beta_n}$. Since the problem is linear we can add these to obtain the general sol $y(x,t) = \sum_{n=1}^\infty \left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$; this is a FSS with the bracketed terms being b_n. So to apply the IC $y(x,0) = \phi(x) = \sum_{n=1}^\infty A_n \sin\frac{n\pi x}{L}$ we just calculate the FSS of ϕ $A_n = \frac{2}{L}\int_0^L \phi(x)\sin\frac{n\pi x}{L}dx$, or we can use the orthogonality of these sols to obtain exactly the same expression.

$\frac{\partial}{\partial t}y(x,0) = \psi(x) = \sum_{n=1}^\infty \left(-A_n\frac{n\pi c}{L}\sin\frac{n\pi ct}{L} + B_n\frac{n\pi c}{L}\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$; at $t = 0$ this is $\sum_{n=1}^\infty B_n \frac{n\pi c}{L}\sin\frac{n\pi x}{L}$ and again we can easily find $B_n = \frac{2}{n\pi c}\int_0^L \psi(x)\sin\frac{n\pi x}{L}dx$; as an exercise the reader should find the deflection of an initially plucked string $\phi(x) = \frac{x}{d}$ for $0 \le x < d$, $\frac{L-x}{L-d}$ for $d \le x \le L$.

Note we can generalise SL theory to complex y_n, p, q, w as is done in the QM course

3.4 Inhomog BVPs

SL theory can also be used to solve inhomog problems of the form $\left(\mathcal{L}-\tilde{\lambda}w\right)y = f(x)$ where $\mathcal{L}$ is an SL operator and $\tilde{\lambda}$ not an eval of it (i.e. not λ_n for any n; this is analagous to finding a CF and PI); let $f = wF$ and say the problem is on $[a, b]$ w/ appropriate BCs. Expand $F(x) = \sum_{n=1}^{\infty} A_n Y_n(x)$ and try to find a sol of the form $y = \sum_{n=1}^{\infty} B_n Y_n(x)$; we have $\sum_{n=1}^{\infty} B_n \left(\mathcal{L}Y_n - \tilde{\lambda}wY_n\right) = w\sum_{n=1}^{\infty} A_n Y_n$ i.e. $\sum_{n=1}^{\infty} B_n \left(\lambda_n w Y_n - \tilde{\lambda}wY_n\right) = w\sum_{n=1}^{\infty} A_n Y_n$, and now it is clear why we need $\tilde{\lambda} \neq \lambda_n \forall n$ as otherwise the bracket would be 0. Multiplying by Y_m and integrating the RHS becomes $\sum_{n=1}^{\infty}\int_a^b wA_nY_nY_m dx = A_m$ by orthogonality, while the RHS similarly becomes $B_m\left(\lambda_m - \tilde{\lambda}\right)$. So $B_m = \frac{A_m}{\lambda_m-\tilde{\lambda}} = \frac{\int_a^b wFY_m dx}{\lambda_m-\tilde{\lambda}} = \frac{\int_a^b Y_m f dx}{\lambda_m-\tilde{\lambda}}$, so $y = \sum_{m=1}^{\infty}\int_a^b \frac{Y_m(\xi)f(\xi)}{\lambda_m-\tilde{\lambda}} d\xi Y_m(x)$; again this is remniscent of a GF. If we try and solve $\frac{d^2G}{dx^2} + \beta\frac{dG}{dx} + \gamma G = \delta(x-\xi)$ in this form, take δ as being $wF = f$ and then $\delta(x-\xi) = \sum_{n=1}^{\infty} C_n Y_n(x), C_m = \int_a^b w(x) y_m(x)\delta(x-\xi)dx = w(\xi)Y_m(\xi)$ by the sampling property so $\delta(x-\xi) = \sum_{n=1}^{\infty} w(\xi)Y_n(\xi)Y_n(x)$; initially this might seem wrong because the δ function is symmetric in x, ξ, but in fact this $= \sum_{n=1}^{\infty} w(x)Y_n(x)Y_n(\xi)$, since $\frac{w(x)}{w(\xi)}\delta(x-\xi) = \frac{w(\xi)}{w(\xi)}\delta(x-\xi) = \delta(x-\xi)$ by the sampling property.

For the finite string and $w = 1$ $\delta(x-\xi) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}$; we can see that in general $\delta(x-\xi) = \sum_{n=1}^{\infty} w(\xi)Y_n(\xi)Y_n(x)$ has the correct sampling property: if $g(x) = \sum_{m=1}^{\infty} D_m(x)Y_m(x)$ then $\int_a^b g(x)\delta(x-\xi)dx = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_mY_n(\xi)\int_a^b w(x)Y_n(x)Y_m(x)dx = \sum_{m=1}^{\infty} D_mY_m(\xi) = g(\xi)$. Recall that the GF $G(x;\xi)$ for this problem satisfies $\left(\mathcal{L}-\tilde{\lambda}w\right)G = \delta(x-\xi)$; putting $w(x)F(x) = f(x) = \delta(x-\xi)$ we have $wF = w(x)\sum_{n=1}^{\infty} Y_n(x)Y_n(\xi) = w\sum_{n=1}^{\infty} A_nY_n(x)$ so $A_n = Y_n(\xi)$. Then $G(x;\xi) = \sum_{m=1}^{\infty} B_mY_m(x) = \sum_{m=1}^{\infty}\frac{Y_m(\xi)Y_m(x)}{\lambda_m-\tilde{\lambda}}$; $y = \sum_{n=1}^{\infty} Y_n(x)\left(\frac{\int_a^b Y_n(\xi)f(\xi)d\xi}{\lambda_n-\tilde{\lambda}}\right)$ with the bracketed term being B_n, and this $= \int_a^b\left(\sum_{n=1}^{\infty}\frac{Y_n(\xi)Y_n(x)}{\lambda_n-\tilde{\lambda}}\right)f(\xi)d\xi$ with the bracketed term being the GF.

Part III

PDEs (using sep of vars)

1 Classification of PDEs (non-examinable)

2nd order linear PDEs have a general form $a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y) = 0$ with $c \neq 0$ anywhere ($u_x = \frac{du}{dx}$ etc.); then let $\xi = x + \beta y, \nu = x + \delta y$ for $\beta \neq \delta$ and $u_x = \zeta_x u_\xi + \eta_x u_\eta = u_\xi + u_\eta, u_y = \beta u_\xi + \delta u_\eta$ and the eqn becomes $\left(a + 2b\beta + c\beta^2\right)u_{\xi\xi} + 2\left(a + b(\delta+\beta) + c\beta\delta\right)u_{\xi\eta} +$

$\left(a+2b\delta+c\delta^2\right)u_{\eta\eta} = G\left(u,u_\xi,u_\eta,\xi,\eta\right)$ where the RHS is some horrible but non-second-order function. If we choose β,δ to be the roots of $a+2b\lambda+c\lambda^2$ (i.e. equal to $\lambda_i = -b\pm\sqrt{b^2-ac}$) this becomes $\frac{2}{c}\left(ac-b^2\right)u_{\xi\eta} = G$, the canonical form. There are three situations: $b^2 > ac$ gives the λ_i real, the eqn is called hyperbolic, and these define characteristics $\xi = x+\lambda_1 y, \eta = x+\lambda_2 y$. If $b^2 < ac$ the λ_i are cplx and the eqn is called elliptic, and if $b^2 = ac$ there is a repeated λ and the eqn is parabolic; respective examples are the wave eqn, laplace's eqn and the diffusion eqn. For example for the wave eqn $y_{tt} - cy_{xx} = 0$ the characteristics are $x+ct, x-ct$.

2 Laplace's Eqn

This has the general form $\nabla^2\psi = 0$ in some domain D subject to some BCs on δD; if ψ is given these are Dirichlet conds and $\exists!$ sol, if $\left(\vec{n}\cdot\nabla\right)\psi$ is given these are Neumann conds and the sol is unique up to an additive constant. We consider sols in various geometries:

2.1 3D Cartesian Coords

$\psi_{xx}+\psi_{yy}+\psi_{zz} = 0$; we separate vars as $\psi = X(x)\,Y(y)\,Z(z)$; this is legit as if we find a sol by this method we know it is unique; substituting into the DE $\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z} = 0$; as before each term must be const so we have two indep consts $\frac{X''}{X} = -\lambda_l, \frac{Y''}{Y} = -\mu_m$ and then $\frac{Z''}{Z} = \lambda_l+\mu_m$. We need to find the evals λ_l,μ_m and assoc efuns $X_l, Y_m, Z_{l,m}$; then GS is $\psi = \sum_{l,m} a_{lm}X_l(x)\,Y_m(y)\,Z_{l,m}(z)$ with the a_{lm} determined by BCs.

As an example, consider heat conduction in a semi-infinite rectangular bar $0\le x\le a, 0\le y\le b, 0\le z$ heated at the end "plate"; $\psi_t = \kappa\nabla^2\psi$, the steady sol is $\kappa\nabla^2\psi = 0$. Our BCs are $\psi(x,y,0) = 1$ as this end is heated, $\psi(0,y,z) = \psi(a,y,z) = \psi(x,0,z) = \psi(x,b,z) = 0$ as the surroundings are cold and $\psi\to 0$ as $z\to\infty$ since this is the only way it makes physical sense; the astute reader may be concerned about the edges of the front face $(x,0,0)$ and similar, but we shall see these are in fact ok. We solve $X'' = -\lambda X$ with $X(0) = X(a) = 0$, finding $\lambda_l = \frac{l^2\pi^2}{a^2}, X_l = \sqrt{\frac{2}{a}}\sin\frac{l\pi x}{a}$ for $l = 1,2,\ldots$ (we normalize for convenience though since we will be multiplying by a constant this will be irrelevant); similarly $\mu_m = \frac{m^2\pi^2}{b^2}, Y_m = \sqrt{\frac{2}{b}}\sin\frac{m\pi x}{b}$ for $m = 1,2,\ldots$. Now we solve the z-dep: $Z'' = \left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right)\pi^2 Z$; $\psi\to 0$ as $z\to\infty$ so$Z = \beta\exp\left(-\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}}\pi z\right)$ so $\psi(x,y,z) = \frac{2}{\sqrt{ab}}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}a_{lm}\sin\frac{m\pi y}{b}\sin\frac{l\pi x}{a}\exp\left(-\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}}\pi z\right)$; a_{lm} are determined by the end plate cond so $1 = \frac{2}{\sqrt{ab}}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}a_{lm}\sin\frac{m\pi y}{b}\sin\frac{l\pi x}{a}$; this looks like the generalisation of a half-range FSS. This means the edges previously mentioned are fine - such a sin series is also a sol of the end plate temperature being -1 for $-a<x<0, 0<y<b$, -1 on $0<x<a, -b<y<0$, 1 on $-a<x<0, -b<y<0$ and similarly in a "checkerboard" pattern in place of the end of the bar; at $x = 0$ (and $x = a$, and $y = 0, b$) there is a discontinuity from 1 to -1 so the sin series here will take the value 0 which is precisely what the other applicable BC says it does.

We therefore solve using orthogonality; multiplying by $\frac{2}{\sqrt{ab}} \sin \frac{p\pi x}{a} \sin \frac{q\pi x}{b}$ (since this is the normalized eigenfunction; we could equally well express the function as $\sum_{l=1}^{\infty} \sum_{m=1}^{\infty} A_{lm} \sin \frac{m\pi y}{b} \sin \frac{l\pi x}{a}$ but that would make this calculation messier) and integrating over the plate $\frac{2}{\sqrt{ab}} \int_0^b \int_0^a (1) \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b} = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} a_{lm} \left(\frac{2}{b} \int_0^b \sin \frac{m\pi y}{b} \sin \frac{q\pi y}{b} \right) \left(\frac{2}{a} \int_0^a \sin \frac{n\pi x}{a} \sin \frac{p\pi x}{a} dx \right) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} a_{lm} \delta_{mq} \delta_{lp} = a_{pq}$ (note that the bracketed 1 in the first expression is the temperature on the plate; any function with a FSS would work equally well) so $a_{pq} = \frac{2}{\sqrt{ab}} \frac{ab}{\pi^2 pq} \left(1 - (-1)^p\right) \left(1 - (-1)^q\right) = a_{pq}$ so $a_{pq} = 0$ if either of p, q even, $\frac{8\sqrt{ab}}{\pi^2 qp}$ for p, q odd and $\psi(x, y, z) = \frac{16 \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sin\left((2m-1)\frac{\pi y}{b}\right) \sin\left((2l-1)\frac{\pi x}{a}\right) e}{\pi^2 (2l-1)(2m-l)}$ where $k_{lm}^2 = \frac{(2l-1)^2}{a^2} + \frac{(2m-1)^2}{b^2}$.

As $z \to \infty$, k_{lm} increases faster for higher values of l, m; the solution becomes dominated by the $l = 1, m = 1$ component or mode.

Notice that for the square bar $a = b$ the evals are degenerate ($k_{lm} = k_{ml}$) but the efuncs remain orthogonal; this is analagous to the situation where we have a real symmetric matrix with a double eval, e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has evals $1, 1, 2$ but its evecs e_1, e_2, e_3 are still orthog.

2.3 Laplace's eqn in plane polars

$x = r\cos\theta, y = r\sin\theta$ and $\nabla^2 \psi = 0$ becomes $\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0$; as before we separate variables, this time as $R(r)\Theta(\theta)$, then we have $\frac{\Theta''}{\Theta} + \frac{r}{R}(rR')' = 0$, so put $\frac{\Theta''}{\Theta} = -\lambda$, $\frac{r}{R}(rR')' = \lambda$; λ is the separation constant (evalue); as an example consider a circular ring of radius 1 but bent upwards and down so that its height above the plane at each point is some $f(\theta)$ and a soap film stretched across it; how does the film hang? Our BCs here are that $f(2\pi + \theta) = f(\theta)$ and $\psi(0, \theta)$ must be bounded. We use three steps:

We solve $\Theta'' = -\lambda\Theta$ with periodic BCs, period 2π, to determine the possible vals of λ and assoc efuncs; $\lambda = n^2, \Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$ for $n = 1, 2, \ldots$. We also must allow $n = 0$; $\Theta_0(\theta) = \frac{1}{2}a_0 + b_0\theta$ which must simply be $\frac{1}{2}a_0$ by periodicity; we take the $\frac{1}{2}$ for convenience so we get an FS at the end.

Next we substitute our possible λ to obtain $\frac{r}{R_n}(rR_n')' = \lambda_n$; for $n > 0$ this gives $r(rR_n')' - n^2 R_n = 0$; we try $R \propto r^\beta$, then we have $\beta^2 - n^2 = 0$, so $R_n(r) = c_n r^n + d_n r^{-n}$; ψ is bounded at 0 so this is $c_n r^n$; if we wanted to look at outside rather than inside the ring we would take the other solution (we can look for a sol for both, but will simply get the two different types of sol inside and outside the ring); when $n = 0$ $(rR_0')' = 0$ so $R_0' = \frac{d_0}{r}$ and $R_0 = c_0 + d_0 \ln r$; by regularity when $r = 0$ we have $d_0 = 0$ and redefining our constants $\psi(r, \theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n$.

Finaly we find a_n, b_n by the BC $\psi(1,\theta) = f(\theta)$; we have $f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta$; this is just an FS, f has period 2π so $a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta, b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$. Notice that as before the higher harmonics are localised, this time close to the boundary $r = 1$; for large n, $r^n \to 0$ quickly away from this boundary.

2.4 Laplace's eqn in spherical polars, Legendre polys

$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$; $dV = r^2 \sin\theta dr d\theta d\phi$, $r = |(x,y,z)|$, $0 \le \phi < 2\pi, 0 \le \theta \le \pi$. $\nabla^2\psi = 0$ becomes $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0$. While it is possible and indeed very worthwhile to solve this fully, we will consider only axisymmetric disturbances, i.e. there is no dependence on ϕ, $\psi = R(r)\Theta(\theta)$ and multiplying the above by $\frac{r^2}{R\Theta}$ we have $\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$; as always we put this $= \lambda + -\lambda$ and $(\sin\theta\Theta')' + \lambda\sin\theta\Theta = 0, \left(r^2 R\right)' - \lambda r = 0$; we use a three step method as before.

The sol to our eqn for Θ is not immediate, so we use a common trick; we substitute $x = \cos\theta$ (not to be confused with the co-ordinate x), then $\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx}$ and the eqn becomes $-\sin\theta\frac{d}{dx}\left(\sin\theta\left(-\sin\theta\frac{d\Theta}{dx}\right)\right) + \lambda\sin\theta\Theta = 0$; assuming everything is well behaved this becomes $-\frac{d}{dx}\left(\left(1-x^2\right)\frac{d\Theta}{dx}\right) = \lambda\Theta$; this is an SL form defined on $[-1,1]$ (recall $x = \cos\theta$ for $\theta \in [0,\pi]$) with $w = 1, q = 0, p = 1 - x^2$; this is called Legendre's eqn. We try to find a sol bounded on $[-1,1]$ by series $\Theta = \sum_{n=1}^{\infty} a_n x^n$; we have $\left(1-x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \lambda\Theta = 0$ so substituting and equating coeffs, $a_{n+2} = \frac{n(n+1)-\lambda}{(n+1)(n+2)}a_n$, and we have two lin ind sols given by $a_0 \ne 0, a_1 = 0$ and $a_0 = 0, a_1 \ne 0$. As before we use the fact that our $x = \cos\theta$ series must be bounded; our lin ind sols are $a_0\left(1 + \frac{-\lambda}{2!}x^2 + \frac{-\lambda(6-\lambda)}{4!}x^4 + \frac{-\lambda(6-\lambda)(20-\lambda)}{6!}x^6 + \ldots\right), a_1\left(x + \frac{2-\lambda}{3!}x^3 + \frac{(12-\lambda)(2-\lambda)}{5!}x^5 + \ldots\right)$ which by the ratio test converge for $|x| < 1$ but diverge for $x = \pm 1$, so for the series to remain finite here it must terminate; $a_{m+2} = 0$ for some m so $\lambda_m = m(m+1)$ are the eigenvals; our sols are then polys of deg m, the Legendre Polys; these are called $P_m(x)$ up to scaling; by convention these are scaled so $P_m(1) = 1$ rather than being normalized; in fact $\int_{-1}^{1} P_m^2(x)\,dx = \frac{2}{2m+1}$. The first few $m = 0,1,2,3,4$ have $\lambda = 0,2,6,12,20$ and $P_m(x) = 1, x, \frac{3x^2-1}{2}, \frac{5x^3-3x}{2}, \frac{35x^4-30x^3+3}{8}$; of course in terms of θ these are $1, \cos\theta, \frac{3\cos^2\theta-1}{2}, \ldots$. The m roots of P_m are all in the interval $[-1,1]$.

The P_m are efuncs for distinct evals of an SL problem so orthog; $w = 1$ from the eqn so $\int_{-1}^{1} P_n(x) P_m(x) = 0 \forall n \ne m$. We can use Legendre polys as a basis for bounded funcs $f(x)$ on $[-1,1]$; $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$ where $a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x) P_n(x)\,dx$.

Returning to the original problem, $\Theta_n(\theta) = P_n(\cos\theta)$ with $\lambda = n(n+1)$; for the r dependence $\left(r^2R'\right)' - \lambda R = 0$ so $\left(r^2R'\right)' - n(n+1)R = 0$. Trying $R \propto r^\beta$ we obtain $\beta(\beta+1)r^\beta - n(n+1)r^\beta = 0$ with sols $\beta = n, -(n+1)$; this is reminiscent of the circular domain problem above. So $\psi(r,\theta) = \sum_{n=0}^{\infty}\left(a_n r^n + \frac{b_n}{r^{n+1}}\right)P_n(\cos\theta)$.

Finally we determine a_n, b_n by applying our BC $\psi(r_0, \theta) = f(\theta)$ and asking whether we want sols for $r < r_0$ or $r > r_0$; as before if we ask for a sol for both we will just get two sols in different form for inside and outside which match up on the boundary.

For example, if we are solving for $r \leq 1$ with $\psi(1, \theta) = f(\theta)$ regularity at $r = 0$ gives $b_n = 0 \forall n$, so $f(\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta)$; as above we can find $a_n = \frac{2n+1}{2} \int_{-1}^{1} F(x) P_n(x) \, dx$ where $F(x) = f(\theta)$ when $x = \cos\theta$. See Exs3q3.

2.5 Generating Fn

Consider an electric field w/ point charge e one unit away from the origin in the z direction. Say we are at some $\vec{r} = (r, \theta)$ in axisymmetric spherical polars; the potential here is $\psi(r, \theta) = \frac{1}{\rho}$ where ρ is the distance from $\vec{r}$ to e; $\rho = |\vec{r} - \hat{z}|$ so $|\rho|^2 = (\vec{r} - \hat{z}) \cdot (\vec{r} - \hat{z})$ (where $\hat{z}$ is the unit vector in the z direction) which is $r^2 + 1 - 2\vec{r} \cdot \hat{z}$ so $\rho^2 = r^2 + 1 - 2r\cos\theta$ and $\psi(r, \theta) = \frac{1}{\sqrt{1 - 2r\cos\theta + r^2}}$, but we know $\nabla^2 \psi = 0$ everywhere except $\hat{z}$ since there is no charge density anywhere else. We look for sols to this, which we know take the form $\psi(r, \theta) = \sum_{n=0}^{\infty} \left(a_n r^n + \frac{b_n}{r^{n+1}}\right) P_n(\cos\theta)$, with $r < 1$, so $b_n = 0$ by regularity at the origin; the sol is unique so we must have $\sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta) = \frac{1}{\sqrt{1 - 2r\cos\theta + r^2}}$ or $\sum_{n=0}^{\infty} a_n r^n P_n(x) = \frac{1}{\sqrt{1 - 2rx + r^2}}$; squaring both sides and integrating between -1 and 1 the LHS here becomes $\sum_{m=0}^{\infty} a_m r^m \sum_{n=0}^{\infty} a_n r^n \int_{-1}^{1} P_n(x) P_m(x) \, dx = \sum_{n=0}^{\infty} a_n^2 r^{2n} \int_{-1}^{1} (P_n(x))^2 \, dx$ and the RHS is $\int_{-1}^{1} \frac{1}{1 - 2rx + r^2} dx = \frac{1}{2r} \int_{-2r}^{2r} \frac{1}{1 + r^2 - y} dy$ where $y = 2rx$; this is $-\frac{1}{2r} \log\left(1 + r^2 - y\right) |_{-2r}^{2r} = \frac{1}{2r}\left(\log\left(1 + r^2 + 2r\right) - \log\left(1 + r^2 - 2r\right)\right) = \frac{1}{r}\left(\log(1 + r) - \log(1 - r)\right)$; $\log(1 + r) = r - \frac{r^2}{2} + \frac{r^3}{3} - \ldots, \log(1 - r) = -\left(r + \frac{r^2}{2} + \frac{r^3}{3} + \ldots\right)$ so this is $\frac{2r}{r} + \frac{2r^3}{3r} + \frac{2r^5}{5r} + \cdots = \sum_{n=0}^{\infty} \frac{2}{2n+1} r^{2n}$, which from before $= \sum_{n=0}^{\infty} a_n^2 r^{2n} \int_{-1}^{1} (P_n(x))^2 \, dx$ so choosing P_n "normalized" so that the integral is $\frac{2}{2n+1}$ we have $a_n = 1 \forall n$ in our sol to Laplace's eqn and $\sum_{n=0}^{\infty} P_n(x) r^n = \frac{1}{\sqrt{1 - 2rx + r^2}}$; this is the generating function since the reader may check by examples that $\frac{d^n}{dt^n}\left(\frac{1}{\sqrt{1 - 2tx + t^2}}\right) |_{t=0} = n! P_n(x)$.

2.6 Laplace's eqn for large r

Say we place a neutral conducting sphere of rad r_0 at the origin in a previously uniform electric field, and want to solve for the new pertubed electrostatic potential V.

$E = -\nabla V$, $\nabla^2 V =$ the charge density $-\frac{\rho}{\rho_0}$ which is here 0 since there are no charges in the far field.

The original field is $E = E_0$ so originally $V_0 = -E_0 z$ (we take the field in the z direction for convenience).

After pertubation we know $\nabla^2 \psi = 0$ so $\psi(r, \theta) = \sum_{n=0}^{\infty} \left(a_n r^n + \frac{b_n}{r^{n-1}}\right) P_n(\cos\theta)$ [using axisymmetric spherical polars]; we want $\psi \to -E_0 z = V_0$ as $r \to \infty$; note that this is unbounded. So we take $a_n = 0$ for $n = 0$ and $n \geq 2$ but $a_1 = -E_0$;

our conditions only define V up to a constant so we can wlog take $V = 0$ on the line $z = 0$ i.e. $\theta = \frac{\pi}{2}$; since the sphere is conducting we then have $V = 0$ on $r = r_0$; so V on $r = r_0$ is $-E_0 r_0 P_1(\cos\theta) + \frac{b_0}{r_0} + \frac{b_1}{r_0^2} P_1(\cos\theta) + \cdots = 0 \forall\theta$ so $b_0 = 0$ and by orthogonality [I don't quite follow this] we have $b_n = 0 \forall n \geq 2$ and $b_1 = E_0 r^3$ so the full sol is $V = -E_0 r P_1(\cos\theta)\left(1 - \frac{r_0^2}{r^3}\right)$.

2.7 Connection with electrostatic multipoles

Consider charges placed nonuniformly but axisymmetrically inside a sphere. What is the induced potential?

$V(r,\theta) = \sum_{n=0}^{\infty} b_n r^{-(n+1)} P_n(\cos\theta)$; in the absence of a uniform field, V must be bounded as $r \to \infty$ so $a_n = 0$. The $\frac{b_0}{r}$ cpt from $n = 0$ is called the monopole cpt; it behaves indistinguishably from a point charge at the origin, it is completely isotropic (of course it is the combined result of the total net charge at $r = 0$); when $n = 1$ we get $\frac{b_1 \cos\theta}{r^2}$, a dipole field as we would get from two charges $\pm q$ close together; then the $n = 2$ term is $b_2 \frac{\left(3\cos^2\theta - 1\right)}{2r^3}$ which is the quadripole field as from a square of charges with diagonally opposite ones having the same charge, and so on.

This shows the utility of uniqueness; we can construct the sol any way we want and then gain useful intuition and properties from the way we constructed it.

3 Wave Eqn

$u_t = c^2 u_{xx}, u(x,0) = \phi(x), \frac{\partial u}{\partial t}(x,0) = \psi(x)$. The characteristics are $\xi = x + ct$ representing motion to the left and $\eta = x - ct$ for motion to the right. $\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$, $\frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}$, so the eqn becomes $c^2\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} - 2\frac{\partial^2}{\partial \xi \partial \eta}\right) u = c^2\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + 2\frac{\partial^2}{\partial \xi \partial \eta}\right) u$ so $\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0$ i.e. $\frac{\partial u}{\partial \eta} = f(\eta)$ so $u = \int^{\eta} f(u)\,du + g(\xi) = f(\eta) + g(\xi)$ for some η, ξ; sols propagate along characteristics (η const or ξ const) without changing shape; these are lines $\xi = \xi_0 = x + ct$ i.e. $t = \frac{\xi_0 - x}{c}$ and similarly for η. In particular, discontinuities can propagate along characteristics e.g. if $\psi(x) = 0, \phi(x) = H(x)$; recall if $\psi = 0$ then $u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct))$ so $u = \frac{1}{2}(H(x+ct) + H(x-ct))$; at $t = 0$ we have $u = 0$ for $x < 0$, 1 for $x > 0$ while at later time t we have $u = 0$ for $x < -ct$, $\frac{1}{2}$ for $-ct < x < ct$ and 1 for $x > ct$; the discontinuities are moving away from the origin at speed c; shocks propagate along characteristics i.e. at speed c

Consider a wave $y = \Re\left(Ae^{i\omega\left(t - \frac{x}{c}\right)}\right)$ for $A \in \mathbb{C}$; $|A|$ is the amplitude, this is a wave of freq ω travelling to the right at speed c; $\arg A = \phi$ is the phase; in reals the eqn becomes $|A|\cos(\omega t - kx + \phi)$ where k is the wavenumber $\frac{2\pi}{\lambda}$ where λ is wavelength so $\omega = ck$.

Consider the finite string problem $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial t^2}$ with $y(0,t) = y(L,t) = 0 \forall t$ and $y(x,0) = \phi(x), \frac{\partial y}{\partial t}(x,0) = \psi(x), c = \frac{T}{\mu}$ i.e. $\frac{\text{tension}}{\text{mass per unit length}}$.

$y(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L}\right) \sin \frac{n\pi x}{L}$ where $A_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} dx, B_n = \frac{2}{n\pi L} \int_0^L \psi \sin \frac{n\pi x}{L} dx$; KE which is normally $\frac{1}{2}mv^2$ will then be $K = \int_0^L \frac{1}{2}\mu \left|\frac{\partial y}{\partial t}\right|^2 dx$; the PE of a small elt is tension $\times$ extension $= T(\delta y - \delta x) = T\left(\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} - 1\right)\delta x$ [I think δ(length) was meant by δy] so $PE = V = T\int_0^L \left(\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} - 1\right) dx$; $\frac{\partial y}{\partial x}$ is small so by binomial series this is approximately $\frac{T}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx$ so the total energy is $E = K + V = \frac{1}{2}\mu\int_0^L \left(\frac{\partial y}{\partial t}\right)^2 + \frac{T}{\mu}\left(\frac{\partial y}{\partial x}\right)^2 dx$ which since $\frac{T}{\mu} = c^2$ is $\frac{1}{2}\mu\int_0^L \left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{\partial y}{\partial x}\right)^2 dx$.

We then find $K = \frac{1}{2}\mu\frac{L}{2}\sum_{n=1}^{\infty} \frac{n^2\pi^2c^2}{L^2}\left(A_n^2 \sin^2 \frac{n\pi ct}{L} + B_n^2 \cos^2 \frac{n\pi ct}{L} - 2A_nB_n \sin \frac{n\pi ct}{L} \cos \frac{n\pi ct}{L}\right)$ [unchecked from now on] and $V = \frac{1}{2}\mu c^2 \frac{L}{2}\sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2}\left(A_n^2 \cos^2 \frac{n\pi ct}{L} + B_n^2 \sin^2 \frac{n\pi ct}{L} + 2A_nB_n \cos \frac{n\pi ct}{L} \sin \frac{n\pi ct}{L}\right.$ so $E = \frac{\mu c^2\pi^2}{4L}\sum_{n=1}^{\infty} n^2\left(A_n^2 + B_n^2\right)$; note that this is indep of time as we would expect from the physics; also contrast this with Parseval's T.

$A_0 = 0$ since the ends are fixed at 0; the period of oscillation will be that of the fundamental $n = 1$ as all others divide it so it is $\frac{2\pi}{\omega} = \frac{2\pi L}{\pi c} = \frac{2L}{c}$.

If we average over the period, we find (this is immediate from the expressions for K, V as $\frac{c}{2L}\int_0^{\frac{2L}{c}} \sin^2 \frac{n\pi ct}{L} dt = \frac{c}{2L}\int_0^{\frac{2L}{c}} \cos^2 \frac{n\pi ct}{L} dt = \frac{1}{2}$ and $\frac{c}{2L}\int_0^{\frac{2L}{c}} \sin \frac{n\pi ct}{L} \cos \frac{n\pi ct}{L} dt = 0$) $\overline{K} = \frac{c}{2L}\int_0^{\frac{2L}{c}} K dt = \overline{V} = \frac{c}{2L}\int_0^{\frac{2L}{c}} V dt$ - we have an equipartition between KE and PE.

3.2 Wave Reflection and Transmission

Consider a string w/ mass per unit length μ_- for $x < 0$ and μ_+ for $x > 0$, with corresponding tentions τ_-, τ_+. $c_\pm = \sqrt{\frac{\tau_\pm}{\mu_\pm}}$, not generally the same for $+, -$. We have three waves: the incoming wave $Ie^{i\omega\left(t - \frac{x}{c_-}\right)}$, transmitted wave $Te^{i\omega\left(t - \frac{x}{c_+}\right)}$ and reflected wave $Re^{i\omega\left(t + \frac{x}{c_-}\right)}$ where I, T, R are complex since in general not only the amplitude but also the phase of the wave can change on transmission/reflection. We are generally interested in the real parts $\Re\left(Ie^{i\omega\left(t - \frac{x}{c_-}\right)}\right)$ etc (Notation: $I_R = \Re(I)$ etc); we can wlog take the incident wave to have zero phase so this specific wave is $I_R \cos\left(\omega\left(t - \frac{x}{c_-}\right)\right)$; the reflected wave is $|R|\cos\left(\omega\left(t + \frac{x}{c_-}\right) + \phi_R\right)$ where ϕ_R is the phase, similarly for the transmitted wave.

The BCs at $x = 0$ are that by resolving forces horizontally, since we are assuming the deflections are small $\tau_+ = \tau_- = \tau$; the string does not break so $y(0-) = y(0+)$; conservation of energy gives $\mu_- |I|^2 = \mu_+ |T|^2 + \mu_- |R|^2$; this gives us an idea of the "Impedance" or "Conductivity" of the problem. We therefore [?] have $I + R = T$; wlog we take $I = 1$ and then $1 + R = T$ i.e. $1 + R_R = T_R, R_i = T_i$. We have $R_R = |R| \cos\phi_R, R_1 = |R| \sin\phi_R$ and sim for T.

Finally we resolve vertically; the vertical forces are balanced on each side because the point $x = 0$ has no inertia; note this differs on the Exs where we have a point mass at $x = 0$. So $\tau \frac{\partial y}{\partial x} |_{x=0_+} = \tau \frac{\partial y}{\partial x} |_{x=0_-}$, so $\frac{1}{c_-} + \frac{R}{c_-} = -\frac{T}{c_+}$; taking the cplx part we have $\frac{R_i}{c_-} = -\frac{T_i}{c_+}$ which combined with the above gives $R_i = T_i = 0$ so $\sin\phi_R = \sin\phi_T = 0$; taking the real part and combining with the above $R = \frac{c_+ - c_-}{c_+ + c_-}, T = \frac{2c_+}{c_+ + c_-}$. This is consistent with the energy BC; $\mu_\pm \sum \frac{1}{c_\pm^2}$ as $\frac{\tau}{\mu_\pm} = c_\pm^2$.

Consider the limiting cases; $\mu_+ = \mu_-$ gives $c_+ = c_-, R = 0, T = 1 = I$ and we have clean transmission, while $\frac{\mu_+}{\mu_-} >> 1$ gives $c_+ << c_-$, $R \approx -1 = -I$ and $T \approx 0$; this gives us pure reflection just as when we have a fixed end; $\phi_R = \pi$, the reflection is 180 degrees out of phase. A very light string for $x > 0$, $\frac{\mu_+}{\mu_-} << 1$ gives $T \approx 2 = 2I, R \approx 1 = I$.

4 Heat Eqn

$\frac{\partial\theta}{\partial t} - D\frac{\partial^2\theta}{\partial x^2} = 0$ where D is the diffusion coeff; we need an initial cond $\theta(x, 0) = \phi(x)$ and for infinite domains we take $\theta \to 0$ as $|x| \to \infty$, for a finite domain we generally have $\theta = \theta_0$ at $x = 0$, $\theta = \theta_L$ at $x = L$ but could have other conds e.g. $\frac{\partial\theta}{\partial x}$ given at $x = 0, L$.

On infinite domains FTs are very powerful; we take the FT of the heat eqn wrt x and obtain $\frac{\partial\tilde{\theta}}{\partial t} = -Dt^2\tilde{\theta}$ so $\tilde{\theta} = \tilde{\theta}(k, 0)\, e^{-Dk^2 t}$; $\tilde{\theta}(k, 0) = \tilde{\phi}$ so we use the convolution Thm; from exs we know $f(x) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$ has FT $\tilde{f}(k) = e^{-Dk^2 t}$ so $\theta(x, t) = \int_{-\infty}^{\infty} \frac{\phi(u)}{\sqrt{4\pi Dt}} e^{\left(\frac{-(x-u)^2}{4Dt}\right)} du$; this is the GS for $\theta(x, t)$ provided θ has an FT which will definitely be the case for $\theta \to 0$ as $|x| \to \infty$. Next we consider some particularly instructive choices of ϕ, namely e^{-ax^2} for $a > 0$ a Gaussian initial cond and $\delta(x)$, localised forcing as from a laser.

First the Gaussian initial pusle, $\phi(x) = \theta_0 e^{-ax^2}$. Then $\theta = \int_{-\infty}^{\infty} \frac{\theta_0}{\sqrt{4\pi Dt}} e^{\left(-\frac{1+4aDt}{4Dt}u^2 + \frac{2xu}{4Dt} - \frac{x^2}{4Dt}\right)} du = \frac{\theta_0}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{1+4aDt}{4Dt}\left(u^2 - \frac{2xu}{1+4aDt} + \frac{x^2}{(1+4aDt)^2}\right)} e^{-\frac{ax^2}{1+4aDt}} du = \frac{\theta_0}{\sqrt{4\pi Dt}} e^{-\frac{ax^2}{1+4aDt}} \int_{-\infty}^{\infty} e^{-\frac{1+4aDt}{4Dt}\left(u - \frac{x}{1+4aDt}\right)^2} du$; substituting $v = \sqrt{\frac{1+4aDt}{4Dt}} \left(u - \frac{x}{1+4aDt}\right)$ the integral becomes $\int_{-\infty}^{\infty} \sqrt{\frac{4Dt}{1+4aDt}} e^{-v^2} dt$ and $\theta(x, t) = \theta_0 e^{-\frac{ax^2}{1+4aDt}} \frac{1}{\sqrt{\pi(1+4aDt)}} \int_{-\infty}^{\infty} e^{-v^2} dv$; the integral is $\sqrt{\pi}$ so this is $\frac{\theta_0 e^{-\frac{ax^2}{1+4aDt}}}{\sqrt{1+4aDt}}$; the Gaussian spreads out and "flattens" (its peak becomes lower) as t increases.

Second we take $\phi = \theta_0 \delta(x)$, then $\theta = \frac{\theta_0}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \delta(u)\, e^{\frac{(x-u)^2}{4Dt}} du = \frac{\theta_0}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$;

at $t = 0$ we have the spike but at any time after this we have a Gaussian, at first very tall and narrow, but again spreading out and flattening as t increases. Note that the information that the bar was heated is transmitted instantaneously; θ becomes nonzero on the whole domain for $t > 0$. Of course this is not true in the real physical situation.

We note that $\frac{x^2}{4Dt}$ appears to be a fundamental group. However, the characteristics in the heat eqn are $t = 0$; discontinuities at $t = 0$ do not propagate to $t > 0$, as shown by the example above, in contrast to the wave eqn.

4.2 Similarity soln in heat eqn

Consider setting $\eta = \frac{x}{2\sqrt{Dt}}$ in the heat eqn; $\frac{\partial}{\partial t} = \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -\frac{x}{4D^{\frac{1}{2}}t^{\frac{3}{2}}}\frac{\partial}{\partial \eta} = -\frac{\eta}{2t}\frac{\partial}{\partial \eta}$, $\frac{\partial}{\partial x} = \frac{1}{2\sqrt{Dt}}\frac{\partial}{\partial \eta}$ so $\frac{\partial^2}{\partial x^2} = \frac{1}{4Dt}\frac{\partial^2}{\partial \eta^2}$ and substituting into the heat eqn we obtain $-\frac{\eta}{2t}\frac{\partial}{\partial \eta}\theta = \frac{D}{4Dt}\frac{\partial^2}{\partial \eta^2}\theta$ so $-\eta\frac{\partial \theta}{\partial \eta} = \frac{1}{2}\frac{\partial^2 \theta}{\partial \eta^2}$; let $X = \frac{\partial \theta}{\partial \eta}$, then $-2\eta X = \frac{\partial X}{\partial \eta}$ so $X = ce^{-\eta^2}$ and $\theta = c_3\frac{2}{\sqrt{\pi}}\int^{\eta} e^{-u^2}du = c_3\frac{2}{\sqrt{\pi}}\int^{\frac{x}{2\sqrt{Dt}}} e^{-u^2}du = c_3 erf\left(\frac{x}{2\sqrt{Dt}}\right)$ where $erf\,(y) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$; notice that if $x \sim t^{\frac{1}{2}}$ the solution doesn't change - this is characteristic of a diffusion process, and an example of a similarity variable.

4.3 Finite domain heat eqn

Note the erf/similarity sol has no information from the BCs; it can be useful as an early time approximation in finite domains e.g. on Exs have the infinite domain prob $\theta\,(x, 0) = \theta_0 H\,(x)$; we find $\theta = \frac{\theta_0}{2}\left(1 + erf\left(\frac{x}{2\sqrt{Dt}}\right)\right)$. How can we solve the finite domain problem $\theta\,(x, 0) = \theta_0$ on $0 \le x \le L$, 0 on $-L \le x < 0$ w/ Bcs $\theta\,(L, t) = \theta_0, \theta\,(-L, t) = 0$? We have a steady state sol $\theta_s = \theta_0\left(\frac{x+L}{2L}\right)$; it is easier to solve the problem for $\hat{\theta} = \theta - \theta_s$, which has homog BCs.

Assume $\hat{\theta} = X\,(x)\,T\,(t)$; define a new var $Y = x + L$ which ranges from 0 to $2L$, then $\hat{\theta} = Y\,(y)\,T\,(t)$; sep vars, $T' = -D\lambda T, Y'' = -\lambda Y$ for some λ; we have $Y = A\cos\sqrt{\lambda}y + B\sin\sqrt{\lambda}y$, $Y\,(0) = 0 \Rightarrow A = 0, Y\,(2L) = 0 \Rightarrow \lambda = \frac{n^2\pi^2}{4L^2}$; then $T' = -\frac{Dn^2\pi^2}{4L^2}T$ so $T = e^{-\frac{Dn^2\pi^2}{4L^2}t}$; the GS is $\theta = \theta_0\frac{x+L}{2L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi(x+L)}{2L}e^{-\frac{Dn^2\pi^2}{4L^2}t}$; the IC for $\hat{\theta}$ is $\hat{\theta}\,(x, 0) = \theta_0\left(H\,(x) - \frac{x+L}{2L}\right)$; we find the b_n are 0 for odd n and $\theta = \frac{\theta_0(x+L)}{2L}\sum_{n=1}^{\infty}\frac{L}{n\pi}(-1)^n \sin\frac{n\pi(x+L)}{L}e^{-\frac{Dn^2\pi^2 t}{4L^2}} = \frac{\theta_0(x+L)}{2L} + \sum_{n=1}^{\infty}\frac{\theta_0}{n\pi}\sin\frac{n\pi x}{L}e^{-\frac{Dn^2\pi^2 t}{L^2}}$; graphing we see the infinite sol is only a good approx when $t << \frac{L^2}{D}$; once $t \sim \frac{L^2}{D}$ the transient cpts are very small compared to the steady state; $\frac{\partial \theta}{\partial t} \approx 0$ so $D\frac{\partial^2 \theta}{\partial x^2} = 0$ and θ is linear in x.

Part IV
Calculus of Variations

4.1 Motivation

Say we have points $A = (x_1, y_1), B = (x_2, y_2)$. The distance from A to B along a path, a $y(x)$ s.t. $y(x_1) = y_1, y(x_2) = y_2$, is $S = \int_A^B ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx$ (s is an arclength paramater); we want to find the min of S over all possible paths. S depends not on x but on $y(x)$, so S is a functional i.e. a function of a function.

Examples

For a sphere, using spherical polars and taking $r \equiv 1$ where a path is given by $\phi(\theta)$; now $S = \int_{\theta_1}^{\theta_2} \sqrt{1 + \phi'^2 \sin^2 \theta} d\theta$ as can be seen (as with all of these) by considering infinitesimal elements. This is used for minimizing the distance a plane flies - the minimal paths are [arcs of] great circles.

The brachistochrone problem - the travel time for a bead moving on a wire from A to B is $T = \int dt = \int_A^B \frac{ds}{v}$ where v is speed; we use the fact that PE+KE is constant; assuming the bead is a point of mass m starting from rest we have $\frac{1}{2}mv^2 + mgy = mgy_1$; this is an example of applying a constraint. Then $v = \sqrt{2g(y_1 - y)}; ds = \sqrt{1 + y'^2}$ as before. So we seek $y(x)$ which minimizes $T = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \sqrt{\frac{1+y'^2}{y_1 - y}} dx$; note that unlike above the integrand here depends on $y(x)$ as well as $y'(x)$.

4.2 Stationary points of a func

A smooth func $f(\vec{x})$ of several vars $x_1, \ldots, x_n$ has a local extremum at $\vec{x} = \vec{a}$ if $\nabla f |_{\vec{x}=\vec{a}} = \vec{0}$ (recall ∇f is a vector). Using a Taylor expansion about $\vec{x} = \vec{a}$ we have $f(\vec{x}) = f(\vec{a}) + (\vec{x} - \vec{a}) \cdot \nabla f |_{\vec{x}=\vec{a}} + \frac{1}{2!}(x_i - a_i)(x_j - a_j) \frac{\partial^2 f}{\partial x_i \partial x_j} [+ \ldots]$ (using the summation convention); the second term here is 0 so $f(\vec{x}) - f(\vec{a}) = \frac{1}{2!}(x_i - a_i)(x_j - a_j) \frac{\partial^2 f}{\partial x_i \partial x_j} + \ldots$; specifically in the 2D case where $\vec{a} = (a_1, a_2), \vec{x} = (x, y)$ this is $\frac{(x-a_1)^2}{2!} \frac{\partial^2 f}{\partial x^2} + (x - a_1)(y - a_2) \frac{\partial^2 f}{\partial x \partial y} + \frac{(y-a_2)^2}{2!} \frac{\partial^2 f}{\partial y^2} + \ldots$ (assuming the function is well behaved so $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$); therefore the local behaviour of $f(\vec{x})$ near $\vec{a}$ is determined by the symmetric mat $M_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$, called the Hessian matrix. Since it is symmetric we can diagonalize it w/ real evals $\lambda_1, \ldots, \lambda_n$; wrt axes in which M is diagonal $\vec{x} = (\widetilde{x_1}, \ldots, \widetilde{x_n})^T$, then $f(\vec{x}) - f(\vec{a}) = \frac{\lambda_1}{2!}(\widetilde{x_1} - \widetilde{a_1})^2 + \frac{\lambda_2}{2!}(\widetilde{x_2} - \widetilde{a_2})^2 + \cdots + \frac{\lambda_n}{2!}(\widetilde{x_n} - \widetilde{a_n})^n$ [plus higher order terms]; now the character of the stationary [turning in lectures but this is wrong] point becomes clear:

1. If all the λ_i are > 0, $f(\vec{x}) > f(\vec{a})$ in all directions away from $\vec{a}$ so $\vec{a}$ is a local minimum.
2. If all the λ_i are < 0 $\vec{a}$ is similarly a local maximum
3. If some λ_i are < 0 and some > 0 $\vec{a}$ is a saddle; $f(\vec{x})$ goes up from $f(\vec{a})$ in some directions and down in others
4. If some $\lambda_i = 0$ we have to consider higher order derivatives.

In 2D we have $\det M = \lambda_1 \lambda_2, trM = \lambda_1 + \lambda_2$ so these conditions become

1. Minimum: $\det M > 0, trM > 0$
2. Maximum: $\det M > 0, trM < 0$
3. Saddle: $\det M < 0$
4. Need to consider higher derivs when $\det M = 0$

Examples

$f(x, y) = x^2 + y^2$ has $\nabla f = (2x, 2y) = \vec{0}$ only at $x = y = 0$; $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has $\det M > 0, trM > 0$ so this is a minimum.

$g(x, y) = x^2 - y^2$ has $\nabla f = (2x, 2y) = \vec{0}$ only at $x = y = 0$; $M = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ has $\det M < 0$ so this is a saddle.

Notes

Note that $\nabla f = 0$ only yields the local and not global maxima/minima; the min/max of f in a given region may be at a turning point or on a boundary.

If f is a harmonic function i.e. a sol of Laplace's eqn $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ then $trM = 0$ so if $\det M \neq 0$ then any turning point is a saddle; in a domain D the max and min of and harmonic function f are always on the boundary.

If $\nabla f \neq \vec{0}$ at the point (x, y) then f increases and decreases most rapidly in the direction of ∇f ; more generally as seen in Exs4 the rate of change of f in the direction $\hat{n}$ is $\hat{n} \cdot \nabla f$ so largest when $\hat{n} \parallel \nabla f$.

Examples

Find the stationary point of $f(x, y) = x^3 + y^3 - 3xy$; we have $\nabla f = (3x^2 - 3y, 3y^2 - 3x)$ which is $\vec{0}$ when $y = x = 0$ or $y = x = 1$. $M = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$; at 0 this is $\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ with evals ± 3; $(0, 0)$ is a saddle; we can find the evecs are $(1, -1)$ corresponding to $+3$, so f goes up in this direction; we call this the steepest ascent, and $(1,1)$ corresponding to -3 so f goes down in this direction; we say this is the steepest descent. At $(1,1)$ $M = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ w/ evals $9, 3$ so this is a minimum.

4.3 Stationary values subject to constraints

We are often interested in extremizing some $f(\vec{x})$ subject to the constraint that some $g(\vec{x}) = 0$. For example, say we want to find the smallest circle centred at $(0,0)$ i.e. minimize $f(x,y) = x^2 + y^2$ subject to $g(x,y) = y - \left(x^2 - 1\right) = 0$ (since a point of the circle must lie on the parabola). We can solve this by direct elimination; $f = x^2 + y^2 = x^2 + \left(x^2 - 1\right)^2 = x^4 - x^2 + 1$, then $\frac{df}{dx} = 4x^3 - 2x$, $= 0$ when $x = 0, \pm\frac{1}{\sqrt{2}}$ and $\frac{d^2 f}{dx^2} = 12x^2 - 2$ which is 4 at $x = \pm\frac{1}{\sqrt{2}}$ so these are minima [while $x = 0$ is a maximum]; at these $y = -\frac{1}{2}$ and $f = \frac{3}{4}$ so the min rad. is $\frac{\sqrt{3}}{2}$. However, this technique is difficult for harder problems and does not generalize to higher dimensions.

Lagrange Multipliers

Define a new func $F(x,y,\lambda) = f(x,y) + \lambda g(x,y)$; now extremize this over these three variables; $\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0, \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0, \frac{\partial F}{\partial \lambda} = g = 0$. In our example $2x - 2\lambda x = 0, 2y + \lambda = 0, y - x^2 + 1 = 0$; eliminating λ between the first two eqns $2x - 2x(-2y) = 0$ so $x = 0$ or $y = -\frac{1}{2}$; if $x = 0$ $y = -1$, if $y = -\frac{1}{2}$ $x = \pm\frac{1}{\sqrt{2}}$ (we can also find λ but this is not necessary to solve the original problem). When using LMs it is best not to look at the Hessian of F in $\mathbb{R}^3$, because a constrained maximum of $f(x,y)$ is actually a saddle of $F(x,y,\lambda)$; we just calculate that f is 1 at the first of these possibilities, $\frac{3}{4}$ at the second so $x = \pm\frac{1}{\sqrt{2}}$ are the minima.

More generally, to extremize $f(x_1, \ldots, x_n)$ subject to the k constraints $g_1(x_1, \ldots, x_n) = \cdots = g_k(x_1, \ldots, x_n) = 0$, introduce LMs $\lambda_1, \ldots, \lambda_k$ and define $F(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n) = f + \lambda_1 g_1 + \cdots + \lambda_n g_n$ (all applied to $(x_1, \ldots, x_n)$). No perform an unconstrained extremization of F (wrt all variables) by $\frac{\partial F}{\partial x_i} = 0 \forall 1 \leq i \leq n, \frac{\partial F}{\partial \lambda_i} = 0 \forall 1 \leq i \leq k$, giving $n + k$ eqns; eliminate the λ_i and then solve for $x_1, \ldots, x_n$ (as before, the values of the λ_i are irrelevant to the sol of the actual problem).

For example, say we are building a cuboidical garden shed against a fixed straight vertical wall; we want to minimize the area of shed walls we have to build (i.e. all faces other than the bottom ($z = 0$) and the one against the wall ($y = 0$) subject to fixed volume; say the shed is $x \times y \times z$. Then we want to minimize $f(x,y,z) = xy + xz + 2yz$ subject to $g(x,y,z) = xyz - V = 0$. $n = 3, k = 1, F(x,y,z,\lambda) = xy + xz + 2yz + \lambda(xyz - V)$; $\frac{\partial F}{\partial x} = 0 = y + z + \lambda yz, \frac{\partial F}{\partial y} = 0 = x + 2z + \lambda xz, \frac{\partial F}{\partial z} = 0 = x + 2y + \lambda xy, \frac{\partial F}{\partial \lambda} = 0 = xyz - V$; the second and thifd of these eqns give $(z - y)(2 + \lambda x) = 0$ so $\lambda x = -2$ or $z = y$, but $\lambda x = -2 \Rightarrow x + 2z - 2x = 0 \Rightarrow x = 0$ which gives a contradiction since $V > 0$; $z = y \Rightarrow 2y + \lambda y^2 = 0$; $y \neq 0$ so $\lambda y = -2 \therefore 2y = x$; then $2y^3 = V$ so $y = z = \sqrt[3]{\frac{v}{2}}, x = 2\sqrt[3]{\frac{v}{2}}$ and the minimum surface area is $f = 6\left(\frac{V}{2}\right)^{\frac{2}{3}}$.

4.4 Euler-Lagrange Eqns

This is a simple case but we can understand everything important from it: extremise the functional $s(y) = \int_{x_1}^{x_2} f(x, y, y')\,dx$ where $y(x)$ is a dependent variable, x being the independent variable. We want to know how s depends on our choice of path from x_1 to x_2; consider a specific path $y(x)$ and perturb away from it by a small abount $y(x) \to y(x) + \epsilon\eta(x)$, with $\eta(x_1) = \eta(x_2) = 0$ since the endpoints are fixed. Then $s(y + \epsilon\eta) = \int_{x_1}^{x_2} f(x, y + \epsilon\eta, y' + \epsilon\eta')\,dx$; taking a Taylor expansion this is $\int_{x_1}^{x_2} f(x, y, y')\,dx + \epsilon\int_{x_1}^{x_2} \frac{\partial f}{\partial y}\mid_{x,y,y'} \eta + \frac{\partial f}{\partial y'}\mid_{x,y,y'} \eta' dx + O\left(\epsilon^2\right)$. In order for the path to extremize $s(y)$ we require the "first variation" δs of s to be 0 i.e. $s(y + \epsilon\eta) + s(y) + O\left(\epsilon^2\right)$; this is analogous to finding the stationary points of an ordinary univariate f by $f(x + \epsilon) = f(x) + O\left(\epsilon^2\right)$. So we require $\int_{x_1}^{x_2} \frac{\partial f}{\partial y}\mid_{x,y,y'} \eta + \frac{\partial f}{\partial y'}\mid_{x,y,y'} \eta' dx = 0$; integrating by parts this $=$ $\int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta dx + \left[\frac{\partial f}{\partial y}\eta\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta dx$ and the middle term is 0 since $\eta = 0$ at the endpoints. So, since this must be the case $\forall\eta$, we require (this is a necessary condition for y to extremize s, and as we may see later not a sufficient one) that the E-L eqns are satisfied: $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} \equiv 0$. Note the distinction between partial and total derivatives; $\frac{\partial f}{\partial y}$ is taken holding x, y' constant and treating x, y, y' as IV, wheras $\frac{d}{dx}$ is a full x derivative, remembering that $y = y(x)$.

Example

$f(x, y, y') = x\left(y'^2 - y^2\right)$; $\frac{\partial f}{\partial y} = -2xy$, $\frac{\partial f}{\partial y'} = 2xy'$, $\frac{\partial f}{\partial x} = y'^2 - y^2$, but $\frac{df}{dx}$ is by the chain rule $\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''$; in this specific case this is $y'^2 - y^2 - 2xyy' + 2xy'y''$.

There are two very important simplified cases; if y is absent from f i.e. f only depends directly on x, y' then $\frac{\partial f}{\partial y} = 0$ and the eqn becomes $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ i.e. taking a first integral $\frac{\partial f}{\partial y'} = c$ constant.

For the second case consider $\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right)$; using the chain rule this is $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' - y''\frac{\partial f}{\partial y'} - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$; by the E-L eqn this last term is $-y'\frac{\partial f}{\partial y}$ and this becomes $\frac{\partial f}{\partial x}$. So if f does not depend on x explicitly (so $\frac{\partial f}{\partial x} = 0$) $\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$, or integrating $f - y'\frac{\partial f}{\partial y'} = c$ constant.

Now we return to the motivating examples to solve them:

4.5 Examples of use of E-L Eqns

4.5.1 Shortest distance between 2 points on a plane

$s = \int_{x_1}^{x_2} \left(1+y'^2\right)^{\frac{1}{2}} dx$; $f(x,y,z) = \sqrt{1+y'^2}$; there is no direct y dependence so $\frac{\partial f}{\partial y'} = c = \frac{2y'}{2\sqrt{1+y'^2}} \Rightarrow y'^2 = c^2\left(1+y'^2\right) \therefore \left(c^2-1\right)y'^2 = c^2$ [surely $-c^2$] so $y' = k$ constant $\left(\sqrt{\frac{c^2}{c^2-1}}\right)$ so $y = kx + d$, w/ k, d determined by the endpoints: $y = y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$; this is the eqn of a straight line.

4.5.2 Shortest distance between 2 pts on a sphere

As above, $S = \int_{\theta_1}^{\theta_2} \sqrt{1+\sin^2\theta\phi'^2}d\theta$; $f(\theta,\phi,\phi')$ has no direct ϕ dependence so $\frac{\partial f}{\partial \phi'} = k = \frac{\sin^2\theta\phi'}{\sqrt{1+\sin^2\theta\phi'^2}}$ so $\phi'^2 = \frac{k^2}{\sin^2\theta(\sin^2\theta-k^2)}$; using the substitution $\cot\theta = t$ the reader may verify this gives $\frac{\pm\sqrt{1-k^2}}{k}\cos(\phi-\phi_1) = \cot\theta$, the eqn of a geodesic or great circle.

4.5.3 Brachistochrone

As above $T = \frac{1}{\sqrt{2g}}\int_0^{x_2}\frac{\sqrt{1+y'^2}}{\sqrt{-y}}dx$ (taking the start point to be $(0,0)$; $\sqrt{-y}$ is OK since the bead can never rise above 0) i.e. $f(x,y,y') = \frac{\sqrt{1+y'^2}}{\sqrt{-y}}$; there is no explicit x dependence here so $f - y'\frac{\partial f}{\partial y'} = \frac{1}{k}$ (a general constant), i.e. $\frac{1}{k} = \frac{\sqrt{1+y'^2}}{\sqrt{-y}} - \frac{y'}{\sqrt{-y}}\left(\frac{2y'}{2\sqrt{1+y'^2}}\right) = \frac{1}{\sqrt{-y(1+y'^2)}}\left(1+y'^2-y'^2\right)$ so $\sqrt{-y\left(1+y'^2\right)} = k$, $-y\left(1+y'^2\right) = k^2$ so [$\frac{\partial y}{\partial x}$ in lectures] $\frac{dy}{dx} = \pm\sqrt{\frac{k^2}{-y}-1} = \pm\sqrt{\frac{-k^2-y}{y}}$ so $x = \pm\int\sqrt{\frac{-y}{k^2+y^2}}dy$. Substitute $y = -k^2\sin^2\frac{\theta}{2} = \frac{k^2}{2}(\cos\theta-1)$ so $dy = -k^2\cos\frac{\theta}{2}\sin\frac{\theta}{2}d\theta$ and $x = \mp\int\frac{k\sin\frac{\theta}{2}}{k\sqrt{1-\sin^2\frac{\theta}{2}}}k^2\cos\frac{\theta}{2}\sin\frac{\theta}{2}d\theta = \mp\frac{k^2}{2}\int 1-\cos\theta d\theta = \mp\frac{k^2}{2}(\theta-\sin\theta)+c$. When $x=0, y=0 \Rightarrow \theta = 0 \Rightarrow c = 0$. We are interested in $+ve$ x for $-ve$ y so we take the $+$ sign; $x = \frac{k^2}{2}(\theta-\sin\theta), y = -k^2\sin^2\frac{\theta}{2} = \frac{k^2}{2}(\cos\theta-1)$. These are the eqns fro a cycloid, the curve traced out by a point on [the circumference of] a wheel of radius $\frac{k^2}{2}$ as it rolls along the x axis (from below). The minimum occurs when $\theta = \pi$; at $\theta = \theta_2$ $y_2 = -\frac{k^2}{2}\sin^2\frac{\theta}{2}$; for $\theta_2 > \pi$ the min transit time route passes through a point below y_2, which is a counterintuitive result.

When $\theta = 2\pi$ $y_2 = 0 = y_1$; see Exs4Q4 [??]. The constant k^2 can be determined from $y = 0$ when $x = 0$ and $y = y_2$ when $x = x_2$.

4.5.4 Minimum surface area

Consider circular wires in the planes $x = \pm L$ w/ respective radii a, b centred on the x axis. We look for the axisymmetric surface joining them with min. surface area [and this will be the surface of a soap film between them]. So we want to minimize $A = \int_{-L}^{L} 2\pi y ds = \int_{-L}^{L} 2\pi y\sqrt{1+y'^2}dx$; $f(x, y, y') = 2\pi y\sqrt{1+y'^2}$; there is no direct x dependence so $f - y'\frac{\partial f}{\partial y'} = c = y\sqrt{1+y'^2} - \frac{\frac{1}{2}\times 2yy'y'}{\sqrt{1+y'^2}}$ so $\frac{y(1+y'^2-y'^2)}{\sqrt{1+y'^2}} = k$ i.e. $\frac{y}{\sqrt{1+y'^2}} = k$. So $y' = \pm\sqrt{\frac{y^2}{k^2}-1}$ meaning $x = \pm\int \frac{1}{\sqrt{\frac{y^2}{k^2}-1}}dy$; using $y = k\cosh\theta$ we have $dy = k\sinh\theta d\theta, x = \pm k\theta - c, \frac{y}{k} = \cosh\frac{x+c}{k}$; c, k are determined by the BCs. If $a = b$ we have $c = 0$ by symmetry so $\frac{a}{k} = \cosh\frac{L}{k}$; we must solve this implicitly to find k. Put $z = \frac{L}{k}$ and then we must solve $\frac{az}{L} = \cosh z$; graphing both sides we see there is no sol for $\frac{a}{L}$ small. This makes sense from the physics - no bubble can form between small rings far apart. For a critical value, there is one sol, a min, and for larger values there are two sols, 1 min and 1 saddle. The critical case has $\cosh z$ and $\frac{a_c z}{L}$ tangent at some z_c so $\cosh z_c = \frac{a_c z_c}{L}, \sinh z_c = \frac{a_c}{L}$ and so $1 = \cosh^2 z_c - \sinh^2 z_c = \frac{a_c^2 z_c^2}{L^2} - \frac{a_c^2}{L^2}$ so $z_c = \sqrt{\frac{L^2}{a_c^2}+1}$ and $\sinh\sqrt{\frac{L^2}{a_c^2}+1} = \frac{a_c}{L}$; numerically we find $\frac{a_c}{L} \approx 1.5089$ so the rings must be quite close relative to their size for a bubble to form.

4.6 Two examples from Physics

4.6.1 Fermat's Principle

Light travels along the path between two points which requires the least time; work in 2D and say light (or sound) travels along the path or ray $y = y(x)$ with speed $c(x, y)$. The time of travel is $T = \int_{x_1}^{x_2} \frac{1}{c}ds = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{c(x,y)}dx$. If c is a func of x alone then $f(x, y, y') = \frac{\sqrt{1+y'^2}}{c(x)}$ with no direct y dependence so $\frac{\partial y}{\partial y'} = k = \frac{y'}{c\sqrt{1+y'^2}}$. Consider the angle made by a ray launched from x_1, y_1; the angle θ of the direction of the ray above the horizontal always has $\tan\theta = y'$, so $k = \frac{\frac{\sin\theta}{\cos\theta}}{c\sqrt{1+\frac{\sin^2\theta}{\cos^2\theta}}}$ i.e. $\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$ at any two points - this is Snell's law.

4.6.2 Dynamics using Hamilton's Principle

We define the Lagrangian by $L = T - V$ (yes, $-$) where T is the KE of a system, and V the PE. Action is defined by $\int L dt$. Hamilton's principle is that motion is such as to minimize the action. In general $L = L(t, y, \dot{y})$ so the E-L eqns become $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$. Consider a particle in 1D: $T = \frac{1}{2}m\dot{y}^2, V = V(y)$, then $\frac{d}{dt}(m\dot{y}) + V'(y)$ so $m\ddot{y} = -V'$ just as in Newton's second law. More generally if $V = V(y), T = T(y, \dot{y})$ with no explicit time dependence (which is intuitively true for the physical situation) we have $L - \dot{y}\frac{\partial L}{\partial \dot{y}} = c$ i.e. $T - V - \dot{y}\frac{\partial T}{\partial \dot{y}} = c$ so if T is quadratic in $\dot{y}$, as for velocity e.g. $T = \frac{1}{2}m(y)\dot{y}^2$ then $\dot{y}\frac{\partial T}{\partial \dot{y}} = 2T$ so $T - V - 2T = c$ i.e. $T + V = c$, which is of course conservation of energy.

4.7 Extensions of the E-L Eqns

4.7.1 Higher derivatives

To extremize $I(y) = \int_a^b f\left(x, y, y', \ldots, y^{(n)}\right) dx$ we proceed as above; introduce a pertubation $y \to y + \epsilon\eta$ with $\eta(x), \eta'(x), \ldots, \eta^{(n-1)}(x) = 0$ at $x = a, b$. Integrating by parts repeatedly on the perturbed form of $I(y)$ we obtain the necessary condition $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^{(n)}}\right) = 0$; see ExsQ8 for more on this subject.

4.7.2 Several DVs

To extremize $I(\vec{y}) = \int_a^x f(x, y_1, y_2, \ldots, y_n, y_1', \ldots, y_n') dx$ (where the y_i are all functions of x) we of corse perturb by $y_i \to y_i + \epsilon\eta_i \forall 1 \le i \le n$ w/ $\eta_i(a) = 0 = \eta_i(b) \forall i$; $I(\vec{y} + \epsilon\vec{\eta}) - I(\vec{y}) = \epsilon \int_a^b \sum_{i=1}^n \frac{\partial f}{\partial y_i}\eta_i + \sum_{i=1}^n \frac{\partial f}{\partial y_i'}\eta_i' dx + O\left(\epsilon^2\right) = \epsilon \int_a^b \sum_{i=1}^n \eta_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right)\right) dx$; the η_i are not linearly dependent so to extremise fully we must have $\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \forall i$. If $\frac{\partial f}{\partial x} = 0$ it is possible to show that $\sum_{i=1}^n y_i' \frac{\partial f}{\partial y_i'} - f =$ some constant c, a single condition, since $\frac{d}{dx}\left(f - \sum_i y_i' \frac{\partial f}{\partial y_i'}\right) = 0$ by the same argument as before.

An example of this is light in an inhomogenous medium in 3D; the speed of light is $c(x, y, z)$ along a path given by $y(x), z(x)$; we want to minimize the time of travel $T = \int \frac{1}{c} ds = \int \frac{\sqrt{1+y'^2+z'^2}}{c(x,y,z)} dx = \int f(x, y, z, y', z') dx$; $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$, $\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{z y'}\right) = 0$, but if c is a function of y, z alone then $\frac{\partial f}{\partial x} = 0$ so this becoes $y' \frac{\partial f}{\partial y'} + z' \frac{\partial y}{\partial z'} - f = k$ ($\frac{\partial c}{\partial y'} = \frac{\partial c}{\partial z'} = 0$); $\frac{\frac{1}{2\sqrt{1+y'^2+z'^2}} y' 2y'}{c} + \frac{z'^2}{c\sqrt{1+y'^2+z'^2}} - \frac{\sqrt{1+y'^2+z'^2}}{c} = k$ so $k = \frac{-1-y'^2-z'^2+y'^2+z'^2}{c\sqrt{1+y'^2+z'^2}}$ or $-k = \frac{1}{c\sqrt{1+y'^2+z'^2}}$; if we let θ be the angle of the light ray away from the x axis we have $\sin\theta = \frac{1}{\sqrt{1+y'^2+z'^2}}$, so again this is Snell's law.

Another example is the motion of a particle is 2D under a central force; we have DVs $r(t), \theta(t)$ and $L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)$; $\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$; $\frac{\partial L}{\partial \theta} = 0$ so this $\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = c$ constant i.e. $mr^2\dot{\theta} = c$; this is conservation of angular momentum $r \times \vec{p}$; also $\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$ so $m\ddot{r} = mr\dot{\theta}^2 - V'$; this is Newton's law of motion. $\frac{\partial L}{\partial t} = 0$ so $\dot{r}\frac{\partial L}{\partial \dot{L}} + \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = c$ constant; this gives conservation of energy $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) = c$.

4.7.3 Several IVs

To extremize $J(\phi) = \int_{\mathcal{D}} f(x, y, z, \phi, \phi_x \phi_y, \phi_z)\, dxdydz$ by variations of ϕ we introduce a pertubation $\phi \to \phi + \epsilon\eta(x, y, z)$ w/ $\eta = 0$ on $\partial\mathcal{D}$; $J(\phi + \epsilon\eta) - J(\phi) = \epsilon \int_{\mathcal{D}} \eta \frac{\partial f}{\partial \phi} + \frac{\partial f}{\partial \phi_x}\eta_x + \frac{\partial f}{\partial \phi_y}\eta_y + \frac{\partial f}{\partial z}\eta_z dxdydz$; we define $\nabla\eta = (\eta_x, \eta_y, \eta_z)$ as usual, and $\frac{\partial f}{\partial(\nabla\phi)} = \left(\frac{\partial f}{\partial \phi_x}, \frac{\partial f}{\partial \phi_y}, \frac{\partial f}{\partial \phi_z}\right)$, then this is $\epsilon \int_{\mathcal{D}} \eta \frac{\partial f}{\partial \phi} + \frac{\partial f}{\partial(\nabla\phi)} \cdot \nabla\eta dxdydz$ which by the prod rule is $\epsilon \int_{\mathcal{D}} \eta \frac{\partial f}{\partial \phi} + \epsilon \int_{\mathcal{D}} \nabla \cdot \left(\frac{\partial f}{\partial(\nabla\phi)}\eta\right) dxdydz - \epsilon \int_{\mathcal{D}} \left(\nabla \cdot \frac{\partial f}{\partial(\nabla\phi)}\right) \eta dxdydz = \epsilon \int_{\mathcal{D}} \eta \frac{\partial f}{\partial \phi} dxdydz + \epsilon \int_{\partial\mathcal{D}} \eta \frac{\partial f}{\partial(\nabla\phi)} \cdot dS - \epsilon \int_{\mathcal{D}} \left(\nabla \cdot \frac{\partial f}{\partial(\nabla\phi)}\right) \eta dxdydz$; the second term is 0 since $\eta = 0$ on $\partial\mathcal{D}$ so this is $\epsilon \int_{\mathcal{D}} \eta \frac{\partial f}{\partial \phi} dxdydz - \epsilon \int_{\mathcal{D}} \left(\nabla \cdot \frac{\partial f}{\partial(\nabla\phi)}\right) \eta dxdydz$; this must be the case $\forall\eta$ so $\frac{\partial f}{\partial \phi} - \nabla \cdot \left(\frac{\partial f}{\partial(\nabla\phi)}\right) = 0$.

Example

Soap film in 3D; say the surface has deflection $\phi(x, y)$ so energy is $\iint \frac{1}{2}|\nabla\phi|^2 dxdy$; we want to minimise this energy so $f = \frac{1}{2}|\nabla\phi|^2$ in the above; $\frac{\partial f}{\partial \phi} = 0$, $\frac{\partial f}{\partial(\nabla\phi)} = \nabla\phi$ so the condition gives $\nabla^2\phi = 0$ (Note: it is very important to remember that a sol of the EL eqns is not necessarily an extremum; satisfying them is a necessary but not sufficient condition. The simplest example of this is that when minimizing the distance between two points N, L on a sphere both arcs of the [great?] circle through them are sols of the EL eqns even though the longer one is not an extremum.

4.8 Integral constraints

A typical problem of this sort is to extremize $I[y] = \int_a^b f(x, y, y')\, dx$ subject to a constraint $\int_a^b G(x, y, y')\, dx = k$ constant. We use a Lagrange multiplier as before; extremize $J[y] = \int_a^b f dx + \lambda\left(\int_a^b G dx - k\right)$; k is a constant so plays no role in the variations so we can simply extremize $\int_a^b f + \lambda G dx$. The EL eqns are $\frac{d}{dx}\left(\frac{\partial}{\partial y'}(f + \lambda G)\right) - \frac{\partial}{\partial y}(f + \lambda G) = 0$; an example problem is the static chain (by which we mean massive string) hanging under gravity, w/ mass per unit length μ; say hanging between $(-a, 0)$ and (a, y_2). We want to minimize the PE $\int_{-a}^a \mu g y\sqrt{1 + y'^2} dx = \int_{-a}^a f dx$ subject to fixed length $2L = \int_{-a}^a \sqrt{1 + y'^2} dx = \int_{-a}^a G dx$; μ, g, L are constants. $\frac{\partial}{\partial x}(f + \lambda G) = 0$ so $f + \lambda G - y'\frac{\partial}{\partial y'}(f + \lambda G) = k$ constant; we find $y' = \pm\sqrt{\left(\frac{\mu g y + \lambda}{k}\right)^2 - 1}$ $\therefore$ $y = \frac{k}{\mu g}\cosh\left(\mu g \frac{x + L}{k}\right) - \frac{\lambda}{\mu g} c$ with c, k, λ determined using the endpoint condition, e.g. if $y_2 = 0$ then $c = 0$ by symmetry. When $x = \pm a$ $y = 0$ so $\lambda = k\cosh\frac{\mu g a}{k}$. Finally we apply the constraint: $\int_{-a}^a \sqrt{1 + y'^2} dx = 2L$ so $\frac{k}{\mu g}\sinh\frac{\mu g a}{k} = L$. Provided $L > a$ we can find one realistic sol; it is called the catenary.

Example

Motion of a pendulum: say we have a pendulum of mass m constained by a (massless inextensible) wire of length l to swing in an arc; use 2D polars with θ taken to be the [signed] angle away from equilibrium, then the constraint is $r-l=0$. The Lagrangian $L=T-V=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+mgr\cos\theta$ (defining the PE $V=0$ when $\theta=\frac{\pi}{2}$); here $G=r$. Conlider $L+\lambda\left(G-l\right)$ or indeed $L+\lambda G$; $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=\lambda$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$ [check lots] so $\frac{d}{dt}\left(m\dot{r}\right)-mr\dot{\theta}^2-mg\cos\theta=\lambda$, $\frac{d}{dt}\left(mr^2\dot{\theta}\right)+mgr\sin\theta=0$. Applying the constraint $r=l$ we have $\dot{r}=0$ so $ml\dot{\theta}^2+mg\cos\theta=-\lambda$. This is the statement of the balance of the forces; λG is the term applying the constraint $r=l$ which physically represents the tension in the wire, which is what balances forces so the wire doesn't stretch. We then have $ml^2\ddot{\theta}+mgl\sin\theta=0$ which is the general form(not merely an approximation for small angles) of the eqn of motion for a pendulum, e.g. for $\theta << l$ $\sin\theta\approx\theta$ so $\ddot{\theta}\approx-\frac{g}{l}\theta$ and we have SHM.

Of course, as with much of this course we could have solved this particular problem much more easily directly. However, the power of this method is that it applies in general, including to situations where ever writing down the forces to resolve in order to solve the problem would be almost impossible.