Graph Theory

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This course is about graphs. The good thing about studying graphs is that they are discrete, simple objects. They have no hidden structure - if we talk about e.g. a group of order p, it takes clever maths to work out this is actually cyclic, wheras with a graph, all of the structure is obvious. There are almost no prerequisites from previous courses; this could almost have been a IB course, and whenever we state a theorem it is clear what it means. The downside is that there is no obvious way to start the proof of a theorem - we can't take a basis, we can't take $\epsilon > 0$, we have to think of something clever.

The only slight link to any previous course is that if you liked the optimization course, you should like this.

A theorem referred to by a single number (e.g. "Theorem 4") refers to the theorem of that number in the current section (e.g. theorem 2.4 if this was in section 2)

0.1 Outline

There are seven chapters: an introduction, conectivity and matching (the most IA/IB-like chapter), extremal problems (the heart of graph theory, and quite hard; however, though many of the theorms look like their proofs must be really ugly, these proofs then turn out to be quite elegant), colourings, Ramsey theory, Random Graphs (the other hard chapter; this section is useful for proving that certain types of graph exist, by proving that a randomly generated graph has a nonzero probability of having the property we want) and algebraic methods (which are good for proving taht certain types of graph don't exist).

0.2 Prerequisites

There are essentially none; the reader should understand the concepts of $\binom{n}{r}$, $\mathbb{R}^2$, eigenvalues, and the mean and variance of a probability distribution; it is nice but not essential to have seen "max flow min cut".

0.3 Books

A book should not be necessary for this course; what is covered here is what is examinable. There is one good book for this course (Bollboa's Modern Graph Theory (also sometimes seen under its old title Graph Theory - An Introductory Course), which gives you the right "feel" for which things are important, easy, or hard), one decent book (Dieltel's Graph Theory, which is less good at this "feel" aspect, but a good book), and many bad books (the best of which is Bondy and Murty's Graph Theory with Applications).

1 Introduction

A graph is a pair (V, E) where V is a set and $E \subset V^{(2)} := \{\{x, y\} : x, y \in V\}$ the set of unordered pars from V; we will take V finite unless otherwise stated. Note that this definition means there are no "loops" (edges from a vertex to itself), no multiple edges between the same pair of vertices, and no directed edges. We call $V(G) := V$ the vertex set of G and $E(G) := E$ the edge set of G. The order of G is $|G| = |V(G)|$; the size of G is $e(G) = |E(G)|$. We often write $x \in G$ to mean $x \in V(G)$.

Some examples are the empty graph E_n for which $V = \{x_1, \ldots, x_n\}, E = \emptyset$, the complete graph K_n for which $V = \{x_1, \ldots, x_n\}, E = V^{(2)}$, the path P_n of length n, for which $V = \{x_1, \ldots, x_{n+1}\}, E = \{x_i x_{i+1} : 1 \leq i \leq n\}$ (beware that some books take P_n to have n vertices), and the cycle C_n of length $n \geq 3$, for which $V = \{x_1, \ldots, x_n\}, E = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\}$ [of course we work with the familiar "drawing" versions of graphs, not these cumbersome sets, but the limitations of typesetting engulf me].

Graphs $G = (V, E), H = (V', H')$ are isomorphic if there exists as bijection $f : V \to V'$ such that $xy \in E \Leftrightarrow f(x)f(y) \in E'$. We say H is a subgraph of G if $V \subset V'$ and $E \subset E'$, e.g. C_n is a subgraph of K_n. For $xy \in E$ we write $G - xy$ for the graph $(V, E - \{xy\})$; similarly $G + xy$.

If $xy \in E$ we say x and y are adjacent or neighbours; the neighbourhood of x is $\Gamma(x) = \{y \in V : xy \in E\}$; the degree of x is $d(x) = |\Gamma(x)|$.

If $V(G) = \{x_1, \ldots, x_n\}$ then the degree sequence of G is $d(x_1), \ldots, d(x_n)$; the maximum degree $\Delta(G)$ and minimum degree $\delta(G)$ are defined in the obvious way. If $d(x) = k \forall x \in G$ we say G is regular of degree k, e.g. C_n is 2-regular and K_n is $(n-1)$-regular.

In a graph G, an $x - y$ path is a set $x = x_1, \ldots, x_k = y$ (with $k \geq 1$) of distinct vertices of G such that $x_i x_{i+1} \in E \forall 1 \leq i \leq k-1$; it has length $k - 1$. We say G is connected if $\forall x, y \in V \exists x - y$ path in G; informally, G is "in one piece". It is important to remember that not all graphs are connected.

We write $x \sim y$ if $\exists$ an $x - y$ path; this is an equivalence relation, though note that we cannot simply concatenate paths as this might give a path which crossed itself, which would not be allowed (instead we e.g. take the part of the $x - y$ path up until it first reaches a node in the $y - z$ path, and then the remainder of the $y - z$ path, as an $x - z$ path for transitivity). A walk is a sequence $x_1, \ldots, x_k$ such that $x_i x_{i+1} \in E \forall 1 \leq i \leq k-1$; clearly G has an $x - y$ walk iff it has an $x - y$ path.

Trees

These are an important concept; many important techniques first become aparrent when we apply them to trees, and later on we will often use trees as the base cases for induction.

A graph is acyclic if it contains no cycle. A tree is a connected acyclic graph.

1.1 Proposition

For G a graph, TFAE:

G is a trees

G is minimal connected (i.e. G is connected but $G - xy$ is disconnected $\forall xy \in E$

G is maximal acyclic (which means the obvious thing)

For a) implies b), G is connected by definition; were $G - xy$ connected for some xy then we have an $x - y$ path P, but then $P + yx$ is a cycle in G, contradiction.

For b) implies a), G is connected; if it has a cycle C then choose $xy \in C$ and $G - xy$ is connected, because for any $a, b \in V$ we have an $a - b$ path in V, but if it uses the edge xy we can replace this with $C - xy$ and have an $a - b$ walk in $G - xy$, contradiction.

For a) implies c), we have an $x - y$ path P in G so $G + xy$ contains the cycle $P + yx$.

For c) implies a), if G is not connected then take x, y in different components of G, then $G + xy$ remains acyclic, contradiction.

A vertex $x \in$ a tree T is called a leaf or end vertex if $d(x) = 1$.

1.2 Proposition

Any tree T with $|T| \geq 2$ contains a leaf: let $P = x_1, \ldots x_k$ be a longest path in T Then $\Gamma(x_k) \subset P$ by maximality of P, but $\Gamma(x_k) \cap P = \{x_{k-1}\}$ as T is acyclic, so $d(x_k) = 1$. This "longest path" technique is useful when we want to take a vertex which is intuitively "far away from the centre" of the graph; also note taht the proof actually shows we have at least two leaves.

An alternative proof is that if T has no leaf we will "go for a walk"; choose $x_1x_2 \in E$, then repeatedly choose $x_{k+1} \in \Gamma(x_k) \setminus \{x_{k-1}\}$; this must repeat since T is finite, so we have a cycle.

For a graph G with $W \subset V$ we write $G[V]$ for the subgraph spanned by W, with vertex set W and edge set $E(G) \cap W^{(2)}$. For $x \in V$ we write $G - x$ for $G[V \setminus \{x\}]$.

1.3 Proposition

For T a tree on $n \geq 1$ vertices, $e(T) = n - 1$: we induct on n, the $n = 1$ case is trivial and for T on $n \geq 2$ vertices we let x be a leaf, then $T - x$ is a tree on $n - 1$ vertices so $e(T - x) = n - 2$ by the induction hypothesis, so $e(T) = e(T - x) + 1 = n - 1$.

Many books say that all proofs of this proposition are inductive, and indeed it is hard to see how we could arrive at a number like $n-1$ without using induction; however, we can do a noninductive proof using the idea of a spanning tree; a spanning tree T of a graph G is a subgraph of G with $V(T) = V(G)$ which is a tree. Clearly every connected G has a spanning tree, by removing edges until it is minima connected and applying proposition 1.1. We can then proove the above proposition by showing that any connected G has a spanning tree T of $n - 1$ edges; then we are done since if G is a tree then the only spanning tree of G is G itself (e.g. by minimal connectedness):

For $x, y \in G$ define the distance from x to y $d(x, y)$ is the minimal length of any $x - y$ path. Now to construct our spanning tree we fix $x_0 \in G$, then for each other $x \in V$ we let a be a shortest $x - x_0$ path; say it is $xx' \ldots x_0$. Then let T consist of all of the xx'; this is certainly $n - 1$ edges, and T is connected: for any x we take $xx'x'' \ldots x_0$, and this is a path to x_0. T is acyclic since we note that each vertex has only one edge connecting it to a vertex closer (or even equally close) to x_0 (that is, there is precisely one $y \in \Gamma(x)$ with $d(x_0, y) \leq d(x_0, x)$. So, if T had a cycle C, choose $x \in C$ of maximum distance from x_0, say $d(x, x_0) = k$, but then both neighbours of x (in C) are at distance $\leq k$ from x_0, a contradiction.

Of course, there is an inductive step therein, but it is deeply buried: we only know the path $xx'x'' \ldots$ reaches x_0 by induction.

A forest is an acyclic graph; thus G is a forest iff every component of G is a tree. For G connected, $xy \in E$ is a bridge if $G - xy$ is disconnected; thus a connected graph G is a tree iff every edge is a bridge. For G connected, $x \in G$ is a cutvertex if $G - x$ is disconnected; clearly if G has a bridge then (for $|G| > 2$) it has a cutvertex, but the converse is false.

1.4 Bipartite Graphs

A graph G is bipartite on vertex classes V_1, V_2 if V_1, V_2 partition V (i.e. $V_1 \cup V_2 = V, V_1 \cap V_2 = \emptyset$) and $E(G) \subset \{xy : x \in V_1, y \in V_2\}$ (i.e. there are no edges inside V_1 or V_2. For example, a path is bipartite by alternating its vertices between V_1 and V_2 (we shall see below that all trees are bipartite); we define the complete bipartite graph $K_{n,m}$ by $|V_1| = n, |V_2| = m, E = \{xy : x \in V_1, y \in V_2\}$.

Proposition: G is bipartite iff it has no odd cycle. Define a circuit in G isa closed walk, i.e. a walk $x_1, \ldots x_l$ where $x_l = x_1$. Note that if G has an odd circuit then it has an odd cycle; if $x_1 \ldots x_k x_1$ is an odd circuit and $x_i = x_j$, wlog $1 \leq i < j \leq k$, then one of $x_i x_{i+1} \ldots x_j$ and $x_j x_{j+1} \ldots x_k x_1 \ldots x_i$ is odd, then induct on k. Then for the forward implication the vertices of any cycle must alternate between V_1 and V_2, for the reverse wlog take G connected (since if each component of G is bipartite, so is G); fix $x_1 \in G$ and put $V_1 = \{x \in G : d(x, x_1) \text{ even}\}, V_2 = \{x \in G : d(x, x_1) \text{ odd}\}$. Now if we had $x, y \in V_1$ or $x, y \in V_2$ with $xy \in E$ then xy together with the shortest paths from x_1 to x, y forms an odd circuit.

1.5 Planar Graphs

A graph G is planar if it can be drawn in the plane without crossing edges; a plane graph is such a drawing. Rigorously, for $x, y \in \mathbb{R}^2, x \neq y$ a polygonal arc from x to y is a finite union of (closed) straight line segments $\overline{x_1 x_2} \cup \overline{x_2 x_3} \cup \cdots \cup \overline{x_{k-1} x_k}$ with $x_1 = x, x_k = y$ that are disjoint except for $\overline{x_i x_{i+1}} \cap \overline{x_{i+1} x_{i+2}} = \{x_{i+1}\}$. For G a graph with vertex set $\{V_1, \ldots, V_n\}$ a drawing of G consists of distinct points $x_1, \ldots, x_n \in \mathbb{R}^2$ together with a polygonal arc P_{ij} between x_i, x_j for each $V_i V_j \in E$ such that $P_{ij} \cap P_{kl} = \emptyset$ for i, j, k, l distinct and $P_{ij} \cap P_{ik} = \{x_i\}$. For $x, y \in \mathbb{R}^2 \setminus G$ (by which we really mean a drawing of G) we write $x \sim y$ if $\exists$ a polygonal arc in $\mathbb{R}^2 \setminus G$ from x to y; the components (equivalence classes) of this are called the faces. The boundary or a face consists of $G \cap$ its closure; note that it need not be a cycle or even connected; also the faces on each side of an edge are not necessarily distinct.

We'll assume various facts about $\mathbb{R}^2$ without proof; for example, a cycle has two faces, and the boundary of a face consists of vertices and (whole) edges. These facts are all obvious by induction on the total number of line segments, since we insist that our edges are polygonal arcs.

Remarks: Every tree is planar, with exactly one face; induction va removing a leaf.

The following theorem is not as obvious as it might seem, since we can have distinct drawings of the same graph; consider drawing a hexagonal 6-cycle with vertices at the left, right, top left, top right, bottom left, bottom right, then connecting top left to bottom left and top right to bottom right; we can do this with both edges outside the hexagon or one outside and one inside, and these are genuinely different drawings as e.g. the first contains a face with 6 edges and the second does not. However, in fact the number of faces is fixed:

1.6 Theorem: Euler's Formula

For G a connected plane graph with n [¿0] vertices, m edges and f faces then $n-m+f=2$; note this is not true if G is not connected, e.g. E_n: If G has no cycles then G is a tree so $m=n-1$ and $f=1$ so $n-m+f=2$ as required; otherwise G has a cycle so choose an edge e in this cycle; then $G-e$ is connected and has n vertices, $m-1$ edges and $f-1$ faces (really, because C divides the plane into two), so we have the result by induction. This theorem is great for showing that many graphs are not planar.

Theorem: For G a plane graph with $n \geq 3$ vertices and m edges, $m \leq 3n-6$; this is a very small bound since a general graph will have $O(n^2)$ edges. Note that this is the best possible bound; if we draw a line with $n-2$ points on it, one point a above the line at the left end and one point b below at the left end, and join each of a and b to each of the other vertices this has $3n-6$ edges. We wlog take G connected (by adding edges if necessary), then $n-m+f=2$, but if we sum over faces F the number of edges in the boundary of F the result is $\geq 3f$ as each face has at least 3 edges in its boundary (taking $m \geq 3$), but the result is $\leq 2m$ as each edge is counted at most twice, so $3f \leq 2m$ i.e. $f \leq \frac{2m}{3}$ so $n-m+\frac{2m}{3} \geq 2$ and $\frac{m}{3} \leq n-2$ as required.

Corollary: K_5 is not planar, since $n=5, m=10$ but $10 \nleq 3 \times 5 - 6$. Hence any graph containing K_5 is non-planar; in fact we have more; if we define a <u>subdivision</u> of a graph G to be a graph obtained by replacing edges of G with (disjoint) paths, then any graph containing a subdivision of K_5 is not planar, as otherwise K_5 itself would be planar.

Proposition: $K_{3,3}$ is not planar; this is a well known puzzle; note that $m \leq 3n-6$ for this graph. However, the graph is "triangle-free"; that is none of its faces can ever be triangles, so if drawn in the plane we must have ≥ 4 edges on the boundary of each face; substituting and repeating the above we have $m \leq 2(n-2)$ which is false for this graph. More generally, if we define the <u>girth</u> of a graph to be the length of its shortest cycle (or 0 if there is no cycle) then if G is planar of girth $\geq g$ then $m \leq \max(\frac{g}{g-2}(n-2), n-1)$ (the $n-1$ is "silly" but necessary to make the result valid for very small graphs).

Corollary: If G contains a subdivision of K_5 or $K_{3,3}$ then G is not planar. This leads to <u>Kuratovsky's Theorem</u>, which states that the converse is true; if G does not contain any subdivision of K_5 or $K_{3,3}$ then it is planar. The proof of this is not hard, but ugly; it would take at least 2 lectures to cover all the cases of what can go wrong when attempting to draw G. However, the result can be used without proof to satisfy us that if G does not contain a subdivision of K_5 or $K_{3,3}$ then we will always be able to draw it.

2 Connectivity and Matchings

Matchings are the most important thing we can do with bipartite graphs; in one sense they are what they were invented for, and they have many real applications if you're of the perverse persuation that cares about such things. Connectivity is a notion of "how connected" a graph is; it is normal for this to be hard to understand.

For G a bipartite graph with vertex classes X, Y a matching from X to Y is a set $\{xx^|prime : x \in X\}$ of edges of G such that $x \mapsto x'$ is injective, or equivalently a set of $|X|$ independent edges (i.e. edges with no vertices in common). When do these exist?

We use the "matchmaker" terminology: X are boys, Y are girs, xy is an edge if x knows y; then the problem becomes pairing up each boy with a girl he knows [this is reversed from the usual way of labelling the sets].

Clearly we cannot always find a matching; if $d(x) = 0$ for some $x \in X$ then we will not be able to match x; similarly if $\Gamma(x_1) = \Gamma(x_2) = \{y\}$ then we cannot have a maching. For $A \subset X$ write $\Gamma(A)$ for $\bigcup_{x \in X} \Gamma(x)$, then it is clearly a necessary condition that $|\Gamma(A)| \geq |A| \forall A \subset X$; this is called Hall's condition. Then we have:

2.1 Theorem (Hall's Theorem or Hall's Marriage Theorem)

If G is a bipartite graph with vertex classes X, Y then G has a matching from X to Y iff Hall's condition holds; the forward implication is trivial, and we have two equally nice proofs for the reverse:

Proof 1: we induct on $|X|$; the $|X| = 1$ case is trivial [and the $|X| = 0$ one more so]. For $|X| > 1$ consider the unhelpful-seeming question: do we have $|\Gamma(A)| > |A| \forall A \subset X$ (other than $A = X, \emptyset$). If this is so, we can take any $x \in X$ and $y \in \Gamma(x)$, then Hall's condition clearly holds on $G' = G - x - y$ so we have a matching from $X \setminus \{x\}$ to $Y \setminus \{y\}$ so by also matching $x \to y$ we have a matching for G. Otherwise, we have some $A \subsetneq X$ with $|\Gamma(A)| = |A|$; let $G' = G[A \cup \Gamma(A)], G'' = G[(X \setminus A) \cup (Y \setminus \Gamma(A))]$. We clearly have a matching in G' from A to $\Gamma(A)$, since $\forall B \subset A$, $\Gamma_G(B) = \Gamma_{G'}(B) \subset \Gamma(A)$ so Hall's condition holds on G' and we have one by induction. For $B \subset X \setminus A$, consider $A \cup B$; we have $|\Gamma_G(A \cup B)| \geq |A \cup B| = |A| + |B|$ so $|\Gamma_G(A \cup B) \setminus \Gamma(A)| \geq |B|$ i.e. $|\Gamma_{G''}(B)| \geq B$ so Hall's condition holds and we have a matching on G'' also; combinind these we have a matching on G.

Proof 2: Form a directed network by adding a source s joined to each $x \in X$ by an edge of capacity 1, a sink t joined to each $y \in Y$ by an edge of capacity 1, and make the edges of G directed from X to Y with capacity ∞ (i.e. some large positive integer). Then an integer-valued flow of size $|X|$ is precisely a matching from X to Y, so by the integrality theorem of max-flow min-cut we just need to show that every cut has capacity $|X|$. So, given a cut (which recall is a set of vertices containing s and not containing t), write it as $\{s\} \cup A \cup B$ with $A \subset X, B \subset Y$, and we need to show that the capacity of the edges out of this is $\geq |X|$. We can wlog take $\Gamma(A) \subset B$, otherwise the capacity is ∞, then there are $|X| - |A|$ edges flowing out of the cut from s and $|B|$ edges flowing out of B, so the capacity is $|X| - |A| + |B| \geq |X| - |A| + |B|$ since $|B| \geq |A|$.

A <u>matching of defficiency d</u> in a bipartite graph with vertex classes X, Y consists of $|X| - d$ independent edges.

2.2 Corollary (Defect Hall)

It is surprising that this is merely a corollary and there is no need to go back through the proof of Hall to obtain this result - Hall "generalizes itself", which is a good sign in a theorem: For G a bipartite graph with vertex classes X, Y, G has a matching of deficiency d from X to Y iff $|\Gamma(A)| \geq |A| - d \forall A \subset X$; note that this is a generalization which reduces to Hall in the case $d = 0$. The forward implication is of course trivial; for the reverse, form G' by adding d points to Y, each joined to all of X, then $|\Gamma_{G'}(A)| \geq |A| \forall A \subset X$, so by Hall there is a matching in G', which gives a matching of deficiency d in G.

2.3 Corollary

Let $S_1, \ldots, S_n$ be sets; a <u>transversal</u> for $S_1, \ldots, S_n$ consists of distinct points $x_1, \ldots, x_n$ with $x_i \in S_i$. The sets have a transversal iff $|\bigcup_{i \in A} S_i| \geq |A| \forall A \in S_1 \cup \cdots \cup S_n$: the forward implication is trivial, for the reverse wlog take all the S_i finite, then form a bipartite graph by $X = \{1, \ldots, n\}$ (or possibly copies of these, if the numbers themselves are elements of some of the S_i), $Y = S_1 \cup \cdots \cup S_n$, $i \in X$ joined to $j \in Y$ if $j \in S_i$. Then a transversal is precisely a matching from X to Y, but for any $A \subset X$ we have $|\Gamma(A)| = |\bigcup_{i \in A} S_i| \geq |A|$ so we have the result by Hall.

This is really an alternate formulation of Hall; a matching in a bipartite graph G with vertex classes $X = \{x_1, \ldots, x_n\}, Y$ is precisely a transversal for $\Gamma(x_1), \ldots, \Gamma(x_n)$. Of course we also have a defect form: $\exists$ a transversal for all but d of $S_1, \ldots, S_n$ iff $|\bigcup_{i \in A} S_i| \geq |A| - d \forall A \in \{1, \ldots, n\}$, with the proof exactly as for Hall.

A typical application of Hall: Hall turns up in many areas of maths, often where we would not expect it. Let G be a finite group, H a subgroup therof; we have the left cosets $L_1, \ldots, L_k = g_1 H, \ldots, g_k H$ of H in G, with $k = \frac{|G|}{|H|}$; we also have the right cosets $R_1, \ldots, R_k = Hg'_1, \ldots, Hg'_k$, which are not in general the same. Can we choose a set of representatives of the left cosets that are also representatives of the right cosets? i.e. $g_1, \ldots, g_k$ such that $g_1 H, \ldots, g_k H$ are precisely the left cosets and $Hg_1, \ldots, Hg_k$ are precisely the right cosets. To do this we just need to label the L_i, R_i such that $L_i \cap R_i \neq \emptyset \forall i$, as then we can just take $g_i \in L_i \cap R_i \forall i$, so we seek a matching in a graph G from $X = \{L_1, \ldots, L_k\}$ to $Y = \{R_1, \ldots, R_k\}$ (with possibly some sets being copies if there are sets which are both left and right cosets), where L_i is joined to R_j if they intersect. So by Hall, STP $|\Gamma(A)| \geq |A| \forall A \subset X$, but this is obvious since all the L_i and R_j have the same size $|H|$, so any union of n left cosets cannot be covered by $< n$ right cosets. Note that this proof is not only nonobvious from group theory, but cannot even be translated into the language of group theory, since the proof of Hall requires induction, and we cannot induct by removing a coset as this would give something which is no longer a group.

Connectivity

This is the idea of "how connected" a graph is; a [large] tree is connected but "only barely", in that if we remove a point we disconnect it; a cycle G is connected and has $G - x$ connected $\forall x \in G$, but we can remove two points to disconnected; a cube (that is the graph of 8 vertices arranged as the corners of a cube with the obvious edges) is even better.

For G connected with $|G| > 1$, the connectivity $\kappa(G)$ of G is the smallest $|S|$ for which we can have $S \subset V_(G)$ such that $G - S$ is disconnected or a single point (this last condition being so that the "right thing" happens with small graphs; the graph of a single point is technically connected, but we want any cycle to have connectivity 2 (including the 3-cycle), and removing all the neighbours of a point "should" disconnect the graph; see below). We say G is k-connected if $\kappa(G) \geq k$; then G is k-connected iff no set of size $< k$ disconnects G or makes it a single point, or equivalently $|G| > k$ and no set of size k disconnects G. Examples are that a tree T is not 2-connected, cycles are 2-connected but not 3-connected, the cube is 3-connected, and K_n is $(n-1)$-connected. Note that we can have $\kappa(G - x) > \kappa(G)$, e.g. where G is a cycle with an extra point x connected to one vertex of the cycle.

Remark: as referenced above, we always have $\kappa(G) \leq \delta(G)$, as we can choose $x \in G$ with $d(x) = \delta(G)$, then $S = \Gamma(x)$ has $G - s$ either disconnected or $= \{x\}$.

We know G is connected if $\forall a, b$ there is an $a - b$ path in G; it would be nice if we had for G k-connected, $\forall a, b \exists k$ independent $a - b$ paths (by which we mean their vertices other than a, b are disjoint). For G connected, $a \neq b$ vertices, we say $S \subset V(G)$ separates a from b (or is an $a - b$ separator) if a and b are in different componets of $G \setminus S$ (i.e. every $a - b$ path meets S).

2.4 Theorem (Menger's Theorem)

This is "the" theorem on connectivity, and quite subtle: for G a graph, $a \neq b$ non-adjacent vertices of G, if all $a - b$ separators have size $\geq k$ then there is a family of k independent $a - b$ paths.

First, some remarks: the converse is trivial, since any separator must contain at least one point from each of the k paths. An equivalent form of this result is therefore that the minimum size of an $a - b$ separator = the maximum number of independent $a - b$ paths. We need to have the non-adjacency condition as otherwise the theorem is false [Imre is lying here]. Menger is actually a generalization of Hall: for a bipartite graph G on X, Y with $|\Gamma(A)| \geq |A| \forall A \subset X$, form G' by adding vertices s joined to all of X, t joined to all of Y; then a matching is precisely a family of $|X|$ independent $s - t$ paths, so if we have Menger then we only need to proove every separator has size $\geq |X|$; if $S = A \cup B$ for $A \subset X, B \subset Y$ is a separator then $\Gamma(X - A) \subset B$ so $|A| + |B| \geq |A| + |\Gamma(X - A)| \geq |A| + |X - A|$ and we have Hall. Thus, the proof for this theorem cannot be "easy" (because prooving Hall itself was hard); in particular we cannot just take a point from each path in the maximum independent set of such and assume this forms a separator, because in general it will not.

A direct proof: let k be the minimal size of any $a - b$ separator; we will now find k independent paths. If this is not possible, take a minimal counterexample (we are doing "grown-up induction" here); in particular take a minimal k and a minimal $e(G)$ for this k. Let S be an $a - b$ separator with $|S| = k$. We first consider the case where $(\star) S \nsubseteq \Gamma(a), S \nsubseteq \Gamma(b)$: form G' from G by replacing the component of $G \setminus S$ containing a by a single point a' joined to all of S. Then $e(G') < e(G)$, and in G' there is no $a' - b$ separator of size $< k$, as any such would also be an $a - b$ separator in G. So by minimality we have k independent $a' - b$ paths in S, i.e. k paths $B_1, \ldots, B_k$ from b to S, disjoint except at b; we can similarly have k paths $A_1, \ldots, A_k$ from a to S, disjoint except at a, and it can never be that an A_i intersects a B_j as then these would form an $a - b$ path missing S. So, since S consists of precisely k points, putting these paths together we have k independent $a - b$ paths.

Now, the case where $(\star)$ cannot hold: every $a - b$ separator S of size k is a subset of $\Gamma(a)$ or $\Gamma(b)$. Now we have $k \geq 2$ as the theorem is true for $k = 1$; we also cannot have any $x \in \Gamma(a) \cap \Gamma(b)$, as if so then all $a - b$ separators S in $G - x$ have size $\geq k - 1$ since $S \cup \{x\}$ separates a and b in G, so by minimality there are $k - 1$ independent $a - b$ paths in $G - x$, so together with axb we have k paths as required. Now take a shortest $a - b$ path, $ax_1x_2 \ldots x_rb$ with $r \geq 2$. Now consider $G - x_1 - x_2$; This has a separator S of size $k - 1$ by minimality; if it had a separator of size $k - 2$ we would have a separator of G of size $k - 1$. So $S \cup \{x_1\}, S \cup \{x_2\}$ are separators in G; since $x_1 \notin \Gamma(b)$ we have $S \cup \{x_1\} \subset \Gamma(a)$; similarly $S \cup \{x_1\} \subset \Gamma(b)$ so $S \subset \Gamma(a) \cap \Gamma(b) = \emptyset$, a contradiction.

As an alternative proof we can apply the vertex-capacity form of max flow min cut: form a directed network by replacing each edge xy with directed edges $\overrightarrow{xy}, \overleftarrow{xy}$, and give each vertex capacity 1; then an integer-valued flow of size k is exactly a family of k independent $a - b$ paths, and by the integrality form of max-flow min-cut we just need that every vertex cut-set has size $\geq k$, i.e. every $a - b$ separator has size $\geq k$.

2.5 Corollary

This is sometimes also called Menger's Theorem, as are results 6 and 7 below.

For a graph G, $|G| > 1$, G is k-connected $\Leftrightarrow \forall a \neq b \in G \exists k$ independent $a - b$ paths; for the reverse implication we certainly have G connected, and $|G| > k$; no set of size $< k$ can disconnect G, as if so we could choose a, b in different components under this. For the forward implication if a, b non-adjacent we are done by Menger: G is k-connected so no set of size $< k$ can separate a and b. If a, b adjacent let $G' = G - ab$ [arrgh, it's painful to watch], then G' is certainly $k - 1$ connected so by Menger we have $k - 1$ independent $a - b$ paths is G; together with ab these form a set as required.

For G connected, $|G| > 1$, the edge connectivity $\lambda(G)$ of G is the smallest size of a $W \subset E(G)$ such that $G \setminus W$ is disconnected; we say G is k-edge-connected if $\lambda(G) \geq k$. So (always assuming $|G| > 1$) G is 1-edge-connected iff it is connected, 2-edge-connected iff it has no bridge (and note this is different from being 2-connected), and so on. So e.g. $\lambda(C) = 2$ for a cycle C. $\lambda(G) \leq \delta(G)$ since removing all the edges joined to any one point disconnects the graph.

2.6 Theorem (Edge form of Menger)

For G a connected graph and a, b distinct vertices therof, the minimum size of $W \subset E(G)$ separating a from b (i.e. such that a, b lie in different components of $G - W$) = the maximum number of edge-disjoint $a - b$ paths. We could just proove this by max-flow min-cut, but the following idea is more useful in general: the line graph $L(G)$ of a graph G has vertex set $E(G)$ with e joined to f if they meet (share a vertex) [in G]. A path is $L(G)$ gives rise to a path in G and vice versa. Form G' from $L(G)$ by adding vertices a', b' and edges $a'e$ for any $e \in G$ with $a \in e$, similarly $b'e$. Then $\kappa(G') = \lambda(G)$ [Imre claims that this statement is redundant; I am unconvinced], since $\lambda(G)$ was already $\leq \delta(G)$ so adding a', b' did not break this equality, so we are done by vertex Menger applied to G'.

2.7 Corollary

For G a graph, $|G| > 1$, G is k-edge-connected iff $\exists k$ disjoint $a - b$ paths [$\forall a \neq b \in G$]; the reverse implication is obvious, the forward by the edge form of Menger.

3 Extremal problems

This is the heart of graph theory; we ask when a graph contains "something nice"; the first result is very easy, but this will not be so for the rest of the theorems in the chapter.

An Eulerian circuit in a graph G is a circuit passing through each edge exactly once, i.e. $x_1 \dots x_k = x_i$ succh that $\forall xy \in E(G) \exists ! 1 \leq i \leq k-1$ with $xy = x_i x_{i+1}$; we say G is Eulerian if it has such a circuit. Clearly not all graphs are Eulerian, e.g. any graph with a bridge will not be; when is a graph Eulerian?

3.1 Proposition

A connected graph G is Eulerian iff $d(x)$ is even $\forall x \in G$ (hence a general graph G is Eulerian iff all degrees are even and at most one component of G has an edge): for the forward implication if x appears k times in an Eulerian circuit then $d(x) = 2k$. For the reverse we induct on $e(G)$; if $e(G) = 0$ we are done. For G connected with $e(G) > 0$ and $d(x)$ even $\forall x \in G$, suppose G is not Eulerian; take C a longest circuit in G with no repeated edge. Note $E(C) > 0$ since G has a cycle, since $d(x) \geq 2 \forall x \in G$ so G is not a tree. Let H be a component of $G - E(C)$ with $e(H) > 0$ [which must exist as C is not Eulerian], then H is connected and $d_H(x)$ is even $\forall x \in H$ since $d_H(x), d_C(x)$ are both even. So by the induction hypothesis H has an Euler circuit C', but now C, C' are edge-disjoint circuits which share a vertex (H must meet some vertex of C as G is connected), so we can combine them to form a longer circuit than C with no repeated edges, a contradiction.

For G a graph of order n we say G is Hamiltonian if it has a cycle of length n (i.e. a cycle through all the vertices); such a cycle is a Hamiltonian cycle. This is clearly not the case for a graph with a cutvertex. Unlike the question of being Eulerian, there is no "nice" double implication for when a graph is Hamiltonian; one reason to believe no such exists is that there could not be some kind of "parity" condition, as if G is Hamiltonian so is $G + xy$ for any edge $xy \notin G$. So instead we ask how "large" must a graph on n vertices be to ensure that it is Hamiltonian. A silly approach is to ask how many edges we need; this is silly as a single vertex with $d(x) = 1$ will make the graph not Hamiltonian, so if we take G to be the complement (the complement of

$G = (V, E)$ is $\bar{G}| = (V, V^{(2)} \setminus E))$ of $V = \{x_1, \dots, x_n\}, E = \{x_1x_2, \dots, x_1x_{n-1}\}$ then $e(G) = \binom{n}{2} - (n-2)$ but G is not Hamiltonian.

A better question is what value of $\delta(G)$ ensures G is Hamiltonian. As a lower bound, for n even we take two disjoint copies of $K_{\frac{n}{2}}$, then $\delta(G) = \frac{n}{2} - 1$ but G is not Hamiltonian; for n odd we can do slightly better and take two copies of $K_{\frac{n+1}{2}}$ meeting at a single point. Then $\delta(G) = \frac{n-1}{2}$ but G is not Hamiltonian.

3.2 Proposition

For a graph G of order $n \geq 3$, $\delta(G) \geq \frac{n}{2} \Rightarrow G$ Hamiltonian: G is connected since if x, y are non-adjacent then $\Gamma(x), \Gamma(y) \subset V \setminus \{x, y\}$ must meet by the pigeonhole principle. Let $x_1 \dots x_l$ be a longest path in G (we take a path rather than a cycle since it is easier to reason inductively about these - we can simply add a vertex to make a slightly longer one). Note $l \geq 3$ as $|G| \geq 3$ and G connected. WLOG G has no cycle of length l; if $l = n$ and it does we are done, and for $l < n$ if we had an l-cycle C then since G is connected there is some $x \notin C$ adjacent to some $y \in C$, giving us a path on $l + 1$ points (by $C-$ one edge and the edge to x), a contradiction. So $x_1x_2 \notin E$; moreover (and this is the heart of the proof) we cannot have $2 \leq i \leq l$ with $x_1x_i, x_{i-1}x_l \in E$ [as then we have such a cycle $x_1 - x_{i-1} - x_l - x_i - x_1$]. Now $\Gamma(x_1) \subset \{x_2, \dots, x_l\}, \gamma(x_l) \subset \{x_1, \dots, x_{l-1}\}$ by maximality of paths, and $\Gamma(x_1)$ is disjoint from $\gamma_+(x_1) := \{x_i : x_{i-1} \in \Gamma(x_l)\}$, but the sizes of both these sets are $\geq \frac{n}{2}$ and they are both $\subset \{x_2, \dots, x_l\}$, a contradiction.

Note that we didn't use the "full strength" of $\delta(G) \geq \frac{n}{2}$; the proof actually shows that the result holds if $\forall x, y$ non-adjacent, $d(x) + d(y) \geq n$.

3.3 Proposition

For G a graph of order $n \geq 3$, connected with $\delta(G) \geq \frac{k}{2}$ for some $k < n$, G has a path of length k; note that we do need $k < n$ by e.g. $G = K_{\frac{k}{2}+1}$; also we need that G is connected by considering two copies of K_k. The proof is all as for proposition 2: let $x_1 \dots x_l$ be a longest path in G, note $l \geq 3$, suppose $l \leq k$, then wlog G has no l-cycle, so $\Gamma(x_1), \Gamma_+(x_l)$ are disjoint subsets of $\{x_2, \dots, x_l\}$ each of size $\geq \frac{k}{2}$, a contradiction. This was a proposition rather than a theorem, since we have better:

3.4 Theorem

For G a graph of order n, if $e(G) > \frac{n(k-1)}{2}$ then $G \supset P_k$; equivalently if $G \not\supset P_k$ then $e(G) \leq \frac{n(k-1)}{2}$. This bound cannot be improved upon, at least for $k \mid n$; consider the graph consisting of $\frac{n}{k}$ disjoint K_k. We induct on n; for $n \leq 2$ the result is trivial; otherwise, suppose G has $|G| \geq 3$ and does not contain a P_k. Then we want $e(G) \leq \frac{n(k-1)}{2}$; this is linear in n so we can wlog take G connected (otherwise the result holds by inductive hypothesis on its components). If $n \leq k$ we have $e(G) \leq \frac{n(k-2)}{2}$ trivially; otherwise, by proposition 3, G has a vertex of degree $\leq \frac{k-1}{2}$, but then $G - x$ is a graph on $n - 1$ vertices not containing a P_k, so $e(G - x) \leq \frac{(n-1)(k-1)}{2}$. So $e(G) \leq \frac{n(k-1)}{2}$ as required.

Both proposition 2 and theorem 4 are extremal results; they answer the question, how large (in some sense) can a graph be with a certain property?

Often the property in question is not containing some fixed graph. Next we shall consider the graph K_k, e.g. how big can $e(G)$ be for a triangle-free graph?

Turán's Theorem

How many edges can a graph on n vertices have if it does not contain a K_r? For $r = 3$ we'd try $G = K_{a,b}$ where $a + b = n$; clearly $a = b$ or near to this is best i.e. $G = K_{\frac{n}{2},\frac{n}{2}}$ for n even and $K_{\frac{n+1}{2},\frac{n-1}{2}}$ for n odd.

More generally, we say G is k-partite on classes $V_1, \dots, V_k$ if $V_1, \dots, V_k$ partition V and $G[V_i] = \emptyset \forall i$; clearly G k-partite $\Rightarrow G \not\supseteq K_r$. If additionally $E(G) = \{xy : x \in V_i, y \in V_j, i \neq j\}$ then we say G is complete k-partite.

The Turán graph $T_k(n)$ is the complete k-partite graph on vertex classes $V_1, \dots, V_k$ with $\sum |V_i| = n$ with $|V_1|, \dots, |V_k|$ as equal as possible (integers $a_1, \dots, a_k$ are as equal as possible if $|a_i - a_j| \leq 1 \forall i, j$). We certainly have $T_{r-1}(n) \not\supseteq K_r$ since it is $(r-1)$-partite, and T_{r-1} is maximal [i.e. locally maximum] K_r-free; if we add any edge this forms a K_r.

If $k \mid n$ then all classes [in $T_k(n)$] have size $\frac{n}{k}$, so $d(x) = n - \frac{n}{k} = n(1 - \frac{1}{k}) \forall x$; in general the casses have size $\lceil \frac{n}{k} \rceil$ or $\lfloor \frac{n}{k} \rfloor$ and the degrees are $n - \lceil \frac{n}{k} \rceil$ or $n - \lfloor \frac{n}{k} \rfloor$. To form $T_k(n-1)$ from $T_k(n)$ we remove a point from a largest class (i.e. a point of minimum degree) (1), and to form $T_k(n)$ from $T_k(n-1)$ we add a point to a smallest class (2).

3.5 Theorem (Turán's Theorem)

Let G be a graph on n vertices, then $e(G) > e(T_{r-1}(n)) \Rightarrow G \supset K_r$. Clearly this is the best possible bound, since $T_{r-1}(n) \not\supseteq K_r$. If we know G is $(r-1)$-partite we are immediately done by AM-GM or similar, but there is no reason G needs to be $(r-1)$-partite, e.g. $C_5 \not\supseteq K_3$ but is not bipartite. The proof would seem to necessarily be fiddly, since $e(T_{r-1}(n))$ is a complicated and fiddly function of n and r; however, it is not:

We shall strengthen the proposition to make it easier to proove (this may initially seem implausible, but is actually reasonably common when inducting): we shall proove that if $|G| = n, e(G) = e(T(n))$ (writing $T(n)$ for $T_{r-1}(n)$ throughout this proof) and $G \not\supseteq K_r$ then $G \cong T(n)$, which implies the theorem by maximality of $T(n)$. We induct on n; the result is trivial for $n \leq r - 1$. Given G with $|G| = n > r, e(G) = e(T(n)), G \not\supseteq K_r$ choose $x \in G$ with $d(x) = \delta(G)$ and let $G' = G - x$. Then we claim $\delta(G) \leq \delta(T(n))$: we have $\sum_{y \in G} d_G(y) = \sum_{y \in T(n)} d_{T(n)}(y)$ and the $d_{T(n)}(y)$ are as equal as possible. So $|G'| = n - 1, G' \not\supseteq K_r, E(G') = e(G) - \delta(G) \geq E(T(n)) - \delta(T(n)) = e(T(n-1))$ by (1), so $\delta(G) = \delta(T(n))$ and $G' \cong T(n-1)$ by the induction hypothesis. Let the vertex classes of G' be $V_1, \dots, V_{r-1}$; we cannot have $\Gamma(x) \cap V_i \neq \emptyset \forall i$ as then we would have $G \supset K_r$, but $d(x) = n - 1 - \min |V_i|$ by (2) so $\Gamma(x) = \bigcup_{j \neq i} V_j$ for some i with $|V_i|$ minimal, so $G \cong T(n)$ as required. Note that there are actually many nice proofs of this result.

The Problem of Zarankiewicz

This is a bipartite analogue of Turán: how many edges can a bipartite graph G with n vertices in each class have if $G \not\supseteq K_{t,t}$? Write $Z(n,t)$ for this maximum.

3.6 Theroem

Take $t \geq 2$ (we do not care whether the result is true for the trivial $t < 2$ cases). Then $Z(n,t) \leq 2^{n-\frac{1}{t}} t^{\frac{1}{t}} + nt$; in particular $Z(n,t) \leq 2n^{2-\frac{1}{t}}$ for sufficiently large n: Let G be bipartite on vertex classes X, Y with $|X| = |Y| = n$, $G \not\supseteq K_{t,t}$. Let the degrees in x be $d_1, \ldots, d_n$; we shall show that the average degree d in X is $\leq n^{1-\frac{1}{t}} t^{\frac{1}{t}} + t$. We can wlog take $d_i \geq t-1 \forall i$ as if $d_i < t-1$ we can add an edge without creating a $K_{(t,t)}$. For each t-set (set of size t $A \subset Y$, we will double-count the number of $x \in X$ with $A \subset \Gamma(x)$: there can be at most $t-1$ such as $G \not\supseteq K_{(t,t)}$, so the number of (x, A) with $x \in X, A \in \Gamma(x), |A| = t$ is $\leq (t-1)\binom{n}{t}$, but also $= \sum \binom{d_i}{t}$ (this is valid since we took $d_i \geq t-1 \forall i$) so we have $\sum \binom{d_i}{t} \leq (t-1)\binom{n}{t}$.

Now the function $\binom{x}{t}$, defined for real x as $\frac{x(x-1)\ldots(x-t+1)}{t!}$ is a convex fuction of x for $x \geq t-1$ (e.g. write $y = x-t+1$, then this is $\frac{(y+t-1)\ldots(y+1)y}{t!}$, a nonnegative linear combination of powers of y) so by Jensen, $\sum \binom{d_i}{t} \geq n\binom{d}{t}$ so $n\binom{d}{t} \leq (t-1)\binom{n}{t}$. Using very crude approximations, since we only really care that the result is $O(n^{2-\frac{1}{t}})$, we have $\frac{n(d-t+1)^t}{t!} \leq \frac{(t-1)n^t}{t!}$, i.e. $(d-t+1)^t \leq (t-1)n^{t-1}$ so $d-t+1 \leq n^{1-\frac{1}{t}}(t-1)^{\frac{1}{t}}$ and $d \leq n^{1-\frac{1}{t}} t^{\frac{1}{t}} + t$.

We ask whether this is the right value: does $Z(n,t)$ grow like $n^{2-\frac{1}{t}}$ for fixed t? For $t = 2$, for G bipartite and $G \not\supseteq K_{2,2}$ $(= C_4)$, we want to know whether we can make $e(G)$ as large as $cn^{\frac{3}{2}}$; making $e(G)$ grow linearly is trivial (e.g. a $2n$-cycle), but getting it to grow even as fast as $n^{1.01}$ is somewhat difficult; in fact it is possible, though difficult, to find examples of G with $e(G) = cn^{\frac{3}{2}}$; these examples come from algebra, cf projective planes. So the bound is correct for the $t = 2$ case. For $t = 3$ the bound is also correct, though this is even harder to show; for $t \geq 4$ the result remains unknown.

Non-examinable: the Erdős-Stone Theorem

Only the statement of this theorem is examinable: For a fixed graph H write $Ex(n, H)$ for $\max\{e(G) : |G| = n, G \not\supseteq H\}$; then e.g. Turán's theorem is that $Ex(n, K_r) \sim (1 - \frac{1}{r-1})\binom{n}{2}$, or more precisely $\frac{Ex(n,K_r)}{\binom{n}{2}} \to 1 - \frac{1}{r-1}$ as $n \to \infty$. Define that $\frac{e(G)}{\binom{n}{2}}$ is the density of G. We have from the fourth result in this section that $Ex(n, P_k) \sim \frac{n(k-1)}{2}$; how does $Ex(n, H)$ behave for general H as $n \to \infty$. This seems an impossibly general question, but in fact this theorem will answer it.

We have from Turán that $\frac{e(G)}{\binom{n}{2}} > 1 - \frac{1}{r-1} \Rightarrow G \supset K_r$; what if we have $\frac{e(G)}{\binom{n}{2}} >$ e.g. $1 - \frac{1}{r-1} + 0.001$. Write $K_r(t)$ for $T_r(rt)$; informally this is "K_r blown up by t". Then, remarkably, $1 - \frac{1}{r-1} + 0.001 \Rightarrow G \supset K_r(t) \forall t$ (for sufficiently large n [depending on t]), and in general:

Erdős-Stone Theorem: $\forall r, \epsilon, t$, $\frac{e(G)}{\binom{n}{2}} > 1 - \frac{1}{r-1} + \epsilon \Rightarrow G \supset K_r(t)$ for n sufficiently large. As a very vague sketch of the proof, suppose we have G of average degree $\geq (1 - \frac{1}{r-1} + \epsilon)n$. 1) We have a large $H \subset G$ with $\delta(H) \geq (1 - \frac{1}{r-1} + \delta)n'$ where $n' = |H|$ for some $\delta > 0$; the proof of this is similar to the result from the first example sheet that if the average degree of a graph is d then we have H with $\delta(H) \geq \frac{d}{2}$. 2) By induction on r, $H \supset K_{r-1}(t')$ for some large t'; write K for this $K_{r-1}(t')$. 3) We have lots of points in $H \setminus K$ which are joined to $\geq t$ members of each class of K 4) we have at least t of these points joined to the <u>same</u> t-set in each class of K by the pigeonhole principle.

For a given H, choose the least r with H r-partite, e.g. $H =$ the Petersen graph is not bipartite but is 3-partite, so $r = 3$. Then $T_{r-1}(n) \not\supseteq H$ since it is $(r-1)$-partite, so $\frac{Ex(n,H)}{\binom{n}{2}} \geq 1 - \frac{1}{r-1}$. But by Erdős-Stone, $\frac{e(G)}{\binom{n}{2}} \geq 1 - \frac{1}{r-1} + \epsilon \Rightarrow$ for sufficiently large G, G contains any $K_r(t)$, so contains any r-partite graph; in particular it contains H. So $\frac{Ex(n,H)}{\binom{n}{2}} \to 1 - \frac{1}{r-1}$ for the least r such that H is r-partite.

This has almost entirely solved the problem: for most types of H we know $Ex(n,H)$ grows like some multiple of $\binom{n}{2}$. However, for H bipartite we only have that $\frac{Ex(n,H)}{\binom{n}{2}} \to 0$, not the precise growth speed of $Ex(n,H)$, and in fact this is unknown for most H, e.g. it is not known for $H = C_{2n}$ for $n \geq 6$.

4 Colourings

An <u>r-colouring</u> of a graph G is a function $c : V(G) \to [r] := \{1, \ldots, r\}$ such that $c(x) \neq c(y) \forall xy \in E(G)$; the <u>chromatic number</u> $\chi(G)$ of G is the leasst r for which G has an r-colouring.

Examples: $\chi(P_n) = 2, \chi(C_n) = 2$ if n is even, 3 if n is odd, $\chi(K_n) = n$, $\chi(E_n) = 1$ (and $\chi(G) \geq 2 \forall$ non-empty graphs), $\chi(T) = 2$ if T is a tree (inductively via removing a leaf), and $\chi(K_{m,n}) = 2$; in fact any bipartite graph is clearly 2-colourable (i.e. has a 2-colouring), and conversely if c is a 2-colouring of G then G is bipartite by $X = \{x \in G : c(x) = 1\}, Y = \{x \in G : c(x) = 2\}$; similarly G is r-colourable iff it is r-partite. So by the corollary to Erdős-Stone, $\frac{Ex(n,H)}{\binom{n}{2}} \to 1 - \frac{1}{\chi(H)-1}$ as $n \to \infty$.

We clearly have $\chi(G) \leq n = |G|$; we can improve this substantially:

4.1 Proposition

A graph G with $\Delta(G) = \Delta$ has $\chi(G) \leq \Delta + 1$; note this is the best possible bound, by e.g. $G =$ a complete graph or odd cycle: order G as $x_1, \ldots, x_n$ and colour each x_i in turn; when we come to colour x_i, we have at most Δ colours unusable because we already have neighbours of x_i that colour, so we have a remaining colour to colour x_i.

Note that we can have $\chi(G) \ll \Delta$, by e.g. the "star" graph $K_{1,n}$. We could view this proof as an "application" of the <u>greedy algorithm</u>: for a given ordering $x_1, \ldots, x_n$ of $V(G)$, colour each x_i in turn, always using the smallest possible colour. Note that the greedy algorithm may use more than $\chi(G)$ colours to colour a given graph. Also note that although $K \supset K_r \Rightarrow \chi(G) \geq r$ the converse is false (e.g. $C_5 \not\supseteq K_3$); in fact there is no simple formula known for $\chi(G)$.

Although it might make more sense to do the general theorems about colouring now, these can be quite intimidating if we do not have a good "feel" for colouring; therefore we shall first cover:

Colouring Planar Graphs

How many colours do we need to colour a planar graph? 3 is insufficient since e.g. K_4 is planar.

4.2 Proposition (6-colour theorem)

Any planar G is 6-colourable: induct on $|G|$, trivial for $|G| \leq 6$. Given a planar G with $|G| > 6$, we claim $\delta(G) \leq 5$: since $e(G) \leq 3n - 6$ as G is planar, so $\sum_{x \in G} d(x) \leq 6n - 12$, so we have some x with $d(x) \leq 5$. Then $G - k$ is planar, so by the induction hypothesis it has a 6-colouring; then $\Gamma(x)$ has at most 5 colours, so we can colour x.

4.3 Theorem (5-colour theorem)

If G is planar then it is 5-colourable; as before we induct on $|G|$, the result is true for $|G| \leq 5$, and given a planar G with $|G| > 5$ we can take $x \in G$ with $d(x) \leq 5$, and by induction we have a 5-colouring of $G - x$. Therefore, we are done unless $d(x) = 5$ and all 5 colours appear in $\Gamma(x)$; in this case say $\Gamma(x) = \{x_1, x_2, x_3, x_4, x_5\}$ going clockwise around x [for some drawing of G], and wlog take $c(x_i) = i \forall i$. Then, consider whether there is a 1-3 path (a path along which the colours 1 and 3 alternate) from x_1 to x_3. If not, let H be the 1-3 component of x (the set of all vertices reachable from x by 1-3 paths), and we have $x_3 \notin H$; swap the colours 1 and 3 on H, then we still have a colouring of $G - x$, but x_1 is now of colour 3, so we can colour x by colour 1. If we do have such a 1-3 path, we cannot have a 2-4 path from x_2 to x_4 [as x_2, x_4 lie in different components of $G-$ this 1-3 path], so we can swap colours 2 and 4 on the 2-4 component of x_2 and then use colour 2 for x.

The $i - j$ paths here are called <u>Kempe chains</u>; we shall see why when we cover the 4-colour theorem.

Suppose we instead want to colour the <u>faces</u> if a plane graph such that distinct faces sharing an edge have different colours; this problem is called colouring a plane map. Given a plane graph G we form the <u>dual graph</u> G' by taking a vertex in each face of G and joining two vertices if their faces share an edge; this is planar, and colouring this is the same as colouring the faces of G. So by theorem 4.3 every plane map is 5-colourable.

The 4-colour theorem, that if G is planar then G is four-colourable, was finally proven in 1976 by Appel and Haken (though a flawed proof given by Kempe in 1879 stood for 11 years before its mistake was spotted). In the proof of theorem 3 we used the fact that "a vertex of degree 1, 2, 3, 4, or 5" forms an <u>unavoidable</u> set (i.e. every plane G contains at least one element of the set) of <u>reducible</u> configurations (configurations which could not be part of a minimal counterexample); Appel and Haken found a similar set, but of around 1900 configurations, for 4-colouring, making extensive use of computers. Some people believe the existence of these configurations, and thus the 4-colour theorem, is an "accident of nature"; others believe a "nicer" proof "should" be found; as yet, while there are alternative proofs, none of them can avoid the dependence on computers.

[Now we return to colouring general (non-planar) graphs]

We have $\chi(G) \leq \Delta + 1 \forall G$ and we can have equality in this bound by e.g. K_n, or C_n for n odd. We wish to improve this.

Remark: if G is connected and not regular, we can certainly colour it with Δ colours: choose x_n with $d(x_n) < \delta$, then choose x_{n-1} adjacent to x_n (as G connected), $x_{n-2} \in G \setminus \{x_n, x_{n-1}\}$ adjacent to $\{x_n, x_{n-1}\}$ (i.e. adjacent to at least one member of this set) and so on; then run greedy on the order $x_1, \ldots, x_n$; for $i < n$ we have $x_i x_j \in E$ for some $j > i$ [so at most $\Delta - 1$ already-coloured neighbours of x_i] so can colour x_i, and then we can colour x_n, with Δ colours.

4.4 Proposition (Brooks' Theorem)

If G is connected and not a complete graph or odd cycle then $\chi(G) \leq \Delta := \Delta(G)$: by the above remark we can wlog take G regular, and $\Delta \geq 3$ since the $\Delta \leq 1$ case is trivial and for $\Delta = 2$ G is a cycle. Let G be a minimal counterexample($|G|$ minimal); wlog G is 2-connected as if it has a cutvertex x then for each component G_i of $G - x$, $G[V(G_i) \cup \{x\}]$ is Δ-colourable, and by permuting the colours on each of these graphs we can take a single colour for x and combine these colourings to obtain a colouring of G. If G is 3-connected (this is the main case), then choose x_n; we must have some $x_1, x_2 \in \Gamma(x_n)$ with $x_1 x_2 \notin E$ as otherwise we have a $K_{\Delta+1}$ by $x_n \cap \Gamma(x_n)$; since G is connected there can be no other vertices and G is a K_n. Now $G \setminus \{x_1, x_2\}$ is connected as G is 3-connected, so ordering its vertices $x_3, \ldots, x_n$ as in the results with $\forall 3 \leq i \leq n-1 \exists j > i : x_i x_j \in E$; running greedy on $x_1, x_2, \ldots, x_n$ then uses at most Δ colours.

If G is not 3-connected, choose a separator $\{x, y\}$; let G_i be the components of $G \setminus \{x, y\}$ together with x and y. Then each G_i has a Δ-colouring by the above remark, as $d_{G_i}(x) \leq \Delta - 1 \forall i$. If $xy \in E$, x, y have different colours in the colouring of each G_i, so we can combine these to give a Δ-colouring of G; otherwise, if in each G_i we have at least one of $d_{G_i}(x), d_{G_i}(y)$ being $\leq \Delta - 2$ we can recolour such that x, y have different colours in G_i and are done. Otherwise, we have some G_i with $d_{G_i}(x) = d_{G_i}(y) = \Delta - 1$; wlog take $i = 1$. Then we must have $k = 2$ and $d_{G_2(x)} = d_{G_2}(y) = 1$; if we let $\Gamma_{G_2}(y) = \{v\}$ then $\{x, v\}$ is a separator which does not have this problem (since $1 < \Delta - 1$).

The Chromatic Polynomial

This carries more information about G than $\chi(G)$: for any graph G and $t \in 1, 2, 3, \ldots$, let $P_G(t)$ be the number of t-colourings of G (thus $\chi(G)$ is the least t with $P_G(t) > 0$. For example, $P_{K_n}(t) = t(t-1)\ldots(t-n+1)$, by colouring each of the vertices in turn, $P_{E_n}(t) = t^n$, $P_{P_n}(t) = t(t-1)^n$ (recall that P_n has $n+1$ vertices); more generally $P_T(t) = t(t-1)^{n-1}$ for any tree on n vertices T, by induction via removing a leaf. $P_{C_n}(t)$ is less obvious. Note that we did not define that $P_G(t)$ was a polynomial; however, it appears to be one in all these examples; is this always the case?

For a graph G and $e = xy \in E(G)$, the contraction of G by e, $\frac{G}{e}$, is formed by replacing x and y with a new vertex z joined to $\Gamma(x) \cup \Gamma(y)$.

4.5 Lemma

DFor G a graph and $e = xy \in E(G)$, $P_G = P_{G-e} - P_{\frac{G}{e}}$; this is called the <u>deletion-contraction relation</u> or <u>cut-fuse relation</u>: every t-colouring of G is a t-colouring of $G-e$, and conversely, any t-colouring of $G-e$ in which x, y get different colours is a colouring of G, and the t-colourings of G in which x and y get the same colour correspond precisely to the t-colourings of $\frac{G}{e}$. Thus we have the result; in practice we shall use this as the definition of P_G (but we could not have defined P_G by this (together with the base case $G = E_n$), as it is not clear that that would be well defined; we might contract G in a different order and obtain a different result).

4.6 Proposition

$P_G(t)$ is a polynomial in t: we induct on $e(G)$; for $e(G) = 0$ we are done as $P_{E_n}(t) = t^n$; given G with $e(G) > 0$ choose an edge e, then $P_{G-e}, P_{\frac{G}{e}}$ are polynomials by the induction hypothesis, so $P_G = P_{G-e} - P_{\frac{G}{e}}$ is also.

Recall that for T a tree, $P_T(t) = t^n - (n-1)t^{n-1} + \ldots$, which suggests the following:

4.7 Proposition

For G a graph on n vertices with m edges, the leading terms of $P_G(t)$ are $t^n - mt^{n-1}$: we induct on $e(G)$, the $e(G) = 0$ case is done. Given G with $e(G) > 0$, choose an edge e, then we have $P_{G-e}(t) = t^n - (m-1)t^{n-1} + \ldots, P_{\frac{G}{e}}(t) = t^{n-1} + \ldots$, so $P_G(t) = t^n - mt^{n-1} + \ldots$.

In fact we can get other information about G from P_G, e.g. it turns out the third term of $P_G(t)$ is $(\binom{n}{2} -$ the number of triangles of $G) \times t^{n-1}$. Note also that since P_G is a polynomial we can talk about $P_G(t)$ for non-integer real (or even complex) values of t. The 4-colour theorem can be phrased as the statement that any planar G has $P_G(4) > 0$, i.e. P_G does not have a root at 4; there is hope that working from this may eventually yield a "better" proof of the 4-colour theorem , though there is none as yet; however we do know that e.g. $P_G(\phi + 2) \neq 0$ (where $\phi = \frac{1+\sqrt{5}}{2}$).

Edge-Colourings

A k-edge-colouring of a graph G is a map $c : E(G) \to \{1, \ldots, k\}$ such that $c(e) \neq c(f)$ whenever e, f share a vertex; the smallest k for which a k-edge-colouring exists is the <u>edge-chromatic number</u> or <u>chromatic index</u> of G, written $\chi'(G)$ (so $\chi'(G) = \chi(\overline{l(G)})$, but this is in fact quite irrelevant; it doesn't help us calculate $\chi'(G)$ at all). Some examples are $\chi'(C_n) = 2$ if n is even, 3 if n is odd. Note that $\chi'(G)$ can be very different from $\chi(G)$, e.g. the "star" $K_{1,n}$ has $\chi(G) = 2, \chi'(G) = n$.

We clearly have $\chi^{|}prime(G) \geq \Delta(G) \forall G$; for any point of degree Δ, we must use Δ colours to colour the edges around it. The example of a n oddcycle shows we can have $\chi'(G) > \Delta(G)$; however, we have:

4.8 Theorem (Vizling's Theorem)

For any graph G, $\chi^{|}prime(G) = \Delta$ or $\Delta + 1$ (writing Δ for $\Delta(G)$ as usual): we induct on $e(G)$; for $e(G) = 0$ we are done as we can 0-edge-colour. Given G with $e(G) > 0$ choose an edge e and take a $\Delta + 1$-edge-colouring of $G - e$; let $e = xy_1$. At each vertex there is at least one colour not being used (since $d(y) < \Delta + 1 \forall y \in G$). Choose a maximal sequence $y_1, y_2, \dots, y_k$ of distinct vertices and corresponding sequence of colours $c_1, \dots, c_k$ such that xy_i is of colour $c_{i-1} \forall 2 \leq i \leq k$ and c_i is not used at $y_i \forall 1 \leq i \leq k$; we must be able to do this since G is finite. By maximality of this sequence we either have c_k is not used at x or $c_k = c_k$ for some $j < k$. In the first case, recolour by giving xy_i the colour $c_i \forall 1 \leq i \leq k$, then we have a $\Delta + 1$-edge-colouring of G. If $c_k = c_j$ for some $j < k$, we may wlog take $j = 1$, since we can recolour xy_i with colour $c_i \forall 1 \leq i < j$, leaving x_j as the edge we need to colour. Let c be a colour unused at x; if there is no $c - c_1$ path from x to y_i, swap c and c_1 on the $c - c_1$ component H of x, and we can then give xy_1 colour c_1; similarly if there is no $c - c_1$ path from x to y_k, swap c and c_1 on the $c - c_1$ component of y_k, and now we can colour by giving xy_i colour c_i for $1 \leq i \leq k - 1$ and xy_k colour c. So we can assume H contains y_1, y_k, but H has $\Delta(H) \leq 2$ by the definition of an edge-colouring, and $d_H(x) = d_H(y_1) = d_H(y_k) = 1$, but thiis s impossible.

Graphs on Surfaces

We know that any G drawn on a plane (or equivalently, by projection from a point in a face, a sphere) has $\chi(G) \leq 5$; in fact it is ≤ 4. What about graphs drawn on other surfaces? E.g. we find can draw K_7 on a torus.

Define the (technically, compact orientable)<u>surface of genus g</u> to be a sphere with g "handles" "attached" (in fact these are all the surfaces embeddable into $\mathbb{R}^3$, but we will not use this fact; this is not a course in topology). For the plane/sphere we know $n - m + f = 2$ (for G connected), and $n - m + f \geq 2$ for any planar G (by adding edges to make G connected). For G on a torus we can ahve $n - m + f = 2$ by drawing our graph in a small region, but we can also have $n - m + f = 1$ by e.g. drawing C_3 as "loop" going around the hole in the middle of the torus, or even $n - m + f = 0$ by drawing a "bowtie" on 5 vertices by C_3 as before, and then a second "loop" with a common vertex going around the "tube" of the torus. In fact, more generally, any G drawn on the surface of genus g has $n - m + f \geq 2 - 2g = E$, the Euler characteristic. For $m \geq 3$ we have $3f \leq 2m$ as usual; thus $n - m + \frac{2}{3}m \geq E$, so $m \leq 3(n - E))$ (this is true for $m \geq 3$ if $E = 2$, and if $E \neq 2$ it is trivially true for $m < 3$ and thus true $\forall m$).

4.9 Theorem (Heawood's Theorem)

A graph G drawn on a surface of Euler characteristic $E \leq 0$ has $\chi(G) \leq \lfloor \frac{7+\sqrt{49-24E}}{2} \rfloor =: H(E)$ (note $H(2) = 4$, so yet again we "almost" have the 4-colour theorem): let $\chi(G) = k$, we then need to proove $k \leq H(E)$. Choose a G with minimal number of vertices such that $\chi(G) = k$; we have $\delta(G) \geq k - 1$ by minimality of G (if we had a vertex v of degree at most $k - 2$, then $G \setminus v$ is $k - 1$-colourable by minimality of G, but then G is $k - 1$-colourable), and

$n \geq k$. We have $m \leq 3(n-E)$ from above, so $\sum_{x \in G} d(x) = 2m \leq G(n-E)$, so $\delta(G) \leq \frac{6(n-E)}{n} = 6 - \frac{6E}{n}$. Thus $k - 1 \leq \delta(G) \leq 6 - \frac{6E}{n} \leq 6 - \frac{6E}{k}$, since $E \leq 0$. So $k^2 - k \leq 6k - 6E$, i.e. $k^2 - 7k + 6E \leq 0$, so $k \leq \frac{7+\sqrt{49-2E}}{2}$, and as k is an integer we have the result.

Equality holds in this theorem, and in fact it is even possible to draw $K_{H(E)}$ on the surface of Euler characteristic E (≤ 0), a result which took 75 years to proove.

5 Ramsey Theory

The theme of this chapter is "can we find some order, given enough disorder?". Suppose we 2-colour [the edges of] K_5 [i.e. we have some $c : E(K_6) \to \{1, 1\}$, not generally with different colours for edges meeting at a point], our chaos; can we always find a monochromatic K_3 (i.e. a K_3 on which c is constant), a piece of order? Yes: choose $x \in V(K_6)$; we have $d(x) = 5$, so at least 3 edges incident at x have the same colour, wlog xy_1, xy_2, xy_3 are red. Now if any edge $y_i y_j$ is red, we have a red K_3 by $\{x, y_i, y_j\}$; otherwise we have a blue K_3 by $\{y_1, y_2, y_3\}$.

What happens more generally? Is there an n such that any 2-coloured K_n contains a monochromatic K_4? What about a K_5?

In general, we write $R(s)$ for the smallest n (if it exists) such that whenever K_n is 2-coloured in this sense, we have a monochromatic K_S; we shall show this exists and find out roughtly how fast it grows. E.g. the above shows $R(3) \leq 6$; in fact $R(3) = 6$ as we can colour a K_5 by drawing it as a "pentagram" and colouring the outside red and the inside blue; this contains no monochromatic K_3.

We want to induct, but it is hard to "jump" from e.g. "containing a monochromatic K_3" to "containing a monochromatic K_4"; therefore we shall consider the notion of "containing a red K_3 or a blue K_4" as an intermediate step. More generally, for $s, t \geq 2$ we write $R(s, t)$ for the least n (if it exists) such that whenever K_n is 2-coloured we have a red K_s or a blue K_t. Then $R(s) = R(s, s)$; clearly we have $R(s, t) = R(t, s)$ and $R(s, 2) = s$. (We could equivalently define $R(s, t)$ is the smallest n (if it exists) such that every graph G on n points has either $K_S \subset G$ or $K_t \subset \bar{G}$, by identifing red edges with edges of G and blue edges with edges not in G).

5.1 Theorem (Ramsey's Theorem)

$R(s, t)$ exists $\forall s, t$; moreover $R(s, t) \leq R(s-1, t) + R(s, t-1)$ (for $s, t \geq 3$): enough to show that if $R(s-1, t), R(s, t-1)$ exist then $R(s, t) \leq R(s-1, t) + R(s, t-1)$, then we will have the existence of $R(s, t) \forall s, t$ by induction on $s + t$. Set $a = R(s-1, t), b = R(s, t-1)$; we shall follow a similar approach to that used in the above example. Given a 2-colouring of K_{a+b}, choose $x \in V(K_{a+b})$, then $id(x) = a + b - 1$, so we have $\geq a$ red edges or $\geq b$ blue edges incident with x; in the first case, consider the K_a spanned by the endpoints of a red edges from x; by the definition of a this contains either a red K_{s-1} [which, together with x, forms a red K_s, or a blue K_t; the other case is similar.

Remarks: This means given s, t, any graph on n vertices has either $K_s \subset G$ or $K_t \subset \bar{G}$, for sufficiently large n. Very few of the "Ramsey numbers" are known exactly; see later.

5.2 Corollary

Let $s, t \geq 2$, then $R(s,t) \leq \binom{s+t-2}{s-1}$; in particular $R(s) \leq 2^{2s}$: we induct on $s+t$, if $s=2$ or $t=2$ we are done. Given $s,t \geq 3$, $R(s,t) \leq R(s-1,t)+R(s,t-1) \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}$.

What about more colours? Write $R_k(s_1, \ldots, s_k)$ for the smallest n (if it exists) such that whenever K_n is k-coloured, $\exists$ a K_{s_i} of colour i for some $1 \leq i \leq k$. Then:

5.3 Corollary

Let $R_k(s_1, \ldots, s_k)$ exists $\forall k \geq 1, s_i \geq 2$; proof by "turquoise spectacles": we induct on k, the $k=1$ case is trivial. Given $s_1, \ldots, s_k$ with $k \geq 2$, let $n = R(s_1, R_{k-1}(s_2, \ldots, s_k))$. Then for any k-colouring of K_n, view this as a 2-colouring with colours 1 and $2 \cup \cdots \cup k$; by the definition of n we either have a K_{n_1} of colour 1 or a $K_{R_{k-1}(s_2,\ldots,s_k)}$ coloured with the $k-1$ colours $2, 3, \ldots, k$, so by the definition of $R_{k-1}(s_2, \ldots, s_k)$ we have the result. (Alternatively, we could copy the proof of the theorem, and obtain that $R_k(s_1, \ldots, s_k) \leq R_k(s_1 - 1, s_2, \ldots, s_n) + \cdots + R_k(s_1, \ldots, s_{k-1}, s_k - 1)$.

What about colouring r-sets rather than edges, e.g. for $r=3$ we colour each triangle red or blue (with the same edge possibly appearing in differently coloured triangles); do we get e.g. a 4-set all of whose triangles are the same colour? Note taht this is asking for a much denser chromatic structure than before, and if we tried to find these by "brute force" by hand or even by computer, we would imagine it to be impossible.

For X a set and $r \in \mathbb{N}$ write $X^{(r)} = \{A \subset X : |A| = r\}$; unless stated otherwise we shall take $X = [n] = \{1, 2, \ldots, n\}$. Write $R^{(r)}(s,t)$ for the smallest n (if it exists) such that whenever $X^{(r)}$ is 2-coloured we have either a red s-set or a blue t-set (so $R^{(2)}(s,t) = R(s,t)$; we clearly have $R^{(1)}(s,t) = s+t-1$ by pigeonhole). We have $R^{(r)}(s,t) = R^{(r)}(t,s), R^{(r)}(s,r) = R^{(r)}(r,s) = s$.

5.4 Theorem (Ramsey for r-sets

(Some books call this Ramsey's theorem; this is also true of the next two theorems in this section)

Let $r \geq 1, s,t \geq r$; then $R^{(r)}(s,t)$ exists. The idea behind this proof that is in our proof of teh $r=2$ case (theorem 1) we actually used the $r=1$ case, in order to say that a point with $a+b-1$ edges from it must have a red edges or b blue red edges from it. Now the proof: we induct on r, the $r=1$ case is just pigeonhole. Now given $r>1$, we induct on $s+t$; for $s=r$ or $t=r$ we are done. Then for $s,t>r$, we claim $R^{(r)}(s,t) \leq R^{(r-1)}(R^{(r)}(s-1,t), R^{(r)}(s,t-1)) + 1$: let $a = R^{(r)}(s-1,t), b = R^{(r)}(s,t-1), n = R^{(r-1)}(a,b)+1$. Then given a 2-colouring c of $X^{(r)}$, choose $x \in X$ and let $Y = X \setminus \{x\}$. Then c induces a 2-colouring of $Y^{(r-1)}$ by $c'(A) = c(A \cup \{x\})$, so by the definition of $R^{(r-1)}(a,b)$ we have either a red a-set (under c') or a blue t-set; wlog assume the first case, we have a red a-set Z, i.e. $A \cup \{x\}$ is red $\forall A \in Z^{(r-1)}$. But then by definition of a, Z contains either a blue t-set (under c), in which case we are done, or a red $(s-1)$-set, which together with x forms a red s-set.

Remarks: the result similarly holds for k colours (e.g. by "turquoise spectacles"). The bounds this gives us on $R^{(r)}$ are quite large, since, informally speaking, $R^{(r)}$ is obtained by iterating $R^{(r-1)}$ about $s+t$ times; if we define $f_1, f_2, \cdots : \mathbb{N} \to \mathbb{N}$ by $f_1(x) = 2x$, and for $n \geq 2$, $f_n(x) := f_{n-1}(f_{n-1}(\dots f_{n-1}(1)\dots))$ where we iterate the function x times. So $f_1(x) = 2x, f_2(x) = 2^x, f_3(x) = 2^{2^{\cdot^{\cdot^{\cdot^2}}}}$ where there are x exponents, f_4 we do not have notation for, but we can calculate e.g. $f_4(1) = 2, f_4(2) = 4, f_4(3) = 65536, f_4(4) = 2^{2^{\cdot^{\cdot^{\cdot^2}}}}$ where there are 65536 exponents. Then our bound on $R^{(r)}(s,t)$ is on the order of $f_r(s+t)$; such large bounds are a common feature of such "double induction" proofs.

Infinite Ramsey Theory

Given a 2-colouring of $\mathbb{N}^{(2)}$, can we always find an infinite monochromatic $M \subset \mathbb{N}$? E.g. 1. Colour ij red if $i+j$ is even, blue if $i+j$ odd, then e.g. $M = \{n : n \text{ even}\}$ works, 2. Colour ij red if $\max\{n : 2^n \mid i+j\}$ is even, blue otherwise; then e.g. $M = \{2^2, 2^4, 2^6, \dots\}$ works, 3. Colour ij red if $i+j$ has an even number of (distinct) prime factors, blue otherwise; here it is not even obvious whether M exists.

Note that asking for an infinite red set is much harder than arbitrarily large finite red sets, e.g. if we draw disjoint red $K_2, K_3, K_4, \dots$ and colour all the other edges blue then we have arbitrarily large red sets but no infinite red set.

5.5 Theorem (Infinite Ramsey)

Let $\mathbb{N}^{(2)}$ be 2-coloured, then $\exists$ an infinite monochromatic $M \subset \mathbb{N}$: choose $a_1 \in \mathbb{N}$, then $\exists$ an infinite $B_1 \subset \mathbb{N} \setminus \{a_1\}$ such that all edges from a_1 to B_1 have the same colour c_1. Then choose $a_2 \in B_1$; there then exists an infinite $B_2 \subset B_1 \setminus \{a_2\}$ such that all edges from a_2 to B_2 have the same colour c_2; continuing in the same way we obtain a sequence of points $a_1, a_2, \dots \in \mathbb{N}$ and colours $c_1, c_2, \dots$ such that $a_i a_j$ has colour $c_i \forall i < j$; then we must have infinitely many c_i the same, say $c_{i_1} = c_{i_2} = \dots$; then $M = \{a_{i_1}, a_{i_2}, \dots\}$ is a set as required.

Remarks: similarly the result holds for k colours. The technique here is called a "2-pass" proof. Note that in our example 3. above, no explicit set M is known.

Example: any sequence in $\mathbb{R}$ (or any totally ordered set) has a monotone subsequence: given a sequence $x_1, x_2, \dots$ colour $\mathbb{N}^{(2)}$ by giving ij for $j < j$ colour "up" if $x_i \leq x_j$, "down" if $x_i > x_j$, and apply the theorem.

What about colouring r-sets? e.g. for $r = 3$, if we 2-colour $\mathbb{N}^{(3)}$ by giving ijk for $i < j < k$ colour red if $i \mid j+k$, blue otherwise; $M = \{1, 2, 4, 8, \dots\}$ works.

5.6 Theorem (Infinite Ramsey for r-sets

Let $r \geq 1$, and $\mathbb{N}^{(r)}$ be 2-coloured, then there is an infinite monochromatic $M \subset \mathbb{N}$: we induct on r, the $r = 1$ case is obvious. Given $r1$, given a 2-colouring c of $\mathbb{N}^{(r)}$, choose $a_1 \in \mathbb{N}$; then we have an induced 2-colouring of $(\mathbb{N} \setminus \{a_1\})^{(r-1)}$ by $c'(F) = c(F \cup \{a_1\})$, so we have (by the induction hypothesis) an infinite monochromatic set B_1 for c', i.e. there is a colour c_1 such that every r-set $\{a_1\} \cup F$ for $F \subset B_1$ has colour c_1. Now choose $a_2 \in B_1$; we have an induced colouring of $(B_1 \setminus \{a_2\})^{(r-1)}$ so get an infinite $B_2 \subset B_1$ and colour c_2 such that every r-set $\{a_2\} \cup F$ for $F \subset B_2$ has colour c_2; continuing inductively we obtain

$a_1, a_2, \dots \in \mathbb{N}$ and colours $c_1, c_2, \dots$ such that any r-set $a_{i_1} a_{i_2} \dots a_{i_r}$ (of course with $i_1 < i_2 < \dots < i_r$) has colour c_{i_1}, but infinitelymany of the c_i must agree; say $c_{i_1} = c_{i_2} = \dots$. Then $M = \{a_{i_1}, a_{i_2}, \dots\}$ is a set as required.

Example: (unlike the previous example, this is actually impossible without Ramsey theory) we know that given points $(1, x_1), (2, x_2), \dots \in \mathbb{R}^2$ we can find a subsequence such that the "induced function", the function piecewise linear between these points, is monotone. Can we insist that the induced function is actually convex or concave? Yes: we colour $\mathbb{N}^{(3)}$ by giving ijk the colour "convex" if (j, x_j) lies below the line between the other two points, "concave" otherwise.

Exact Ramsey Numbers

Very few of the non-trivial $R(s,t)$ (i.e. $s, t \geq 3$ are known exactly; the full set of known cases is $R(3,3) = 6, R(34) = 9, R(3,5) = 14, R(3,6) = 18, R(3,7) = 23, R(3,8) = 28, R(3,9) = 36, R(4,4) = 18, R(4,5) = 25$; other than that we know e.g. $43 \leq R(5,5) \leq 49$. Proving the lower bounds on Ramsey numbers is hard, since we need to find a very disordered graph, but by its very nature this is difficult to construct - there must be no pattern, there is no way to use induction to produce such a thing. As an example, we can show $R(4,4) > 17$ by 2-colouring $\mathbb{Z}_{17}^{(2)}$ by giving ij colour red if $i - j$ is a square modulo 17, blue otherwise, but it is hard to verify that this colouring contains no monochromatic K_4, and this example does not extend to any other cases.

The only known number for > 2 colours is $R_3(3,3,3) = 17$; the only known number for r-sets for $r > 2$ is $R^{(3)} * 4, 4) = 13$. Finding these numbers is hard, because it requires us to find the paradoxical notion of the exact amount of disorder required. We might try "throwing a computer at it", but to e.g. show $R(5,5) > 43$ we would need to examine each of the $\binom{43}{5}$ 5-sets in each of $2^{\binom{43}{2}}$ possible colourings, but $2^{\binom{43}{2}} > 10^{250}$ so this is completely implausible.

6 Random Graphs

The results of this section of the course are nice, the first theorem especially so. There will be a lot of estimation in this section, but the reader should not be scared of this; it should always be clear that our estimates bound the quantities in question.

How fast does $R(s)$ grow? We know $R(s) \leq 4^s$; what about a lower bound? It is easy to see $R(s) > (s-1)^2$, by drawing $s-1$ sets of size $s-1$ and colouring the edges within each set red, and the edges between sets blue. In the 1940s it was believed that perhaps this is close to the best bound, and $R(s) = O(s^2)$. It can be shown by clever algebra that $R(s)$ is at least $O(s^3)$, and with some very hard algebra we can construct examples which show $R(s)$ is at least $O(s^4)$. However, we have:

6.1 Theorem (Erdős, 1947)

Let $s \geq 3$, then $R(s) > 2^{\frac{s}{2}}$: choose a colouring of K_n at random, by taking each edge to be red with probability $\frac{1}{2}$, blue otherwise, independently of each other.

Then the probability of a fixed s-set being monochromatic is $2 \times (\frac{1}{2})^{\binom{s}{2}}$, and there are $\binom{n}{2}$ s-sets, so the probability of there being a monochromatic s-set is at most $\binom{n}{s}2^{1-\binom{s}{2}}$. So we must have $R(s) > n$ if $\binom{n}{s}2^{1-\binom{s}{2}} < 1$, i.e. $\binom{n}{s} < 2^{\binom{s}{2}-1}$. But $\binom{n}{s} \le \frac{n^s}{s!}$ and $s! \ge 2^{\frac{s}{2}+1}$ (by induction on s; recall we're taking $s \ge 3$), so we are done if $n^s \le 2^{\frac{s^2}{2}}$, i.e. $n \le 2^{\frac{s}{2}}$.

This is a "random graphs" argument. In this particular case we could rewrite the argument by saying there are $2^{\binom{n}{2}}$ colourings, and a given s-set is monochromatic in $2 \times 2^{\binom{n}{2}-\binom{s}{2}}$ of them, so we are done if $\binom{n}{s}2^{1+\binom{n}{2}-\binom{s}{2}} < 2^{\binom{n}{2}}$; however, this is a bad viewpoint for later in this section, where we will not be taking all graphs to be equally likely. Note the proof gives no hint as to how to construct such a colouring, and in fact no exponential lower bound on $R(s)$ is known - so if we need to find an actual graph on e.g. 10^6 points with no K_{40} in G or $\bar{G}$, the best thing really is to draw a random graph.

We now have $\sqrt{2}^s \le R(s) \le 4^s$; no better bounds are known, and there are heuristic arguments that $R(s)$ should be each of $\sqrt{2}^s, 2^s, 4^s$.

The probability space $G(n,p)$ is defined on the set of all graphs on $\{1,\ldots,n\}$ by choosing each edge independently to be present with probability p, absent otherwise. So e.g. in the proof of theorem 1 we worked in $G(n,\frac{1}{2})$. It can be useful to consider $p \ne \frac{1}{2}$.

Recall that in Zarankiewicz we had $z(n,t) \le 2n^{2-\frac{1}{t}}$ (for t fixed and n large; recall $z(n,t)$ is the size of the largest bipartite graph on n vertices containing no $K_{t,t}$). We would like a lower bound better than the trivial linear bound. We could choose a random bipartite graph G on vertex classe X, Y with $|X| = |Y| = n$ by choosing each edge independently with probability p and choosing p to make the expected number of $K_{t,t}$s < 1; now the number of possible $K_{t,t}$s is $\binom{n}{t}^2$, and the probability a fixed $K_{t,t}$ is contained in G is p^{t^2}, so the expected number of $K_{t,t}$s in G is $\binom{n}{t}^2 p^{t^2} < \frac{1}{4}n^{2t}p^{t^2}$, so taking $p = n^{-\frac{2}{t}}$ we have the expected number of $K_{t,t}$s being $< \frac{1}{4}$, so the probability that G has no $K_{t,t}$ is $> \frac{3}{4}$. Now the expected number of edges is pn^2; then $P(e(G) > \frac{1}{2}pn^2) > \frac{1}{2}$, so there must be some $G \not\supseteq K_{t,t}$ with $e(G) > \frac{1}{2}pn^2 = \frac{1}{2}n^{2-\frac{2}{t}}$. However, we can do better:

6.2 Theorem

$Z(n,t) > \frac{1}{2}n^{2-\frac{2}{t+1}}$ - this is an improvement worth making over the result in the discussion above, because even though it is only important in the small t cases, the small t cases are quite interesting. The idea behind this proof is that if a graph G has m edges and r copies of $K_{t,t}$, we can remove an edge from each coopy f$K_{t,t}$ to obtain a graph with $\ge m-r$ edges and no $K_{t,t}$, so $Z(n,t) \ge m-r$. Choose a random bipartite G (on vertex classes X, Y with $|X| = |Y| = n$), by taking each edge independently with probability p. Let $M = e(G)$ (a random variable) and R be the number of $K_{t,t}$s in G. Then $E(M) = pn^2$ and $E(R)$ is the number of possible $K_{t,t}$s times the probability that a fixed $K_{t,t}$ is contained in G, which comes to $\binom{n}{t}^2 p^{t^2}$. So $E(M-R) = pn^2 - \binom{n}{t}^2 p^{t^2} \ge pn^2 - \frac{1}{2}n^{2t}p^{t^2}$ (recall expectation is linear $E(A+B) = E(A) + E(B)$, without needing to assume the random variables are independent), so take $p = n^{-\frac{2}{t+1}}$, then we have $E(M-R) \ge n^{2-\frac{2}{t+1}} - \frac{1}{2}n^{2t-\frac{2t^2}{t+1}}$; the exponent in the last term is $\frac{2t^2?2t-2t^2}{t+1}$ so our result is $\frac{1}{2}n^{2-\frac{2}{t+1}}$, so there is a graph G with $m-r \geq \frac{1}{2}n^{2-\frac{2}{t+1}}$, and we have the result. This technique is called "modifying a random graph".

Graphs with large $\chi(G)$

To make $\chi(G) \geq k$, we can just take $G \supset K_k$. But we need not have a $K_k \subset G$ to have $\chi(G) \geq k$, by e.g. $G = C_5$. In fact we can have $\chi(G)$ much larger than the clique number. For example, in Theorem 1, the graph G (i.e. the graph consisting of the red edges) was on $n = 2^{\frac{s}{2}}$ vertices with no $K_s \subset G$, and no independent (definition: containing no edges) set of size s, but for any colouring each class is an independent set, so $\chi(G) \geq \frac{2^{\frac{s}{2}}}{s-1}$, which is much greater than the clique number of at most $s-1$. In fact, we can construct triangle-free graphs G with $\chi(G)$ large, though this is not easy.

Could we have large girth, but still have $\chi(G)$ large (e.g. a triangle-free graph has girth at least 4), e.g. girth at least 10 but $\chi(G) \geq 100$? The lecturer claims this appears unlikely; however:

6.3 Theorem

$\forall k, g$, there is a graph G with $\chi(G) \geq k, \mathrm{Girth}(G) \geq g$. The idea is to find a graph G on n vertices such that the number of short cycles (i.e. cycles of length $< g$ is $\leq \frac{n}{2}$ and each independent set has size at most $\frac{n}{2k}$; then we are done, as by removing a vertex from each sthort cycle we obtain a graph H with girth at least g and $\chi(H) \geq \frac{\frac{n}{2}}{\frac{n}{2k}} = k$. So, choose a random $G \in G(n,p)$ where $p = n^{-1+\frac{1}{g}}$ (we will see why this was chosen later). Let X_i be the number of i-cycles in G for $3 \leq i \leq g-1$, and X be the number of cycles of length $< g$, $= \sum X_i$. Then $E(X_i) \leq n^i p^i$, since there are $\leq n^i$ possible i-cycles, each of which would need i edges to form. So $E(X) \leq \sum_{i=3}^{g-1} (np)^i = \sum_{i=3}^{g-1} n^{\frac{i}{g}} \leq gn^{\frac{g-1}{g}} = n\frac{g}{n^{\frac{1}{g}}} < \frac{n}{4}$ for n large, since $\frac{g}{n^{\frac{1}{g}}} \to 0$. Thus $P(X \leq \frac{n}{2}) > \frac{1}{2}$ (as otherwise we have $E(X) \geq \frac{n}{4}$). Let $t = \frac{n}{2k}$ (wlog taking n a multiple of $2k$) and let Y be the number of [sets of size t] that are independent. Then $E(Y) = \binom{ns}{t}(1-p)^{\binom{t}{2}}$, as there are $\binom{n}{t}$ possible t-sets and $\binom{t}{2}$ possible edges in each. This is $\leq n^t e^{-p\binom{t}{2}}$, as $1-x < e^{-x} \forall x \in \mathbb{R}$. In turn this is $\leq \exp(\frac{n}{2k}\log n - n^{-1+\frac{1}{g}}\frac{n^2}{8k^2}) \to 0$ as $n \to \infty$ (because $n \log n - n \times n^{\frac{1}{g}} \to -\infty$), so $E(Y) < \frac{1}{2}$ for n large, so $P(Y=0) > \frac{1}{2}$ and $\exists G \in G(n,p)$ with $X \leq \frac{n}{2}$ and $Y = 0$, as required.

The Structure of a Random Graph

What does a "typical" random graph $G \in G(n,p)$ look like? How do the properties of G vary as p varies?

One interesting question is: does G have no isolated vertices? We would expect the probability of this to be some function increasing in p, but how? In fact we find a "threshold effect"; the probability remains very small below some critical value of p, then rapidly climbs and becomes very close to 1. The "jump" or "threshold" must happen below $p =$constant, since for any constant p, the probability a given vertex is isolated becomes exponentially (in n) small, so the probability of an isolated vertex is also small. So we ask: where does the jump happen?

Probability Digression

Let X be a random variable taking values in $0, 1, 2, \ldots$. To show $P(X = 0)$ is large we show the mean $\mu = E(X)$ is small, as for any t we have $\mu \geq P(X \geq t)t$, so $P(X \geq t) \leq \frac{\mu}{t}$, a property sometimes called "Markov". So $P(X \geq 1) \leq \mu$, so $P(X = 0) \geq 1 - \mu$.

To show $P(X = 0)$ is small, it is not enough to show that μ is large, e.g. $X = 0$ with probability 0.999, 10^{10} with probability 0.001. So we must look at the variance $V = E((X - \mu)^2) = E(X^2) - E(X)^2$. For any t, $P(|X - \mu| \geq t) = P(|X - \mu|^2 \geq t^2) \leq \frac{V}{t^2}$, by Markov; this result is sometimes called "Chebyshev". Thus $P(|X - \mu| \geq \mu) \leq \frac{V}{\mu^2}$; in particular $P(X = 0) \leq \frac{V}{\mu^2}$. So to show $P(X = 0)$ is small, we show $\frac{V}{\mu^2}$ is small.

Suppose X is the number of some event A which occur. Then $\mu = E(X) = \sum_A P(A)$ (note there is no need to assume the A are independent). For variance, we have $E(X)^2 = \sum_{A,B} P(A)P(B)$ (i.e. $\sum_A \sum_B P(A)P(B)$), and $E(X^2) = \sum_{A,B} P(A \cap B)$ (since e.g. $X = \sum_A I_A \therefore X^2 = \sum_{A,B} I_A I_B = \sum_{A,B} I_{A \cap B}$), which is $\sum_{A,B} P(A)P(B|A)$, so the variance $V = \sum_{A,B} P(A)(P(B \mid A) - P(B))$ (note that many terms in this sum are zero, e.g. any terms where A, B are independent).

6.4 Theorem

Let λ be constant, and $G \in G(n,p)$ where $p = \lambda \frac{\log n}{n}$. Then if $\lambda < 1$, then G almost surely has an isolated vertex, and if $\lambda > 1$ then G almost surely has no isolated vertices, where by "almost surely" we mean "with a probability which $\to 1$ as $n \to \infty$" (i.e. $p = \frac{\log n}{n}$ is a "threshold" for the property of having an isolated vertex): let X be the number of isolated vertices of G. Then $\mu = E(X) = n(1-p)^{n-1} = \frac{n}{1-p}(1-p)^n$. For $\lambda > 1$ (the easy case) we have $\mu \leq \frac{n}{1-p} e^{-pn} = \frac{n}{1-p} e^{-\lambda \log n} = \frac{n^{1-\lambda}}{1-p}$, which $\to 0$ as $n \to \infty$ since $\lambda > 1$, so we certainly have $P(X = 0) \to 1$. For $\lambda < 1$ we have $1 - p \geq e^{-(1+\delta)p}$ for any $\delta > 0$, for p sufficiently small (which we will have, since $\frac{\log n}{n} \to 0$), so $\mu \geq \frac{n}{1-p} e^{-(1+\delta)pn} = \frac{n}{1-p} e^{-(1+\delta)\lambda \log n} = \frac{n^{1-(1+\delta)\lambda}}{1-p}$; choosing δ such that $(1+\delta)\lambda < 1$, we have $\mu \to \infty$ as $n \to \infty$. Now for the variance, $V = n(1-p)^{n-1}(1 - (1-p)^{n-1}) + n(n-1)(1-p)^{n-1}((1-p)^{n-2} - (1-p)^{n-1})$, since there are n terms in which $A = B$ and $n(n-1)$ terms in which $A \neq B$ in our expression above for V. This is $\leq \mu + n(n-1)(1-p)^{n-1} p(1-p)^{n-2} \leq \mu + \frac{p}{1-p} n^2 (1-p)^{2n-2} = \mu + \frac{p}{1-p}\mu^2$, so $\frac{V}{\mu^2} \leq \frac{1}{\mu} + \frac{p}{1-p} \to 0$ as $n \to \infty$, so $P(X = 0) \to 0$.

A different kind of "threshold" effect is observed in the clique number of a random graph. Let p be fixed, $0 < p < 1$; then what is the distribution of the clique numbers of $G \in G(n,p)$? We would guess some form of Gaussianesque distribution for the probability of a clique number between 1 and n, but in fact there is an integer a such that the clique number is almost surely a or $a + 1$

6.5 Theroem

Let $0 < p < 1$ be fixed, and let d be a real number with $\binom{n}{d}p^{\binom{d}{2}} = 1$, which must exist by continuity. Then $G \in G(n,p)$ almost surely has clique number $\lfloor d \rfloor$ or $\lceil d \rceil$ or $\lfloor d \rfloor - 1$ (with more work, we can eliminate this last possibility). The proof is non-examinable, since it is horrible; the following is a sketch, the "easily" checked sections are done by expansion and repeated application of Stirling's formula:

Fix an integer k and let X be the number of K_ks in G, so $\mu = E(X) = \binom{n}{k}p^{\binom{k}{2}}$. We need to show that if $k \leq d-1$ then almost surely $X \neq 0$ and if $k \geq d+1$ then almost surely $X = 0$. If $k \geq d+1$, $\mu = \binom{n}{k}p^{\binom{k}{2}}$ which can be easily checked to $\to 0$, so $P(X=0) \to 1$. If $k \leq d-1$, $\mu = \binom{n}{k}p^{\binom{k}{2}}$ can be easily checked to $\to \infty$; now $V = \binom{n}{k}p^{\binom{k}{2}} \sum_{s=2}^{k} \binom{k}{s}\binom{n-k}{k-s}(p^{\binom{k}{2}-\binom{s}{2}} - p^{\binom{k}{2}})$ ($\binom{k}{s}\binom{n-k}{k-s}$ is the number of B meeting A in S points; the $s = 0, 1$ cases give independent events, so those terms are 0), so $\frac{V}{\mu^2} \leq \frac{1}{\mu}\sum_{s=2}^{k}\binom{k}{s}\binom{n-k}{k-s}p^{\binom{k}{2}-\binom{s}{2}}$, which can be checked to be dominated by the first and last terms (i.e. it is $\leq$ a constant times the sum of the first and last terms), so $\leq \lambda(\binom{k}{2}\binom{n-k}{k-2}p^{\binom{k}{2}-1} + 1)$, and the first term in this bracket is at most a constant by the definition of k [check this - I couldn't bear to - it might only grow slower than n], so the whole thing is a constant, and $\frac{V}{\mu^2} \to 0$, so $P(X=0) \to 0$.

7 Algebraic Methods

The diameter of a graph G is $\max_{x,y \in G} d(x,y)$. How "big" can a graph of small diameter be? If G has diameter 1 it must be complete, but if G has diameter 2, how many vertices can it have if it has max degree Δ?

Expanding out from a fixed point x, we see that $|\Gamma(x)| \leq \Delta$, so $|G| \leq 1 + \Delta + \Delta(\Delta - 1)$ (since it consists only of x, at most Δ neighbours therof, and at most $\Delta - 1$ other neighbours of each of these), i.e. $1 + \Delta^2$. If $|G| = 1 + \Delta^2$, then G must be regular; a k-regular graph of diameter 2 on $n = 1 + k^2$ vertices is called a Moore graph (of diameter 2) (equivalently this is a k-regular graph for which $\forall x \neq y \exists !$ path of length ≤ 2 from x to y).

For $k = 2$, C_5 is such a graph; for $k = 3$ the Petersen graph (as seen on the example sheets; the graph consists of a pentagon, a five-pointed star inside it, and five "radial" edges).

For $k = 4$, we find it is impossible to make such a graph. For $k > 4$ it "ought" to remain impossible, but how can we prove this?

For G a graph on vertex set $\{1, \ldots, n\} = [n]$, the adjacency matrix is the $n \times n$ matrix A with $A_{ij} = 1$ if $ij \in E(G)$, 0 otherwise. Clearly A is real and symmetric. The matrix A contains all the information of G, but we might ask: what is the use of viewing it as a matrix? Consider A^2: $(A^2)_{ij} = \sum_k A_{ik}A_{kj}$, which we can see is the number of walks of length 2 from i to j; similarly $(A^3)_{ij}$ is the number of walks of length 3 from i to j, and so on.

$x \mapsto Ax$ is a linear map $\mathbb{R}^n \to \mathbb{R}^n$; thus $(Ax)_i = \sum_j A_{ij}x_j = \sum_{j \in \Gamma(i)} x_j$, so if x is some vector we can draw the graph with numbers x_1 at point 1, x_2 at point 2 etc. Then to calculate Ax, each $(Ax)_j$ is just the sum of the values at the neighbours of j.

So e.g. if A is k-regular then we have $(1,\ldots,1) \mapsto (k,\ldots,k)$, i.e. $(1,\ldots,1)$ is an eigenvector with eigenvalue k. So we look at eigenvectors and eigenvalues. Since A is real and symmetric, it is diagonalisable, and has a basis of eigenvectors; say we have eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and correspond eigenvectors $e_1,\ldots,e_n$ (which we may take to form an orthonormal basis if we like). We often write $\lambda_{\max}$ for λ_1 and $\lambda_{\min}$ for λ_n. Note that $\sum \lambda_i = \mathrm{tr}A = 0$, so $\lambda_{\max} > 0$ and $\lambda_{\min} < 0$ (unless $G = E_n$). To find the eigenvalues is in principle easy in any case, but in practice it is best to "stay awake" rather than just "ploughing through". For example, $G = C_4$ has $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, so $\mathrm{rk}A = 2$ and we have 0 as a double eigenvalue. Then clearly $A(1,1,1,1) = (2,2,2,2)$ from the graph, so 2 is an eigenvalue; the last eigenvalue must be -2, either because the sum of the eigenvalues is 0, or because $A(1,-1,1,-1) = (-2,2,-2,2)$.

Take $x \in \mathbb{R}^n$, say $x = \sum c_i e_i$, with $\|x\| = 1$, i.e. $\sum c_i^2 = 1$. Then $Ax = \sum c_i \lambda_i e_i$, so the inner product (x, Ax) [lecturer's notation for $x \cdot Ax$] is $\sum \lambda_i c_i^2$. So $\min_{\|x\|=1}(x, Ax) = \lambda_{\min}$ (attained by $c_n = 1$ and the other c_i being 0), $\max_{\|x\|=1}(x, Ax) = \lambda_{\max}(\star)$.

7.1 Proposition

For any graph G, i) if λ is an eigenvalues then $|\lambda| \leq \Delta$ ($= \Delta(G)$: choose an eigenvector x for λ and choose i with $|x_i|$ maximal; wlog $x_i = 1$. Then $(Ax)_i = \sum_{j \in \Gamma(i)} x_j$, so $|(Ax)_i| \leq \Delta$, so $|\lambda| \leq \Delta$ ii) for G connected, Δ is an eigenvalue iff G is regular; for the reverse implication let $x = (1,\ldots,1)$, then $Ax = (\Delta,\ldots,\Delta)$. For the forward, from [the proof of] i) we must have $d(x_i) = \Delta$ and $x_j = 1 \forall j \in \Gamma(i)$, then we can repeat this for each $k \in \Gamma(j)$, etc. to get $d(x_k) = \Delta \forall k$ since G is connected iii) for G connected, $-\Delta$ is an eigenvalue iff G is regular and bipartite: for the reverse implications let $x = (1,\ldots,1,-1,\ldots,-1)$ where the 1s are on one vertex class X of G and the -1s are on the other vertex class Y. For the forward, from i) we must have $d(i) = \Delta$ and $x_j = -1 \forall j \in \Gamma(i)$, then repeat for each $k \in \Gamma(j)$ and so on. So we have $d(j) = \Delta \forall j \in G$, and for every $jk \in E$, either $x_j = 1, x_k = -1$ or vice versa. So G is regular and has no odd cycle, so is bipartite. iv) $\lambda_{\max} \geq \delta$: let $x = (1,\ldots,1)$, then $(Ax)_i \geq \delta \forall i$, so $(Ax, x) \geq \delta n = \delta(x, x)$, so $\lambda_{\max} \geq \delta$ by $(\star)$.

Note that from ii), if Δ is an eigenvalue, it has multiplicity 1, as its only eigenvector is $(1,\ldots,1)$.

Eigenvalues can link in to aother graph paramaters, e.g. we know $\chi(G) \leq \Delta + 1$; we can strengthen this to:

7.2 Proposition

For any graph G, $\chi(G) \leq \lambda_{\max} + 1$: we induct on $|G|$, the $|G| = 1$ case is done. Given G with $|G| > 1$, choose $v \in G$ with $d(v) = \delta(G)$. Then we claim $\lambda_{\max}(G \setminus v) \leq \lambda_{\max}$; then we can colour $G \setminus v$ by the induction hypothesis, and then we can colour v, as $d(v) \leq \lambda_{\max}$ by iv), above. Proof of this claim: let B be A with the row and column corresponding to the vertex v removed; wlog these are the last row and column. For any $x = (x_1,\ldots,x_{n-1})$ put $y = (\vec{x}, 0)$, then $(Bx, x) = (Ay, y)$ and so $\lambda_{\max}(G \setminus v) \leq \lambda_{\max}$ by $(\star)$.

Now, we look towards Moore graphs.

A graph G is strongly regular with paramaters k, a, b if G is k-regular, any two adjacent points have exactly a common neighbours, and any two non-adjacent points have exactly b common neighbours, e.g. C_4 has $(2, 0, 2)$, C_5 has $(2, 0, 1)$, and in general a Moore graph of degree k is strongly regular with paramaters $(k, 0, 1)$ (for an example of a strongly regular graph with $a \neq 0$, the graph made by drawing 3 triangles and then joining the corresponding points of each has $(4, 1, 2)$).

7.3 Theorem (Rationality condition for strongly regular graphs)

Let G be a graph on n verticies, strongly regular with parameters (k, a, b), $b > 0$. Then the numbers $\frac{1}{2}(n - 1 \pm \frac{(n-1)(b-a)-2k}{\sqrt{(a-b)^2+4(k-b)}})$ are integers: G is connected as $b > 0$, so k is an eigenvalue with multiplicity 1. What are the other eigenvalues? We have $(A^2)_{ij} = k$ if $i = j$, a if $i \neq j$ and $ij \in E$, b if $i \neq j$ and $ij \notin E$. So $A^2 = kI + aA + b(J - I - A)$ where J is the $n \times n$ matrix all of whose entries are 1.

So $A^2 + (b-a)A + (b-k)I = bJ$. For $\lambda \neq k$ an eigenvalue with eigenvector x we have $x \perp (1, \ldots, 1)$, since we can take our eigenvectors to be orthogonal. So $Jx = 0$; applying both sides of this equation to x, $(\lambda^2 + (b-a)\lambda + (b-k))x = 0$, so $\lambda^2 + (b-a)\lambda + b - k = 0$. So the eigenvalues other than k are $\frac{a-b\pm\sqrt{(b-a)^2+4(k-b)}}{2}$; let these eigenvalues be λ, μ with respective multiplicities r, s. So we have $r + s = n - 1$ since there are n eigenvalues in total, and $\lambda r + \mu s = n - k$ since the eigenvalues must sum to 0. Solving these two equations for r and s gives the two numbers in the statement, but r and s are multiplicities of eigenvalues so must be integers.

7.4 Corollary

If there is a Moore graph of degree k, then $k \in \{2, 3, 7, 57\}$ (we have seen examples C_5 for $k = 2$, the Petersen graph for $k = 3$, and an example can be found by computer for $k = 7$. In the $k = 57$ case it is unknown whether an example exists). By the previous theorem applied to $k = k, n = k^2+1, a = 0, b = 1$, $\frac{1}{2}(k^2 \pm \frac{k^2-2k}{\sqrt{1+4(k-1)}})$ are integers, so either $k^2 - 2k = 0$ or $1 + 4(k-1) = 4k - 3$ is a square. In the first case we have $k = 2$; in the second say $4k - 3 = t^2$. Then we must have $t \mid (k^2 - 2k) = (\frac{t^2+3}{4})^2 - 2(\frac{t^2+3}{4})$; multiplying by 16, we must certainly have $t \mid (t^2 + 3)^2 - 8(t^2 + 3) = t^4 - 2t^2 - 15$, so $t \mid 15$. So we have four possibilities; $t = 1$ gives the impossible case $k = 1$, $t = 3$ gives $k = 3$, $t = 5$ gives $k = 7$ and $t = 15$ gives $k = 57$.

[This would appear to be the end of the course]