Cplx Anal

Profellor Sherlock

July 5, 2008

1

1.1 Introduction

The recommended book for this course is Ahlfors' excellent work; be wary, however, since while it appears to be "informal" each sentence has in fact been written very carefully; every word matters. This is perhaps the best book for one who wants to be good at mathematics. At the other end of the spectrum, Stewart-Tall's book is good for passing exams with, though somewhat pretentious and overcomplicated. In between, Jameson or Whittaker-Watson's now-amusingly titled "Modern Analysis" are good. The preious year's lecture notes are available online; however, do note that the schedules have changed; the lecturer has so far only noticed elements being removed but presumes something has also been added to the course.

1.2 Cplx Differentiation

Notations: $a \in \mathbb{C}, r > 0 \in \mathbb{R}$. Call $D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$, an open disc or ball, also called $B(a,r)$ or $\Delta(a,r)$; this is the cplx analogue of (a,b).

An open subset of $\mathbb{C}$ is a $U \subset \mathbb{C}$ st $\forall A \in U \exists r > 0 : D(a,r) \subset U$; in particular for any a, r, $D(a,r)$ is an open subset of $\mathbb{C}$ ($D(z, r-r') \subset D(a,r)$ where $r' = |z-a|$).

A curve (in $\mathbb{C}$) is a cnts map $\gamma : [a,b] \to \mathbb{C}$ for $[a,b]$ some closed interval in $\mathbb{R}$. General curves can be somewhat counterintuitive (e.g. space-filling curves, such as curves whose image is the entire unit square, exist). We say a curve γ is cntsly diffable or C^1 if $\gamma'(t)$ (at the endpoints, we take this to be the one-sided derivative) exists $\forall t \in [a,b]$ and is cnts.

An open set U is (path-) connected if $\forall z, w \in U \exists$ some curve $\gamma : [0,1] \to U : \gamma(0) = z, \gamma(1) = w$.

Without pf: if an open set U is path-connected then any 2 pts of it can be joined by a curve which is polygonal i.e. made up of [presumably finitely many, otherwise pointless] line segments.

Note $\mathbb{C} \cong \mathbb{R}$ by $(x,y) \mapsto x + iy$

Defn

A <u>domain</u> is a non-empty connected open subset of $\mathbb{C}$ e.g. a disk, or $\mathbb{C}$
some finite set of points .

We are generally interested in functions $f : U \to \mathbb{C}$ where U is an open set or a domain in $\mathbb{C}$.

Given such an f, we can write $f = u(x,y) + iv(x,y)$ where u, v are real-val'd funcs on the domain (identified w/ an open subset of $\mathbb{R}^2$).

Defn

i) $f : U \to \mathbb{C}$ is diffable @ $w \in U$ if $\lim_{z\to w} \frac{f(z)-f(w)}{z-w}$ exists (and in this case it is called the deriv $f'(w)$ of f @ w)

ii) $f : U \to \mathbb{C}$ is holomorphic on U if f is diffable @ every pt of U; we say f is holomorphic @ $w \in U$ if $\exists r > 0 : f$ diffable @ every pt of $D(w,r)$

Terminology

"holomorphic" is used synonymously with "regular", and also sometimes with "analytic".

Cplx differentiation obeys the same formal rules as real differentiation of functions of a single variable (sum, prod, quot, chain, rules, inverse function thm), by the same pfs. For example, a polynomial function $f(z) = \sum_{0\le n\le N} c_n z^n$ is diffable everywhere. A rational function $f(z) = \frac{P(z)}{Q(z)}$ for P, Q polys is diffable everywhere $Q(z) \neq 0$, so is holomorphic on the domain $\mathbb{C}$
{ zeroes of Q}. Also see the later section on power series.

Defn

An entire function is a holomorphic $f : \mathbb{C} \to \mathbb{C}$ e.g. polynomials, and also exp, sin and cos. Compare this with diffability of funcs $\mathbb{R} \to \mathbb{R}^2$; recall that a func $u : U \to \mathbb{R}$ (for $U \subset \mathbb{R}^2$ open) is diffable @ $(c,d) \in U$ if $\exists \lambda, \mu \in \mathbb{R} : \frac{u(x,y)-u(c,d)-\lambda(x-c)-\mu(y-d)}{\sqrt{(x-c)^2+(y-d)^2}} \to 0$ as $(x,y) \to (c,d)$, and if so $Du(c,d)$ defined by $(\lambda,\mu) \in \mathbb{R}^2$ is the derivative of u at (c,d).

1.2.1 T: "Cauchy-Riemann Eqns"

$f : U \to \mathbb{C}$ is diffable @ $w = c + id \in U$ iff the functions $u, v : U \to \mathbb{R}$ are diffable @ (c,d) (as funcs of two real vars) and $u_x(c,d) = v_y(c,d), u_y(c,d) = -v_x(c,d)$. If so, then $f'(w) = u_x(c,d) + iv_x(c,d) = v_y(c,d) - iu_y(c,d)$: from the defn f will be diffable @ $w = (c,d)$ w/ deriv $f'(w) = p + iq$ iff $\lim_{z\to w} \frac{f(z)-f(w)-f'(w)(z-w)}{|z-w|} = 0$ or equivalently (considering the real and imaginary parts separately)

$$\lim_{(x,y)\to(c,d)} \frac{u(x,y)-u(c,d)-p(x-c)+q(y-d)}{\sqrt{(x-c)^2+(y-d)^2}} = 0$$

and

$$\lim_{(x,y)\to(c,d)} \frac{v(x,y)-v(c,d)-q(x-c)+p(y-d)}{\sqrt{(x-c)^2+(y-d)^2}} = 0$$

i.e. iff u is diffable @ (c,d) w/ deriv $(p,-q)$ and v is diffable @ (c,d) w/ deriv (q,p) and we are done.

Defn

The exponential function is $\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$

Prop 1.3.3

i) $\exp(z)$ is an entire function and $\frac{d}{dz}\exp(z) = \exp(z)$; Use the previous Thm, STP that series has rad of conv ∞, but since $\frac{\frac{z^n}{n!}}{\frac{z^{n-1}}{(n-1)!}} = \frac{z}{n} \to 0$ as $n \to \infty \forall z \in \mathbb{C}$, by the ratio test the series converges $\forall z$; differentiating term by term, $\exp' = \exp$

ii) $\exp(z + w) = \exp(z)\exp(w)$; let $a = w + z, g(z) = \exp(a - z)\exp(z)$; this is holomorphic on $\mathbb{C}$ with derivative $-\exp(a - z)\exp(z) + \exp(a - z)\exp(z) = 0$ so must be constant, and thus $= g(0) = \exp(a)$, i.e. $\exp(w + z) = \exp(w)\exp(z)$; putting $w = -z$ we have $\exp(z)\exp(-z) = 1$ so $\exp(z) \neq 0 \forall z$

iii) If $z = x + iy, x, y \in \mathbb{R}, \exp(z) = e^x(\cos y + i \sin y)$ since it is $\exp(x)\exp(iy)$ by ii), then compare with the Taylor series for $e^x, \cos y, \sin y$

iv) $\exp(z) = \exp(w) \Leftrightarrow z - w = 2\pi in$ some $n \in \mathbb{Z}$; by ii), STP $\exp(z) = 1 \Leftrightarrow z = 2\pi in$; let $z = x + iy$, then $\exp(z) = 1 \Leftrightarrow e^x(\cos y + i \sin y) = 1$ and we have the result

v) $\forall w \neq 0 \in \mathbb{C} \exists z \in \mathbb{C} : \exp(z) = w$; this also follows from iii) by expressing w in "polar co-ordinates"

Rk

One can also define the trigonometric and real exponential functions in terms of this definition of the complex exponential and these properties, and then prove their more usual definitions.

Ex

Define for $z \in \mathbb{C}, \sin(z) = \frac{\exp(iz)-\exp(-iz)}{2i}, \cos(z) = \frac{\exp(iz)+\exp(-iz)}{2}$. From ii) we can then easily deduce $\sin(z + w) = \sin z \cos w + \cos z \sin w$ etc.

From now on we shall use the standard (abuse of) notation that $e^z = \exp(z)$.

Logarithm

If $z \in \mathbb{C}$ we say $w \in \mathbb{C}$ is a logarithm of z if $e^w = z$; either z has no logarithm ($z = 0$) or z has infinitely many ($z \neq 0$), since if $e^w = z$ then $e^{w+2\pi in} = z \forall n \in \mathbb{Z}$. In general there is no "canonical" choice for a logarithm of z, so $\log(z)$ is a "multivalued function" of $z \in \mathbb{C} \setminus \{0\}$.

It is often necessary to select a particular logarithm. If $U \subset \mathbb{C}, 0 \notin U$ by a branch of the logarithm we mean a cnts (in fact holomorphic) fn $l : U \to \mathbb{C}$ s.t. $\forall z \in U, l(z)$ is a logarithm of z (or equivalently $\exp(l(z)) = z \forall z \in U$. A standard choice is $U = \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$; we define the principal branch of the logarithm to be the func $\mathrm{Log} : U \to \mathbb{C}$ by $\mathrm{Log}(z) = \ln|z| + i\arg(z)$ where we take $\arg(z) \in (-\pi, \pi)$.

Prop 1.3.4

i) $\forall z \in U, \exp(\mathrm{Log}(z)) = z$ by 1.3.3; $\exp(\mathrm{Log}(z)) = e^{\ln|x|}(\cos \arg z + i \sin \arg z) = z$

ii) $\mathrm{Log}(z)$ is holomorphic on U w/ deriv $\frac{1}{z}$: $\mathrm{Log}(z)$ is cnts on U (since $|z|, \arg z$ are) so we can apply the formula for the derivative of an inverse fn; $1 = \frac{d}{dz}(\mathrm{Log}\, z)\exp(\mathrm{Log}\, z) = \frac{d}{dz}(\mathrm{Log}\, z)z$

iii) $\forall z$ with $|z| < 1$, $\mathrm{Log}(1+z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$: let $l(z) = \sum_{n\geq 1}(-1)^{n-1}\frac{z^n}{n}$; this has rad of conv 1 by ratio test. $l(z)$ is holomorphic on $D(0,1)$ and $l'(z) = \sum_{n\geq 1}(-1)^{n-1}z^{n-1} = \frac{1}{1+z} = \frac{d}{dz}(\mathrm{Log}(1+z))$ so $l(z) - \mathrm{Log}(z)$ is constant on $\{|z| < 1\}$; putting $z = 0$ we find this constant is 0

It is natural to try to extend $\mathrm{Log}(z)$ to a cns func on $\mathbb{C}\backslash\{0\}$ but this is impossible since e.g. $\lim_{\theta\to\pi-}\mathrm{Log}(e^{i\theta}) = \lim_{\theta\to\pi-} i\theta = i\pi$ but $\lim_{\theta\to\pi+}\mathrm{Log}(e^{i\theta}) = \lim_{\theta\to\pi+} i(\theta-2\pi) = -i\pi$. More generally we will see there is no branch of the logarithm on any set of the form $\{z\in\mathbb{C} : 0 < |z| < r\}$. We say that $z = 0$ is a branch point for the logarithm function.

There is a similar picture for "functions" $z \mapsto z^{\frac{1}{n}}$; there is no canonical choice of the nth root of $z \neq 0$ if $n > 1$. The simplest way to deal with this is:

Defn

For $a \in \mathbb{C}$ the principal branch of z^α is $z^\alpha = \exp(\alpha \operatorname{Log} z), z \in U = \mathbb{C}\setminus\{x\in\mathbb{R} : x \leq 0\}$. When $\alpha\in\mathbb{Z}$ this gives the correct function z^α since $\exp(\operatorname{Log} z) = z$.

E.g.

$\alpha = \frac{1}{2}, z^{\frac{1}{2}} = |z|^{\frac{1}{2}}e^{\frac{1}{2}i\arg(z)}$ for $-\pi < \arg(z) < \pi$. From the defn z^α is holomorphic on U w/ $\frac{d}{dz}z^\alpha = \alpha\frac{1}{z}\exp(\alpha\operatorname{Log} z) = \alpha\exp(-\operatorname{Log} z)\exp(\alpha\operatorname{Log} z) = \alpha\exp(\alpha-1)\operatorname{Log} z = \alpha z^{\alpha-1}$. Note that it is **not** generally the case that $(zw)^\alpha = z^\alpha w^\alpha$, or that $\mathrm{Log}(zw) = \mathrm{Log}(z) + \mathrm{Log}(w)$.

1.3 Conformal Mapping

Say $U\subset\mathbb{C}$ open, $w \in C$, $f : U\to\mathbb{C}$ holomorphic.

Suppose $f'(w)\neq 0$. Let $\gamma : [-1,1]\to U$ be a simple $(\gamma(t_1)\neq\gamma(t_2)\forall t_1\neq t_2)$ C curve w/ $\gamma(0) = w, \gamma'(0)\neq 0$. Let $\gamma'(0) = r(\cos\phi + i\sin\phi)$; ϕ is then the angle of the tangent to γ at w. Let δ be the image of γ under f, i.e. $\delta(t) = f(\gamma(t))$, then $\delta'(t) = \gamma'(t)f'(\gamma(t))$ and $\delta'(0) = \arg(\gamma'(0)) + \arg(f'(w)) + 2\pi n$ some $n\in\mathbb{Z}$, i.e. the angle δ makes at w is the angle γ makes at $w+$ a constant indep of γ; the mapping f preserves angles at w. This is said to be conformal

A particular important case is when $f : U\to\mathbb{C}$, f holomorphic on U, $f'\neq 0$ on U and f is a bijection $U\to f(U)$; we say f is a conformal equivalence between U and $f(U)$, or sometimes simply a conformal mapping.

Exercise: Möbius transformations $f(z) = \frac{az+b}{cz+d}, a,b,c,d\in\mathbb{C}, ad-bc\neq 0$ are conformal mappings from the riemann sphere $\mathbb{C}\cup\{\infty\}$ to itself.

Exercise: $n\geq 1$, $f(z) = z^n$ is a conformal equivalence between $\{z\in\mathbb{C}\setminus\{0\} : 0 < \arg(z) < \frac{\pi}{n}\}$ and $\{z\in\mathbb{C} : \mathrm{Im}(z) > 0\}$.

$\exp z = \exp w \Leftrightarrow z = w + 2\pi i n$; $z\mapsto \exp z$ is a conformal equivalence between $\{z\in\mathbb{C} : -\pi < \mathrm{Im}(z) < \pi\}, \mathbb{C}\setminus\{x\in\mathbb{R} : x\leq 0\}$.

Using such functions we can build quite complex conformal equivalences; the important Riemann Mapping Theorem implies if £D£ is a domain in $\mathbb{C}$ bounded by a simple closed curve $\exists$ a conformal equivalence $f : D\to D(0,1)$.

2 Complex Integration

2.1 Integral along curve

If $f : [a,b] \to \mathbb{C}$ is a (for now, continuous) cplx-vald fn on a real interval, def $\int_a^b f(x)dx := \int_a^b \mathrm{Re}(f(x))dx + i\int_a^b \mathrm{Im}(f(x))dx$.

Prop 2.1.1

For $a < b$ and f cnts, $\left|\int_a^b f(x)dx\right| \leq (b-a)\sup_{a\leq x\leq x}|f(x)|$ w/ equality iff $f(x)$ is constant: let $\theta = \arg\int_a^b f(x)dx$ (if the integral is 0 we are done). Let $M = \sup|f(x)|$. $|\int_a^b f(x)dx| = e^{-i\theta}\int_a^b f(x)dx = \int_a^b \mathrm{Re}(f(x)e^{-i\theta})dx$ since the imaginary part is 0. But this is $\leq \int_a^b |f(x)|dx \leq (b-a)M$. For f not identically 0 equality holds iff both these inequalities are equalities; the second is an equality iff $|f(x)| = M\forall x$ i.e. $|f|$ is constant, and the first is iff $e^{-i\theta}f(x) = |f(x)|$, i.e. $\theta = \arg f(x)$ so $\arg f$ is also constant and f is constant.

Let $\gamma : [a,b] \to \mathbb{C}$ cntsly diffable and $\gamma(t) = x(t) + iy(t)$, then $|\gamma'(t)| = \sqrt{x'^2 y'^2}$, so reasonable to define length$(\gamma) = \int_a^b |\gamma'(t)|dt$, since for γ a simple curve this is just the length of its image in $\mathbb{C}$.

Defn

Let $f : U \to \mathbb{C}$ cnts on an open $U \subset \mathbb{C}$ and $\gamma : [a,b] \to U$ a C^1 curve. Def $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$.

Basic properties

Linearity: $\int_\gamma c_1 f_1(z) + c_2 f_2(z)dz = c_1\int_\gamma f_1(z)dz + c_2\int_\gamma f_2(z)dz$.

Additivity: for $a < a' < b, \gamma_1 : [a,a'] \to U, \gamma_2 : [a',b] \to U$ defd by $\gamma_i(t) = \gamma(t)$ have $\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$.

Inverse path: for $\gamma : [a,b] \to U$ def $-\gamma : [-b,-a] \to U$ by $(-\gamma)(t) = \gamma(-t)$, then $\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$.

Reparamaterisation: if $\phi : [a',b'] \to [a,b]$ is C^1 and $\phi(a') = a, \phi(b') = b$, if $\delta = \gamma\circ\phi$ ($\delta(t) = \gamma(\phi(t))\forall t \in [a',b']$) then $\int_\gamma f(z)dz = \int_\delta f(z)dz$ (this is analagous to change of variables in conventional integration). In particular, since we can always find $\phi : [0,1] \to [a,b]$ we can restrict our attention to curves $\gamma : [0,1] \to \mathbb{C}$.

Let $\gamma : [a,b] \to \mathbb{C}$ be cnts and suppose have $a = a_0 < a_1 < \cdots < a_n = b$ s.t. on each interval $[a_{i-1}, a_i]$ γ is cntsly diffable, then we say γ is piecewise C^1 and can def $\int_\gamma f dz = \sum_i \int_{\gamma_i} f(z)dz$ where the $\gamma_i : [a_{i-1}, a_i] \to \mathbb{C}$ are defd by $\gamma_i(t) = \gamma(t)$; by the second property above this does not depend of the way γ has been decomposed.

It is convenient to "add" curves: if $\gamma : [a,b] \to \mathbb{C}, \delta : [c,d] \to \mathbb{C}$ are curves with $\gamma(b) = \delta(c)$ then we define $\gamma + \delta : [a, d-c+b] \to \mathbb{C}$ by $t \mapsto \gamma(t)$ for $a \leq t \leq b$, $\delta(t-b+c)$ for $b \leq t \leq d-c+b$, thus the above γ is simply $\gamma_1 + \cdots + \gamma_n$.

From now on by "curve" we shall mean "piecewise C^1 curve" unless otherwise stated.

Prop 2.1.2

If $f : U \to \mathbb{C}$ is cnts and $\gamma : [a,b] \to U$ is any curve then $\left|\int_\gamma f dz\right| \leq \text{length}(\gamma) \sup_\gamma |f|$ ($\sup_\gamma |f| = \sup_{t\in[a,b]} |f(\gamma(t))|$ exists since $t \mapsto f(\gamma(t))$ is a cnts map on the closed bounded interval $[a,b]$): by additivity we can wlog take γ C^1. Let $M = \sup_\gamma |f|$, then $\left|\int_\gamma f dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \leq \int_a^b |f(\gamma(t))\gamma'(t)|dt \leq M \int_a^b |\gamma'(t)|dt$, which is M length(γ) as required.

Prop 2.1.3

Suppose f, f_n (a sequence for $n \in \mathbb{N}$) are continuous functions on an open $U \subset \mathbb{C}$ and $\gamma : [a,b] \to U$ is a curve such that $f_n \to f$ uniformly on γ (by which we of course mean $\gamma([a,b]) \subset U$), then $\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz$: let $M_n = \sup_\gamma |f_n - f|$; by the def of uniform convergence $M_n \to 0$, and by the previous result $\left|\int_\gamma f - f_n dz\right| \leq M_n \text{ length}(\gamma) \to 0$ as $n \to \infty$.

Thm 2.1.4 (Fundamental Theorem of Calculus)

If $F : U \to \mathbb{C}$ is holomorphic (and F' continuous - but we will later (2.5.2) see this is automatically the case) and $\gamma : [a,b] \to U$ is any curve then $\int_\gamma F'(z)dz = F(\gamma(b)) - F(\gamma(a))$; in particular if γ is closed, that is, $\gamma(a) = \gamma(b)$, then $\int_\gamma F'(z)dz = 0$: take wlog γ C^1 by additivity, then $\int_\gamma F'(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)dt = \int_a^b (F(\gamma(t))'dt = F(\gamma(t)) |_a^b$.

A trivial corollary is that if f is the derivative of a holomorphic function then $\int_\gamma f dz$ depends only on the endpoints of γ.

Example

$f(z) = z^n, \gamma =$ a circle $\gamma(t) = Re^{2\pi i t}, t \in [0,1]$ for some $n \in \mathbb{Z}, R > 0$.If $n \neq -1, z^n = \frac{d}{dz}(\frac{z^n+1}{n+1})$ on $\mathbb{C} \setminus \{0\}$ (and even $\mathbb{C}$ for $n \geq 0$), which contains γ, so by the theorem, $\int_\gamma z^n dz = 0$ for $n \neq -1$. If $n = -1$ we don't know a holomorphic function on $\mathbb{C} \setminus \{0\}$ with derivative $\frac{1}{z}$ (the "obvious" choice, $\operatorname{Log} z$, is only holomorphic on $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$. So we instead compute the integral directly: $\int_\gamma \frac{1}{z}dz = \int_0^1 \frac{1}{Re^{2\pi i t}} 2\pi i Re^{2\pi i t}dt = 2\pi i$. Since this $\neq 0$, there cannot exist an F holomorphic on $\mathbb{C} \setminus \{0\}$ (or even on any open subset of $\mathbb{C}$ containing a circle about the origin $\{|z| = R > 0\}$) with derivative $\frac{1}{z}$, so there is no branch of the logarithm on these sets.

Theorem 2.1.5 (Converse of FTC)

If $f : D \to \mathbb{C}$ is cnts on a domain D and $\int_\gamma f dz = 0$ for every closed curve γ in D then f has an antiderivative on D, i.e. $\exists F : D \to \mathbb{C}$ holomorphic with $F' = f$ on D: pick $a_0 \in D$, and for each $w \in D$ pick γ_w some curve from a_0 to w (which we can do since domains are path-connected). Let $F(w) = \int_{\gamma_w} f dz =$. For any w, pick $r > 0$ so $D(w,r) \subset D$. For $|h| < r$, the line segment from w to $w + h$, $\delta_h : [0,1] \to \mathbb{C}$ given by $\delta_h(t) = w + th$ lies in D. By hypothesis the integral of f around the closed curve $\gamma_w + \delta_h + (-\gamma_{w+h})$ is 0 (and $F(w)$ is well defined, i.e. independent of the choice of γ_w). So $F(w+h) = \int_{\gamma_{w+h}} f(z)dz = \int_{\gamma_w} f(z)dz + \int_{\delta_h} f(z)dz = F(w) + \int_{\delta_h} f(z)dz = F(w) + hf(w) + \int_{\delta_h} f(z) - f(w)dz$ since $\int_{\delta_h} dz = h$.

$\left|\frac{F(w+h)-F(w)}{h} - f(w)\right| = \left|\frac{1}{h}\int_{\delta_h} f(z) - f(w)dz\right| \leq |h^{-1}| \operatorname{length}(\delta_h) \sup_{\delta_h} |f(z) - f(w)| \leq \sup_{|z-w|\leq h} |f(z) - f(w)| \to 0$ as $|h| \to 0$ since f is cnts, so $F'(w) = f(w)$.

We will also need a slight variant of this theorem:

Lemma 2.1.6

Let D be a disc (or, more generally, a convex or starlike domain: D is starlike if $\exists a_0 \in D : \forall z \in D$ the line segment $[a_0, z]$ lies in D, and convex if the line segment $[z_1, z_2]$ lies in $D \forall z_1, z_2 \in D$. Clearly all discs are convex and all convex domains are starlike. The only really interesting starlike domain is $\mathbb{C} \setminus \{x \leq 0\}$, the set on which the principal branch of the logarithm is defined). If $\int_\gamma f dz = 0$ for every triangle γ in D, then f has an antiderivative on D: Define $F(w)$ in the above as $\int_{\gamma_w} f(z)dz$ where γ_w is a line segment from a_0 to w, then the closed curve $\gamma_w + \delta_h + (-\gamma_{w+h})$ is a triangle so by hypothesis the integral of f around it is 0, then we proceed as above.

The astute reader will notice that combined with the FTC, this result implies that for a starlike domain, if the integral around any triangle is 0 then so is that around any closed curve.

2.2 Cauchy's Theorem for a disc

If $f : D \to D$ is holomorphic and $\gamma : [a, b] \to D$ is a closed curve, then under suitable conditions on $\gamma, D, \int_\gamma f dz = 0$.

There are various forms of this theorem depending on the nature of the condition; for now we shall proove a "local version of Cauchy's theorem", the case where D is a disc.

2.2.1 Theorem

If $f : U \to \mathbb{C}$ is holomorphic (for $U \subset \mathbb{C}$ open) and $\Delta \subset U$ a trinagle then $\int_{\partial\Delta} f dz = 0$; by Δ we mean the (solid) triangle with verticies $a, b, c, \partial\Delta$ is the boundary therof viewed as a simple closed curve; say wlog anticlockwise.

The proof of this is by bisection: let $L = \operatorname{Length}(\partial\Delta), I = \left|\int_{\partial\Delta} f(z)dz\right|$. Subdivide $\Delta = \bigcup_{i=1}^4 \Delta^{(i)}$ by bisecting the edges, and note $\sum_{i=1}^4 \int_{\partial\Delta^{(i)}} f(z)dz = \int_{\partial\Delta} f(z)dz$, since each of the internal edges is integrated along once in each direction. Therefore for some $1 \leq j \leq 4, \left|\int_{\partial\Delta^{(i)}} f(z)dz\right| \geq \frac{I}{4}$. Put $\Delta_1 = \Delta^{(j)}$, then similarly bisect Δ_1 to find Δ_2 and so on, so that we have $\Delta = \Delta_0 \supset \Delta_1 \supset \dots$ with $\operatorname{length}(\partial\Delta_n) = \frac{1}{2^n}L, \left|\int_{\partial\Delta_n} f(z)dz\right| \geq \frac{1}{4^n}I$. Consider $\bigcap_{n\geq 0} \Delta_n$ this is a single point $\{w\}$ (it is clearly at most one point since $\operatorname{length}(\Delta_n) \to 0$, and if we pick $w_n \in \Delta_n$ for each n the w_n form a Cauchy sequence so converge to some w, and since Δ_n is closed $\forall n, w \in \Delta_n \forall n$). Since f is holomorphic the function $g(z) = \frac{f(z)-f(w)}{z-w} - f'(w), g(w) = 0$ is a cnts func on D. So $4^{-n}I \leq \left|\int_{\partial\Delta_n} f(z)dz\right| = \left|\int_{\partial\delta_n} f(z) - f(w) - (z-w)f'(w)dz\right|$ since $\int_\gamma dz = \int_\gamma zdz = 0$ for any closed γ by FTC. So this is $\left|\int_{\partial\Delta_n}(z-w)g(z)\right| \leq \operatorname{length}(\partial\Delta_n).\ \sup_{\partial\Delta_n} |(z-w)g(z)| \leq 2^{-n}L \times 2^{-n}L \times \sup_{\Delta_n} g(z) \therefore I \leq L^2 \sup_{\Delta_n} |g(z)| \to 0$ as $n \to \infty$.

It is important to know this holds under an apparently weaker hypothesis:

2.2.2 Theorem

Let $S \subset U$ be a finite set, $f : U \to \mathbb{C}$ continuous on U and holomorphic on $U \setminus S$. Then for any triangle $\Delta \subset U$, $\int_{\partial\Delta} f(z)dz = 0$.

By subdividing Δ we can assume $S = \{a\}, a \in \Delta$. Let $M = \sup_\Delta |f|$. We can choose an arbitrarily small triangle Δ' such that $a \in \Delta', a \notin \partial(\Delta \setminus \Delta')$ (if a is on the edge of Δ, we choose Δ' against the same edge, and similarly). Subdividing Δ into subtriangles and using the previous result we have $\int_{\partial\Delta} f dz = \int_{\partial\Delta'} f dz$. As $\left|\int_{\partial\Delta'} f dz\right| \leq \text{length}(\partial\Delta')M$ and Δ' is arbitrarily smass, this means $\int_{\partial\Delta} f dz = 0$.

2.2.3 Cauchy's Theorem for a disc

Let D be a disc (or starlike domain) and $f : D \to \mathbb{C}$ holomorphic, except possibly on a finite set of points where it remains continuous. Then for every closed curve γ in D, $\int_\gamma f dz = 0$: by 2.2.2 $\int_\gamma f dz = 0$ for every triangle γ, so by 2.1.6 f has an antiderivative F on D, so by FTC $\int_\gamma f dz = 0$ for every closed γ.

2.3 Cauchy integral formula (1st form)

2.3.1 Theorem

Let $D = D(a, r)$ be a disc, $f : D \to \mathbb{C}$ holomorphic. For every $w \in D$ and ρ with $|w - a| < \rho < r$, $f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} dz$ where $\gamma(t) = a + \rho e^{2\pi i t}, t \in [0, 1]$; in an abuse of notation we call this integral $\int_{|z-a|=\rho} \frac{f(z)}{z-w} dz$.

Notice this means if ρ is fixed, the values of $f(w)$ for $|w - a| < \rho$ are determined by those for $|w - a| = \rho$.

To proove this we apply 2.2.3 to $g(z) = \frac{f(z)-f(w)}{z-w}, g(w) = f'(w)$ which is holomorphic on $D \setminus \{w\}$ and continuous at w. So $\int_\gamma \frac{f(z)}{z-w} dz = \sum_{n=0}^\infty f(w) \int_\gamma \frac{(w-a)^n}{(z-a)^{n+1}} dz = \sum_{n=0}^\infty f(w)(w - a)^n \int_{|z|=\rho} \frac{dz}{z^{n+1}} = 2\pi i f(w)$ since $\int \frac{dz}{z^n+1} = 2\pi i$ for $n = 0$ and 0 otherwise; this worked by using the geometric series $\frac{1}{z-w} = \frac{1}{(1-\frac{w-a}{z-a})(z-a)} = \sum_{n=0}^\infty \frac{(w-a)^n}{(z-a)^{n+1}}$. Note this series is uniformly continuous on $\{z \mid |z - a| = \rho\}$ so we can interchange integration and summation.

2.3.2 Corollary

If $f : D(w, R) \to \mathbb{C}$ holomorphic then $\forall r \in (0, R), f(w) = \int_0^1 f(w + re^{2\pi i t})dt$; this is sometimes called the "mean-value property" since it is the statement that $f(w) =$ the mean value of f on the circle $|z - w| = r$. The proof comes from simply putting $a = w$ in 2.3.1 and writing out the integral: $\int_\gamma \frac{f}{z-w} dz = \int_0^1 \frac{f(w+re^{2\pi i t})}{re^{2\pi i t}} 2\pi i r e^{2\pi i t} dt = 2\pi \int_0^1 f(w + re^{2\pi i t})dt$.

2.4 Applications

2.4.1 Theorem (Liouville's Theorem)

Every entire function which is <u>bounded</u> is constant: say f is a bounded entire function, i.e. $f : \mathbb{C} \to \mathbb{C}$ holomorphic and $|f(z)| < M \forall z \in \mathbb{C}$. If $w \in \mathbb{C}, R > |w|$ then $|f(w) - f(0)| = \frac{1}{2\pi} \left|\int_{|z|=R} \frac{f(z)}{z-w} - \frac{f(z)}{z} dz\right|$ by the CIF, $= \frac{1}{2\pi} \left|\int_{|z|=R} f(z) \frac{w}{z(z-w)} dw\right| \leq \frac{1}{2\pi} 2\pi RM \frac{|w|}{R(R-|w|)} \to 0$ as $R \to \infty$, so $f(w) = f(0)$ and f is constant.

This tells us that e.g. $\sin z$ is unbounded on $\mathbb{C}$, since it is an entire function.

2.4.2 Theorem (Fundamental Theorem of Algebra)

Every non-constant polynomial with complex coefficients has a root in $\mathbb{C}$. Let $P(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_0$, of degree $n > 0$. Then $|P(z)| \to \infty$ as $|z| \to \infty$, so $\exists R : |z| > R \Rightarrow |P(z)| > 1$.Consider $f(z) = \frac{1}{P(z)}$; if P has no zero then f is an entire function, $|f(z)| < 1$ for $|z| > R$ and f is continuous so bounded on the closed and bounded set $\{z : |z| \leq R\}$. Thus f is bounded on $\mathbb{C}$, so constant, a contradiction since P is non-constant.

2.4.3 Theorem (Local maximum modulus principle)

If $f : D(a,r) \to \mathbb{C}$ is holomorphic and $|f(z)| \leq |f(a)| \forall z \in D(a,r)$ then f is constant, as by corollary 2.3.2 for any $0 < \rho < r, |f(a)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(a+\rho e^{it})dt\right| \leq \sup_{|z-a|=\rho}|f(z)|$ with equality iff f is constant (proposition 2.1.1). By assumption this is $\leq |f(a)|$ so $t \mapsto f(a+\rho e^{it})$ is constant for every $\rho > 0$, and $|f(z)| = |f(a)| \forall z \in D(a,r)$ which implies f is constant (using the Cauchy-Riemann equations; see the example sheet).

2.5 Taylor Expansion

2.5.1 Theorem

If $f : D(a,r) \to \mathbb{C}$ is holomorphic then f has a convergent power series representation on $D(a,r)$, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ where $c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}}dz$ for any $0 < \rho < r$.

2.5.2 Corollary

If f is a holomorphic function on some open $U \subset \mathbb{C}$ then its derivatives of all orders exist and are holomorphic, as for $a \in U, \exists r > 0$ such that $D(a,r) \subset U$ and by the above theorem f can be represented by a convergent power series on $D(a,r)$, then apply theorem 1.3.2; in particular if f is holomorphic so is f' (as we assumed earlier).

Proof of theorem

By CIF, if $|w-a| < \rho < r$ then $f(w) = \frac{1}{2\pi i}\int_{|z-a|=\rho} \frac{f(z)}{z-w}dz$; recall $\frac{1}{z-w} = \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}$ and this is uniformly convergent for $|z-a| = \rho$ (see proof of CIF), so $f(w) = \frac{1}{2\pi i}\int \sum_{n=0}^{\infty} \frac{f(z)}{(z-a)^{n+1}}(w-a)^n = \sum_{n=0}^{\infty} c_n(w-a)^n$ where $c_n = \frac{1}{2\pi i}\int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}}dz$, since as f is bounded on $\{|z-a| = \rho\}$ the series is uniformly convergent so we can interchange integration and summation.

So for fixed $\rho, 0 < \rho < r$, f is represented by the convergent power series $\sum c_n(z-a)^n$ on $D(a,\rho)$, so it has derivatives of ll orders, with $f^{(n)}(a) = n!c_n$. So c_n is independent of the choice of ρ.

Remark

A function $f : U \to \mathbb{C}$ is said to be <u>analytic</u> if $\forall a \in U \exists D(a,r) \subset U$ such that f can be represented by a convergent power series on $D(a,r)$. The previous theorem shows that this is equivalent to being holomorphic (and many books use the terms interchangeably throughout); however, in <u>real</u> analysis a function $f : (a,b) \to \mathbb{R}^n$

is said to be real analytic if $\forall c \in (a,b) \exists (c-r, c+r) \subset (a,b)$ such that f can be represented on $(c-r, c+r)$ by a power series. Any such f is infinitely differentiable, but the converse is false: there exists (though it is a surprisingly difficult exercise to find such an f) an infinitely differentiable f on $(-1,1)$ such that the Taylor series $\sum \frac{f^{(n)}(0)}{n!}$ only converges at $x = 0$, and also infinitely differentiable $f : (-1,1) \to \mathbb{R}$ whose Taylor series converges but to a different function, e.g. $f(x) = e^{-\frac{1}{x^2}}, f(0) = 0$ has $f^{(n)}(0) = 0 \forall n$.

Henceforth we will use "analytic" and "holomorphic" interchangeably.

Our final application is a "converse" to Cauchy's Theorem:

2.5.3 Corollary (Movera's Theorem) (2.5.3)

Let $f : D = D(a,r) \to \mathbb{C}$ be continuous on a disc such that $\forall \gamma$ closed in D, $\int_\gamma f(z)dz = 0$. Then f is holomorphic: By 2.1.5 $f = F'$ for some holomorphic F, but then F has derivatives of all orders so so does f.

Application: Caroll (2.5.4)

For $U \subset \mathbb{C}$ open, $[a,b] \subset \mathbb{R}$ let $\phi : U \times [a,b] \to \mathbb{C}$ be continuous and such that $\forall s \in [a,b]$ the function $z \mapsto \phi(z,s)$ is holomorphic. Then $g(z) = \int_a^b \phi(z,s)ds$ is holomorphic on U: without loss of generality we can take U to be a disc. Let $\gamma : [0,1] \to U$ be a closed curve, then $\int_\gamma g(z)dz = \int_0^1 \left(\int_a^b \phi(\gamma(t), s)ds \right) \gamma'(t)dt$, but by lemma 2.5.5, below, this is $\int_a^b \left(\int_0^1 \phi(\gamma(t), s)\gamma'(t)dt \right) ds = \int_a^b \left(\int_\gamma \phi(z,s)dz \right) dz$, but the inner integral is 0 by Cauchy's theorem since ϕ is holomorphic for any fixed s. So by Movera's theorem g is holomorphic.

Lemma (2.5.5)

Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous, then the functions $f_1 : x \mapsto \int_c^d f(x,y)dz$, $f_2 : y \mapsto \int_a^b f(x,y)dx$ are continuous on $[a,b]$, $[c,d]$ respectively and $\int_a^b f_1(x)dx = \int_c^d f_2(y)dy$. This is a very simple form of Fubini's Theorem. f is continuous on a compact set so uniformly continuous, i.e. $\forall \epsilon \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x,y) - f(x_0,y)| < \epsilon$ i.e. $|f_1(x) - f_1(x_0)| < \epsilon(d-c)$ so f_1 is continuous, and similary so is f_2.

Now recall that a step function on $[a,b] \times [c,d]$ is a finite linear combination of characteristic functions of rectangles $[a',b'] \times [c',d'] \subset [a,b] \times [c,d]$ [note the steps can go down as well as up]. By the same argument as in the 1D case every continuous function on $[a,b] \times [c,d]$ is a uniform limit of step functions. But any step function is of the form $f(x,y) = g(x)h(y)$ for some step functions g, h, so clearly $\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$ for these and we are done.

Note that this result can be false for an f which is discontinuous at only one point.

Let $f : D(w,R) \to \mathbb{C}$ be holomorphic, and write it as a power series $\sum_{n=0}^\infty c_n(z-w)^n$ for $z \in D)w, R)$. If f is not identically zero (on this D) then we have $c_n \neq 0$ some n; let $m = \min\{n : c_n \neq 0\} \geq 0$. Then $f(z) = (z-w)^m g(z)$ where $g(z) = \sum_{n=0}^\infty c_{m+n}(z-w)^n$, holomorphic on $D(w,R)$ with $g(w) \neq 0$. If $m > 0$ we say f has a zero of order m at $z = w$. Clearly m is the least n such that $f^{(n)}(w) \neq 0$.

Theorem 2.5.6 (Principle of isolated zeroes)

Let $f : D(w,R) \to \mathbb{C}$ holomorphic and not identically zero, then $\exists r > 0$ such that $f(z) \neq 0$ for $0 < |z-w| < r$: if $f(w) \neq 0$ then by continuity $\exists r > 0 : |f(z) - f(w)| < |f(w)| \forall z \in D(w,r)$ so $f(z) \neq 0$ on $D(w,r)$; if $f(w) = 0$ then write $f(z) = (z-w)^m g(z)$ with $m > 0, g(w) \neq 0, g$ holomorphic and apply the previous argument to g.

There are essentially two branches to complex analysis; local analysis concerns the behaviour of functions on discs, with results such as Taylor's theorem, Cauchy's theorem, and the integral formula for discs, while global analysis looks at the behaviour of functions on more general domains, with results like Liouville's theorem and more general forms of Cauchy's theorem we shall see later. The relation between these two areas is given by analytic continuation.

2.6 Analytic Continuation

2.6.1 Theorem (Principle of analytic continuation)

Let $D' \subset D$ be domains and $f : D' \to \mathbb{C}$ be analytic, then there is at most one analytic $g : D \to \mathbb{C}$ such that $f(z) = g(z) \forall z \in D'$ (note that we use "analytic" rather than "holomorphic"; in $\mathbb{R}$ this result only holds for functions with power series, not general infinitely differentiable functions. For example, $f(x) = e^{-\frac{1}{x^2}}$ for $x > 0$ and 0 for $x \leq 0$ is infinitely differentiable and has $f^{(n)}(0) = 0 \forall n \geq 0$). Such a g (if it exists) is said to be an analytic continuation of f to D:

Let $g_1, g_2 : D \to \mathbb{C}$ be analytic continuations of f, then $h = g_1 - g_2$ is analytic on D and $\equiv 0$ on D'. Define $D_1 = \{w \in D : h \equiv 0 \text{ on some } D(w,r), r > 0\}, D_2 = \{w \in D : h^{(n)}(w) \neq 0 \text{ some } n \geq 0\}$; as we saw above $D = D_1 \cup D_2, D_1 \cap D_2 = \emptyset$. Clearly we have both D_1 and D_2 open, so since D is connected one of the D_i is empty; it cannot be D_1 since $D' \subset D_1$ so $D_2 = \emptyset$ and $D = D_1$, so $h \equiv 0$ on D and $g_1 = g_2$ on D.

Combining this and 2.5.6 we have:

2.6.2 Corollary (Identity Theorem)

Let $f, g : D \to \mathbb{C}$ be analytic on a domain D. If $S = \{z \in D : f(z) = g(z)\}$ contains a non-isolated point then $f = g$ on D, i.e. $S = D$, as let $w \in S$ be a non-isolated point i.e. $\forall \epsilon > 0, S \cap D(w,\epsilon) \neq \{w\}$. The function $f - g$ is holomorphic on D and vanishes on S so by 2.5.6 if $D(w,r) \subset D$ then $f - g \equiv 0$ on $D(w,r)$, so by 2.6.1 $f \equiv g$ on D. So if $f, g : \mathbb{C} \to \mathbb{C}$ are analytic then to show $f \equiv g$ it is enough to e.g. show $f = g$ on $\mathbb{R}$.

Remark

In general it is difficult to tell whether $f : D' \to \mathbb{C}$ has an analytic continuation to $D \supset D'$, e.g. $f(z) = \sum_{n=0}^{\infty}$ on $D(0,1)$ has analytic continuation to $\mathbb{C} \setminus \{1\}$ by $\frac{1}{1-z}$, but $\sum z^{n^2}$ has no such analytic continuation (and prooving this is hard).

3 Complex integration (II)

3.1 Winding Numbers

Say $\gamma : [a,b] \to \mathbb{C} \setminus \{w\}$ a closed curve. We want to make sense of "the number of times γ winds around w". Suppose we can srite $\gamma(t) = w + r(t)e^{i\theta(t)}$ with r, θ continuous functions $[a,b] \to \mathbb{R}$ and wlog taking $r(t) > 0$ (we never have $r(t) = 0$ since γ does not pass through w, so $r(t)$ is either always positive or always negative). Then "the angle the line joining w to $\gamma(t)$ sweeps out" is $\theta(b) - \theta(a)$, and we define the winding number (or index) of γ about w to be $[I(\gamma; w) \;=]\frac{\theta(b)-\theta(a)}{2\pi}$; note that this is an integer since $\gamma(b) = \gamma(a)$.

We say that θ is a continuous choice of argument for $\gamma(t) - w$; if θ_1 is another such choice of arcument then $\theta - \theta_1$ is a continuous function with values in $2\pi\mathbb{Z}$ so is constant, and in particular $\theta_1(b) - \theta_1(a) = \theta(b) - \theta(a)$ and our definition is independent of the choice of θ.

However, the existance of θ is nontrivial.

3.1.1 Theorem

If $\gamma : [a,b] \to \mathbb{C}$ is a continuous curve then $\exists$ a continuous $\theta : [a,b] \to \mathbb{R}$ such that $\gamma(t) = w + r(t)e^{i\theta(t)}$ where $r(t) = |\gamma(t) - w|$.

The naive approach would be to set $\theta = \arg(\gamma(t) - w)$ [taking the principal branch of arg], adding or subtracting 2π each time γ crosses the line $\{w - x | x \geq 0\}$; however, even a continuously differentiable γ can cross this line infinitely many times.

We first note that if γ lies in the half-plane $U = \{z \in \mathbb{C} : \mathrm{Re}(z - w) > 0\}$ then the principal branch of the argument is continuous, so we can take $\theta = \arg(\gamma(t) - w)$ for this case. More generally, if the image of γ lies in $\{z : \mathrm{Re})(\frac{z-w}{e^{i\alpha}})\}$ for some fixed α then we can take $\theta(t) = \arg(\frac{\gamma(t)-w}{e^{i\alpha}}) + \alpha$.

For a general γ, we first take wlog $w = 0$ by translation, then (by replacing $\gamma(t)$ by $\frac{\gamma(t)}{|\gamma(t)|}$) we wlog take $|\gamma(t)| = 1 \forall t$; now since γ is continuous on the closed interval $[a,b]$ it is uniformly continuous, so $\exists \epsilon > 0 : \forall s,t \in [a,b], |s - t| < \epsilon \Rightarrow |\gamma(s) - \gamma(t)| < \sqrt{2}$.

Now we can subdivide $[a,b]$ as $a = a_0 < a_1 < \cdots < a_N = b$ with $a_n - a_{n-1} < 2\epsilon \forall n$ so the image under γ of $[a_{n-1}, a_n]$ lies inside a half-plane with boundary a line through 0. By the above, for each n we have $\theta_n : [a_{n-1}, a_n] \to \mathbb{R}$ continuous and such that $\gamma(t) = e^{i\theta_n(t)} \forall t \in [a_{n-1}, a_n]$.

$\theta_{n+1}(a_n) = \theta_n(a_n) + 2\pi B_n$ for some $B_n \in \mathbb{Z}$, so by adding multiples of 2π to θ_n we can arrange that $B_n = 0 \forall n$ and then the θ_n together define a continuous function θ with the required properties.

The relation of this with complex integration is:

3.1.2 Proposition

Let γ be a (piecewise C^1) closed curve, $\gamma : [a,b] \to \mathbb{C} \setminus \{w\}$. Then $I(\gamma; w) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-w}$; note one can also take this to be the definition of the winding number for piecewise C^1 curves, as is done in e.g. Ahlfors, but then we have an almost equal amount of work to the above prooving this is always an integer.

Write $\gamma(t) = w + r(t)e^{i\theta(t)}$; since γ is piecewise continuously differentiable so are $r(t), \theta(t)$ and $\int_\gamma \frac{dz}{z-w} = \int_a^b \frac{\gamma'(t)dt}{\gamma(t)-w} = \int_a^b \frac{r'(t)}{r(t)} + i\theta'(t)dt = [\ln r(t) + i\theta(t)]_a^b = i(\theta(b) - \theta(a)) = 2\pi i I(\gamma; w)$.

3.1.3 Proposition

If $\gamma : [a,b] \to \mathbb{C}$ is a (continuous) closed curve with $\gamma(t) \in D(w_0, R) \forall t$ and $w \notin D(w_0, R)$ then $I(\gamma; w) = 0$; wlog take $w_0, w \in \mathbb{R}$ by linear transformation, then as in the proof of theorem 3.1.1 we have a continuous choice of argument by simply $\theta(t) = \arg(\gamma(t) - w$ so $\theta(b) = \theta(a)$ and $I(\gamma; w) = 0$.

Remark

For piecewise continuously differentiable curves, we could take a sledgehammer approach by applying 3.1.2 and Cauchy's theorem for a disc.

Definition

Let $U \subset \mathbb{C}$ be open.

A closed curve $\gamma : [a, b] \to U$ is homologous to 0 (in U) if $I(\gamma; w) = 0 \forall w \notin U$, e.g. for $U = \mathbb{C} \setminus \{0\}$ a curve about 0 is not homologous to 0, but one not around 0 is.

U is said to be simply connected if every closed γ in U is homologous to 0, e.g. $\mathbb{C} \setminus \{0\}$ is not simply connected, but by 3.1.3 any disc is, and more generally any starlike domain is simply connected. Note that this is not the usual definition (which applies to a general topological space, and uses homotopy), but is equivalent to it for open subsets of $\mathbb{C}$. Also note (without proof) that this definition is equivalent if we consider all continuous γ, all piecewise continuously differentiable γ, or even just all polygonal γ.

It is sometimes useful to generalise the notion of a closed curve slightly: if $U \subset \mathbb{C}$ is open a cycle in U is a formal finite sum of closed curves in U, $\Gamma = \gamma_1 + \cdots + \gamma_n$. If Γ is a cycle and f is continuous on U, we define $\int_\Gamma f(z)dz = \sum_i \int_{\gamma_i} f(z)dz$ and for $w \in \mathbb{C}, w \notin \gamma_i \forall i$ we define $I(\Gamma; w) = \sum_i I(\gamma_i; w)$; we say Γ is homologous to zero in U if $I(\Gamma; w) = 0 \forall w \in U$.

Clearly if γ_i is homologous to 0 in $U \forall i$ so is $\Gamma = \gamma_1 + \cdots + \gamma_n$ but th econverse is not true, e.g. take $U = \{z \in \mathbb{Z} : |z| > 1\}, \Gamma = \gamma_1 + \gamma_2, \gamma_1(t) = re^{it}, \gamma_2(t) = Re^{-it}$, both for $t \in [0, 2\pi]$, with $r, R > 1$. $I(\gamma_1; 0) = 1, I(\gamma_2; 0) = -1 \therefore I(\Gamma; 0) = 0$ and similarly $I(\Gamma; w) = 0 \forall w : |w| \leq 1$ so Γ is homologous to 0 in U even though the γ_i are not; this is why we introduced this notion.

3.2 Cauchy's integral formula (general case)

3.2.1 Theorem

Let D be a domain, γ a closed curve (or cycle) in D homologous to 0 in D, and $f : D \to \mathbb{C}$ holomorphic. Then $\forall w \in D$ not in the image of γ, $\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} dz = I(\gamma; w) f(w)$ (1) and $\int_\gamma f(z)dz = 0$ (2).

Remark

If D is a disc then any closed curve in D is homologous to 0 in D and we recover 2.3.1 (by taking γ to be a circle).

Consider the function on $D \times D$ given by $g(z, w) = \frac{f(z)-f(w)}{z-w}$ for $z \neq w$ and $f'(w)$ for $z = w$. We claim g is continuous; this is clearly true for $z \neq w$, and if $a \in D$ and $D(a, r) \subset D$ then f can be represented by a power series on $D(a, r)$, so by the remark following Theorem 1.3.2 g is continuous on $D(a, r) \times D(a, r)$ and we are done. Also, for fixed z, g is an analytic function of w. We want to show that for $w \in D \setminus \gamma$ $\int_\gamma g(z, w)dz = 0$ so then $\int_\gamma \frac{f(z)}{z-w} dz - 2\pi i f(w) I(g; w) = 0$; consider this integral as a function of w; let $h(w) = \int_\gamma g(z, w)dz$ for $w \in D \setminus \gamma$ and $\int_\gamma \frac{f(w)}{z-w} dz$ for $w \in E = \{w \in$

$\mathbb{C} \setminus \gamma : I(\gamma; w) = 0\}$. We have $D \cup E = \mathbb{C}[\setminus \gamma]$ since γ is homologous to 0 in D, and for $w \in D \cap E$, $\int_\gamma \frac{f(w)}{z-w} dw = 2\pi i f(w) I(\gamma; w)$ which is 0 since $I(\gamma; w) = 0$, so our definition is consistent and h is a function $\mathbb{C}[\setminus\gamma] \to \mathbb{C}$. Since g is continuous on $D \times D$ and analytic in w, by 2.5.4 h is analytic; moreover if R is sufficiently large that $\gamma \subset D(0, R)$ then $\forall w : |w| > R, I(\gamma; w) = 0$ (by 3.1.3) so $|h(w)| = \left|\int_\gamma \frac{f(z)}{z-w} dz\right| \le \frac{\text{length}(\gamma)\sup_\gamma |f|}{|w|-R} \to 0$ as $|w| \to \infty$, so by Liouville (2.4.1) $h(w) = 0$, then for (2) we just apply (1) to the holomorphic function $z \mapsto (z-w)f(z)$ for any $w \notin \gamma$.

3.2.2 Corollary (Cauchy's Theorem for simply connected domains)

If D is simply connected, $\forall$ holomorphic $f : D \to \mathbb{C}$ and closed curves γ in D, $\int_\gamma f(z)dz = 0$ since γ is homologous to 0 in D.

3.3 Laurent Series and singularities

For f holomorphic on $D(a, R)$ we know we can write $f(z) = \sum_{n=0}^\infty c_n(z-a)^n$ for some $c_n \in \mathbb{C}$. For f holomorphic on $D(a, r) \setminus \{a\}$ (a punctured disc; a is an isolated singularity of f) we will show we can express $f(z)$ as $\sum_{n=-\infty}^\infty c_n(z-a)^n$; in fact more is true:

3.3.1 Theorem

For f holomorphic on $A = \{z \in \mathbb{C} : r < |z-a| < R\}$ (an annulus) with $0 \le r < R \le \infty$, (1) f has a convergent series expansion on A, $f(z) = \sum_{n=-\infty}^\infty c_n(z-a)^n$ (*), which we define to be $\sum_{n=0}^\infty c_n(z-a)^n + \sum_{m=1}^\infty c_{-m}(z-a)^{-m}$, for some $c_i \in \mathbb{C}$, (2) $c_n = \frac{1}{2\pi i}\int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz \forall \rho \in (r, R)$ and (3) for $r < \rho' \le \rho < R$ (*) converges absolutely uniformly on $\{z : \rho \le |z-a| \le \rho\} \subset A$. The expansion (*) is called the laurent expansion of f in A.

Particularly, for $r = 0$, $A = D(a, R) \setminus \{a\}$.

For (1), let $w \in A$ and choose ρ_1, ρ_2 with $r < \rho_2 < |w-a| < \rho_1 < R$ and let $\gamma = \gamma_1 + (-\gamma_2)$ where γ_i is the circle $|z-a| = \rho_i$; $I(\gamma; w) = 1 - 0 = 1$. Then γ is homologous to 0 in A, so $f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w} dz == \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{z-w} dz + \frac{-1}{2\pi i}\int_{\gamma_2} \frac{f(z)}{z-w} dz = f_1(w) + f_2(w)$; as in the proof of Taylor series an expand the integral f_1 as a geometric series to get $f_1(w) = \sum_{n=0}^\infty c_n(w-a)^n$ where $c_n = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{(z-a)^{n+1}} dz$. For f_2 we use $\frac{-1}{z-w} = \frac{\frac{1}{w-a}}{1-\frac{z-a}{w-a}} = \sum_{m=1}^\infty \frac{(z-a)^{m-1}}{(w-a)^m}$ on γ_2 since $|\frac{z-a}{w-a}| < 1$, so $f_2(w) = \sum_{m=1}^\infty d_m(w-a)^{-m}$ where $d_m = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(z)}{(z-a)^{-m+1}} dz \forall m \ge 1$ and we have the result. Now consider (*) and $\rho_2 < \rho' \le \rho < \rho_1$. Then the power series $\sum_{n=0}^\infty c_n(z-a)^n$ must have radius of convergence $\ge \rho_1$ so converges uniformly on $\{z : |z-a| \le \rho\}$; similarly write $u = \frac{1}{z-a}$, then $\sum_{n=1}^\infty c_{-n}u^n$ has radius of convergence $\ge \frac{1}{\rho_2}$ so converges uniformly for $|u| \le \frac{1}{\rho_2}$. So (*) converges uniformly for $\rho' \le |z-a| \le \rho$ and then $\int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz = \int_{|z-a|=\rho} \sum_{m=-\infty}^\infty c_m(z-a)^{m-n-1} dz = \sum_{m=-\infty}^\infty c_m \int_{|z-a|=\rho} (z-a)^{m-n-1} dz$ since the series converges uniformly, and this integral is 0 for $m \ne n$, $2\pi i$ for $m = n$ so this is $2\pi i c_n$. Since we can choose ρ_1, ρ_2 arbitrarily close to r, R this holds $\forall \rho \in (r, R)$.

Remark

This proof shows we can write a holomorphic function f on A as $f_1 + f_2$ with f_1 holomorphic on $D(a, R)$ and f_2 holomorphic on $\{z : |z-a| > r\}$.

A function f holomorphic on the "punctured disc" $D^0(a,R) = D(a,R) \setminus \{a\}$ (a special case of A above with $r = 0$) is said to have a an isolated singularity at $z = a$; we want to consider the possibilities for "what can happen at $z = a$".

Classification of isolated singularities

Let $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ be the Laurent expansion of f on $D^0(a,R)$. Then there are three possibilities:

1) $c_n = 0 \forall n < 0$, i.e. the Laurent series is a power series so defines a holomorphic function on $D(a,R)$. We say f has a removable singularity at $z = a$; this typically arises when f is given by an explicit formula which is not defined at $z = a$, e.g. $f(z) = \frac{e^z-1}{z}$ is undefined at $z = 0$ but its power series $f(z) = \sum_1^{\infty} \frac{1}{n!} z^{n-1}$ tells us that by setting $f(0) = 1$ we can make f analytic everywhere.

2) $\exists k > 0$ such that $c_{-k} \neq 0$ but $c_n = 0 \forall n < -k$; in this case we say f has a pole of order k at $z = a$, e.g. $\frac{e^z}{z^3}$ at $z = 0$.

3) $c_n \neq 0$ for infinitely many $n < 0$; we say f has an essential singularity at $z = a$, e.g. $e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}$.

3.3.2 Proposition

(We are still assuming f is holomorphic on $D^0(a,R)$)

f has a removable singularity at $z = a$ if and only if $\lim_{z\to a}(z-a)f(z) = 0$; if we have a removable singularity f is represented by a power series on $D(a,R)$ so $\lim_{z\to a}(z-a)f(z) = 0$, and for the converse define $g(z) = (z-a)^2 f(z)$ for $z \neq a$, 0 at $z = a$. This g is clearly holomorphic on $D^0(a,R)$, and we have $\frac{g(z)-g(a)}{z-a} = (z-a)f(z) \to 0$ as $z \to 0$ so g is holomorphic at $z = a$ with $g'(a) = 0 = g(a)$. Thus g is represented by a power series on $D(a,R)$ with the first two terms 0, so dividing by $(z-a)^2$ f is represented by a power series on $D(a,R)$ and we have the result.

3.3.3 Proposition

f has a pole at $z = a$ if and only if $|f(z)| \to \infty$ as $z \to a$; moreover TFAE: 1) f has a pole of order k at $z = a$. 2) $f(z) = (z-a)^{-k} g(z)$ with g holomorphic at $z = a$ and $g(a) \neq 0$. 3) $f(z) = \frac{1}{h(z)}$ where h has a zero of order k at $z = a$.

The equivalence of 1) and 2) is obvious; $f(z) = c_{-k}(z-a)^{-k} + \ldots$ with $c_k \neq 0$ corresponds with $g(z) = c_{-k} + c_{1-k}(z-a) + \ldots, g(a) = c_{-k} \neq 0$. For the equivalence of 2) and 3), if $f(z) = (z-a)^{-k} g(z)$ then $\frac{1}{f} = (z-a)^k g(z)^{-1}$ and since $g(a) \neq 0$, $g(z)^{-1}$ is holomorphic at $z = a$. Conversely if h has a zero of order k, $h(z) = (z-a)^k q(z)$ some q with $q(a) \neq 0$ so $f(z) = \frac{1}{h(z)} = (z-a)^{-k} q(z)^{-1}$ has a pole of order k.

If f has a pole then by 3) $|\frac{1}{f}| \to 0$ so $|f| \to \infty$ [as $z \to a$]; conversely if $|f(z)| \to \infty$ then $|\frac{1}{f(z)}| \to 0$ and also $f(z) \neq 0$ for $0 < |z-a| < r$ for sufficiently small r, so $\frac{1}{z}$ is holomorphic on $D^0(a,r)$ and by 3.3.2 has a removable singularity at $z = a$ so $\exists$ holomorphic h with $h(z) = \frac{1}{f(z)} \forall z \in D^0(a,r)$ and then $h(a) = \lim_{z\to a} \frac{1}{f(z)} = 0$ so 3) holds.

If f has a removable singularity then $\lim_{z\to a} f(z) = c_0$ exists, so:

3.3.4 Corollary

f has an essential singularity at $z = a$ if and only if $|f|$ has no limit in $R \cup \{\infty\}$ as $z \to a$; cf the second example sheet for this course for the Cassorati-Weierstrauss and Picard theorems.

Remark: "Poles are not really singularities at all"

Consider the Riemann Sphere $\mathbb{C} \cup \{\infty\}$, also written $\hat{\mathbb{C}}$ or $\mathbb{CP}^1$ the "complex projective line". A holomorphic function $D^0(a, R) \to \mathbb{C}$ with a pole at $z = a$ extends to a continuous function $f : D(a, R) \to \mathbb{CP}^1$ by setting $f(a) = \infty$ (this is continuous by 3.3.3). We can regard this f as a "holomorphic mapping" $D(a, R) \to \mathbb{CP}^1$. So when we use the Riemann sphere poles are no longer singularities, and the only "genuine" singularities are the essential ones.

Definition

If D is a domain and $S \subset D$ a set of isolated points, a function $f : D \setminus S \to \mathbb{C}$ holomorphic with only poles at the points of S is said to be meromorphic.

Definition

The residue of $f(z)$ at $z = a$ is $\text{Res}_{z=a} f(z) = c_{-1}$, the coefficient of $\frac{1}{z-a}$ in the Laurent expansion.

The principal part of f at $z = a$ is the series $\sum_{n=-\infty}^{-1} c_n(z - a)^n$; this is the "simplest" expression which we can subtract from f to remove the singularity. If f has a pole of order k (¿0) at $z = a$ then its prncipal part is a polynomial in $(z - a)$, the unique such for which $f - P$ has a removable singularity at $z = a$, $P(z) = \frac{c_{-k}}{(z-a)^k} + \cdots + \frac{c_{-1}}{z-a}$.

Proposition 3.3.7

If γ is a closed curve in $D^0(a, R)$ then $\int_\gamma dz = 2\pi i I(\gamma; a) \text{Res}_{z=a} f(z)$: by uniform convergence of the Laurent expansion on $\{z : \rho_1 \leq z - a \leq \rho_2\}$ for any $0 < \rho_1 \leq \rho_2 < R$ we have $\int_\gamma f(z)dz = \int_\gamma \sum_{n=-\infty}^{\infty} c_n(z - a)^n dz = \sum_{n=-\infty}^{\infty} c_n \int_\gamma (z - a)^n dz = 2\pi i c_{-1} I(\gamma; a)$.

Remark

The fundamental theorem of calculus and its converse imply f has an antiderivative on $D^0(a, R)$ if and only if $\int_\gamma f(z)dz = 0 \forall$ closed curves $\gamma \subset D^0(a, R)$, i.e. if and only if $\text{Res}_{z=a} f(z) = 0$.

Suppose f meromorphic on D and $\{a_1, \ldots, a_m\}$ some of the poles of f on D. Let f_i be the principal part of f at $z = a_i$. Then $f - f_i$ has a removable singularity at $z = a_i$ and f_i is holomorphic on $\mathbb{C} \setminus \{a_i\}$, so $g(z) = f(z) - \sum_{i=1}^{n} f_i(z)$ has removable singularities at all the a_i; we can then proove:

Theorem 3.3.8 (Residue Theorem)

Let f be meromorphic on D and γ a closed curve (or cycle) in D homologous to 0 on D, with f having no poles on γ and only a finite number of poles $z = a$ for which $I(\gamma; a) \neq 0$. Then $\int_\gamma f(z)dz = 2\pi i \sum_{1 \leq i \leq m} I(\gamma; a_i) \text{Res}_{z=a_i} f(z)$. Notice this includes Cauchy's theorem (when f holomorphic) and the Cauchy Integral Formula (by applying it to $\frac{f(z)}{z-w}$ which has residue $f(w)$ at $z = w$) as special cases.

We can wlog assume f holomorphic on $D \setminus \{a_i : 1 \leq i \leq m\}$, since replacing D by $D' = D \setminus \{w \in D : f$ has a singularity at w and $I(\gamma; w) = 0\}$ does not change the hypotheses. Then we saw $f = g + \sum f_i$ where the f_i are the principal parts of f at a_i and g is holomorphic on D. So $\int_\gamma f dz = \int_\gamma g dz + \sum \int_\gamma f_i dz$; by Cauchy's theorem and 3.3.7 this is $0 + 2\pi i \sum I(\gamma; a_i) \operatorname{Res}_{z=a_i} f$.

Remarks

Without changing the proof, we can relax the conditions to allow a function with any isolated singularities, not just poles.

We can show that the set of poles w with $I(\gamma; w) \neq 0$ is always finite for γ homologous to 0 in D: let $V = \{w \in \mathbb{C} : I(\gamma; w) = 0\}$ and as seen on the second example sheet V is open in $\mathbb{C}$; it contains a "neighbourhood of infinity" $\{z : |z| > R\}$ for some R by 3.1.3. Since γ is homologous to 0 in D, $D \cup V = \mathbb{C}$, so the complement $K = \mathbb{C} \setminus V$ is a closed and bounded i.e. compact subset of D; since f has only isolated singularities, by Bolzano-Weierstrauss only finitely many of them are $\in K$. Thus we did not need the hypothesis that only a finite number of poles have $I(\gamma; a) \neq 0$.

For some applications it is useful to have another form of the residue theorem for simple closed curves γ (that is, γ for which $t \neq t' \Rightarrow \gamma(t) \neq \gamma(t')$ if $\{t, t'\} \neq \{a, b\}$): traditionally (and even now in many applied mathematics books) Cauchy's theorem and similar results were formulated as "if f is holomorphic on and inside a simple closed curve γ then $\int_\gamma f dz = 0$" and similar; however this is only well defined if "inside" is; in fact it is, since we have the Jordan Curve Theorem: the complement of a simple closed curve γ is the disjoint union of two domains $D_1 \amalg D_2$, where exactly one of these, wlog D_2, is unbounded, and the bounded component D_1, the "inside" of γ, is simply connected. However, proof of this is difficult; for our purposes we can avoid doing so by using the winding number:

Definition

A closed curve or cycle γ bounds a domain D if $\forall w \in D, I(\gamma; w) = 1$ and $\forall w \notin D \cup \gamma, I(\gamma; w) = 0$.

Suppose γ bounds a domain D and let f be holomorphic on $D \cup \gamma$ (i.e. on an open set $U \supset D \cup \gamma$), then by definition γ is homologous to 0 on U, so we can apply Cauchy's theorem and similar to obtain:

Theorem 3.3.9

Suppose γ bounds D, then:

If f is holomorphic on $D \cup \gamma$ then $\int_\gamma f(z) dz = 0$ (Cauchy's Theorem) and $\int_\gamma \frac{f(z)}{z-w} dz = 2\pi i f(w) \forall w \in D$ (Cauchy's Integral Formula).

If f is meromorphic on $D \cup \gamma$ with no poles on γ then $\int_\gamma f dz = 2\pi i \sum_{\{\text{poles } a \text{ of } f \text{ in D}\}} \operatorname{Res}_{z=a} f(z)$ (Residue Theorem).

[Lecture missed at this point]

Example

$\int_0^\infty \frac{x^\alpha}{x^2+1}dx$ for $0 < \alpha < 1$; we integrate $f(z) = \frac{z^\alpha}{1+z^2}$ and since we want to evaluate this for z real and more generally in the upper half-plane we pick the branch of $z^\alpha = e^{\alpha \log z} = |z|^\alpha e^{i\alpha \arg z}$ for $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$. Then we integrate around the (closed) curve $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where γ_4 is the semicircle $|z| = R$ [in the upper half-plane, paramaterised anticlockwise], γ_2 the semicircle $|z| = R$ [paramaterised clockwise], and γ_1, γ_3 are the real intervals from $-R$ to $-r$ and r to R respectively. Then $\int_{\gamma_3} f(z)dz = \int_r^R \frac{x^\alpha}{1+x^2}dx$. For γ_1, let $(-\gamma_1)(t) = -t$ for $t \in [r, R]$. Then $\int_{\gamma_1} f(z)dz = -\int_r^R \frac{(-x)^\alpha}{1+(-x)^2} \times -1 dx$ which is (*) $e^{\pi i\alpha} \int_r^R \frac{x^\alpha}{1+x^2}dx$. $\left|\int_{\gamma_4} \frac{z^\alpha}{1+z^2}dz\right| \leq \pi R \frac{R^\alpha}{R^2-1} \to 0$ as $R \to \infty$ since $0 < \alpha < 1$, but also $\left|\int_{\gamma_2} \frac{z^\alpha}{1+z^2}dz\right| \leq \pi r \frac{r^\alpha}{1-r^2} \to 0$ as $r \to 0$. We have $\int_\gamma f(z)dz = 2\pi i \operatorname{Res}_{z=i} f(z) = 2\pi i \frac{z^\alpha}{z+i}\ |_{z=i} = \pi i^\alpha$ by the residue theorem, so $\lim_{r\to 0, R\to\infty}(1 + e^{i\pi\alpha} \int_r^R \frac{x^\alpha}{1+x^2}dx = \pi i^\alpha = \pi e^i \pi \frac{\alpha}{2} = \cdots \therefore \int_0^\infty \frac{x^\alpha}{1+x^2}dx = \frac{2\pi}{\cos \pi\alpha}$ [or possibly another similar function; the lecturer was unsure].

This method wouldn't work for (for example) $\int_0^\infty \frac{x^\alpha}{x^4+x+3}dx$, since the step (*) relied on the denominator of the integrand being an even function; however, we could substitute $x \mapsto x^2$ and then proceed as above. Alternatively we could integrate directly around a "key-hole" contour consisting of the lines at angles $\pm\epsilon$ above and below the x-axis, and a large and small circle missing the short sections from angle ϵ to $-\epsilon$.

3.5 The argument principle and Rouche's Theorem

Proposition 3.5.1

Let f have a zero (or pole) of order $k \geq 1$ at $z = a$, then the "logarithmic derivative" $\frac{f'}{f}$ has a simple pole at $z = a$ with residue k (respectively $-k$): we have $f(z) = (z-a)^k g(z)$ with g holomorphic and $\neq 0$ at $z = a$, so $\frac{f'(z)}{f(z)} = \frac{k}{z-a} + \frac{g'(z)}{g(z)}$ with $\frac{g'}{g}$ holomorphic at $z = a$ so we have the result; the proof for poles is similar.

Remark

$\frac{f'}{f}$ is not necessarily $\frac{d}{dz}(\log f(z))$ since $\log f(z)$ need not be well defined.

Theorem 3.5.2 (Argument Principle)

Let γ be a closed curve (or cycle) bounding a domain D and let f be meromorphic on $D \cup \gamma$ with no zeroes or poles on γ. If f has N zeroes and P poles on D, counted with multiplicity (i.e. we count each pole or zero of degree k k times) (by the remark after the Residue Theorem (3.3.8) these numbers are finite) then $N - P = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = I(\Gamma; 0)$ where $\Gamma = f \circ \gamma$ is the image of γ under f. For the first equality we just apply the Residue Theorem to $\frac{f'}{f}$; the sum of residues of this in D is $N - P$ by (3.5.1). For the second equality notice 0 is not on Γ since $f \neq 0$ on γ so $I(\Gamma; 0) = \frac{1}{2\pi i}\int_\Gamma \frac{dw}{w} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$ (this is integration by substitution $w = f(z)$).

Suppose γ is a closed curve $\gamma : [0, 1] \to \mathbb{C}$, then the theorem says $2\pi(N - P)$ is the change in the argument of $f(z)$ as z traces γ, hence the name of the theorem.

Definition

Let f be holomorphic at $z = a$ and non-constant with $f(a) = b$. Then the <u>local degree</u> of f at $z = a$, $\deg_{z=a} f(z)$, is the order of the zero of $f(z) - b$ at $z = a$; note this is a (strictly) positive integer.

Proposition 3.5.3

$\deg_{z=a} f(z) = I(f \circ \gamma; f(a))$ for any circle $\gamma(t) = a + re^{2\pi it}, t \in [0, 1]$ with r sufficiently small: apply the argument principle to $f(z) - f(a)$. Since its zero at $z = a$ must be isolated, $N = \deg_{z=a} f(z)$ for r sufficiently small.

Theorem 3.5.4 (Local Mapping Theorem)

Let $f : D(a, R) \to \mathbb{C}$ be holomorphic and non-constant with $\deg_{z=a} f(z) = d \geq 1$, then if $r > 0$ is sufficiently small, $\exists \epsilon > 0$ such that $\forall w \in \mathbb{C}$ with $|w - f(a)| < \epsilon$, the function $f(z) - w$ has exactly d zeroes in $D(a, r)$, all of them simple if $w \neq f(a)$. For $f(a) = 0, w \neq a$ and $|w - a|$ sufficiently small, the number of solutions of $f(z) = w$ is the order of the zero of f at $z = a$: let $x = f(a)$ and choose $r > 0$ such that both $f(z) - b$ and $f'(z)$ are nonzero for $0 < |z - a| < r$; we can do this since f is non-constant so f' is not identically 0. Let γ be the circle $\gamma(t) = a + re^{2\pi it}, t \in [0, 1]$, then $\Gamma = f \circ \gamma$ is a closed curve not containing b; choose $\epsilon > 0$ such that $D(b, \epsilon)$ doesn't meet Γ. Then by 3.5.2, if $|w - b| < \epsilon$ the number of zeroes of $f(z) - w$ in $D(a, r)$ is $I(\Gamma; w)$, but this is $I(\Gamma; b = d$ for any $w \in D(b, \epsilon)$ by continuity of the winding number (which we have not actually prooven, but is true by continuity of the integral definition of the winding number). Since r is chosen such that $f' \neq 0$ on $D(a, r) \setminus \{a\}$, the zeroes are simple for $w \neq b$.

Corollary 3.5.5 ("Holomorphic mappings are open")

This is a corollary to the Argument Principle: for D a domain and $f : D \to \mathbb{C}$ holomorphic and nonconstant, the images under f of open sets are open sets; we say f is an <u>open</u> mapping: it suffices to proove that $\forall a \in D \exists D(a, r) \subset D$ such that $f(D(a, r)) \supset$ some $D(f(a), \epsilon)$ which follows from the previous result, since if $w \in D(f(a), \epsilon)$ then $f(z) - w$ has at least one zero in $D(a, r)$.

Remark

Exercise: this gives another proof of the maximum modulus theorem (2.4.3).

Theorem 3.5.6 (Rouché's Theorem)

Let γ be a closed curve bounding a domain D. Let f, g be holomorphic on $D \cup \gamma$ and $|f(z)| > |g(z)| \forall z \in \gamma$. Then f and $f + g$ have the same number of zeroes in D [counted with multiplicity] (note the hypothesis implies $f, f + g$ are nonzero on γ: it suffices to proove that $h = \frac{f+g}{f} = 1 + \frac{g}{f}$ has the same number of zeroes as poles in D; applying the argument principle it suffices to proove $I(h \circ \gamma; 0) = 0$, but the hypothesis implies $|h(z) - 1| < 1 \forall z$ on γ, so $h \circ \gamma$ is contained in $D(1, 1)$, which does not contain 0, so $I(h \circ \gamma, 0) = 0$ by (3.1.3).

We can use this to (approximately) locate zeroes of functions:

Example

$P(z) = z^4 + 6z + 3$; on γ = the circle $|z| = 2$, $|z^4| = 16 > 15 \geq |6z+3|$ so (taking $f = z^4, g = 6z+3$) all the zeroes of P have $|z| < 2$. Now consider the circle $|z| = 1$; $|6z| = 6 > 4 \geq |z^2 + 3|$ so $P(z)$ has the same number of zeroes with $|z| < 1$ as $f(z) = 6z$, namely 1, so 3 zeroes of P satisfy $1 < |z| < 2$ while the fourth has $|z| < 1$.

We can also estimate the number of zeroes in a half-plane by taking γ to be a semicircle and letting its radius $\to \infty$; see the example sheet for more on this.

3.6 Uniform Limits of analytic functions

Definition

Let $U \subset \mathbb{C}$ be open and $f_n : U \to \mathbb{C}$ a sequence of functions. We say (f_n) is <u>locally uniformly convergent</u> if $\forall a \in U \exists D(a,r) \subset U$ such that (f_n) converges uniformly on $D(a,r)$.

Examples

$f_n = \frac{1}{1-z^n}$ converges uniformly to $f(z) = 1$ on any $D(0,r)$ with $r < 1$ (since $|f - f_n| = |\frac{z^n}{1-z^2}| \leq \frac{r^n}{1-r^n}$) but (f_n) is <u>not</u> uniformly convergent on $D(0,1)$ since $\sup_{|z|<1} |f - f_n| = \infty$, so (f_n) is locally uniformly convergent on $D(0,1)$ but not uniformly convergent.

Theorem 3.6.1

A sequence of functions $f_n : U \to \mathbb{C}$ is locally uniformly convergent on U if and only if it converges uniformly on every <u>compact</u> subset of U; if $f_n \to f$ uniformly on every compact subset of U then $\forall a \in U$ and $r > 0$ such that $\overline{D(a,r)} = \{z : |z-a| \leq r\} \subset U$ the sequence converges uniformly on $\overline{D(a,r)}$ so also on $D(a,r)$ and (f_n) is locally uniformly convergent on U; conversely suppose $(f_n) \to f$ locally uniformly on U and $K \subset U$ compact, then for each $a \in K \exists r_a 0$ such that $D(a,r_a) \subset U$ and $f_n \to f$ uniformly on $D(a,r_a)$; as K is compact we have some finite set $S \subset K$ such that $K = \bigcup_{a \in S} D(a, r_a)$; then $f_n \to f$ uniformly on K (since if $f_n \to f$ uniformly on $X_1, \ldots, X_d$ then $f_n \to f$ on $\bigcup X_i$.

Theorem 3.6.2

Let (f_n) be a sequence of analytic functions on $U \subset \mathbb{C}$ which is locally uniformly convergent, then $f = \lim f_n$ is analytic on U and $(f_n') \to f'$ locally uniformly on U: let $D = D(a,r) \subset U$ be any disc, then f is continuous since it is a uniform limit of continuous functions, and by Cauchy's theorem $\int_\gamma f_n(z)dz = 0$ for any closed curve γ in D. Since the image of γ is a compact subset of $D \subset D$, $f_n \to f$ uniformly on γ by 3.6.1, so $\int_\gamma f dz = \lim \int_\gamma f_n dz =$ [, so by Movera's theorem f is analytic on any such D, and thus in U. Next, $\forall w \in D(a, \frac{r}{2})$, $|f'(w) - f_n'(w)| = \frac{1}{2\pi} \left| \int_{|z-a|=r} \frac{f(z)-f_n(z)}{(z-w)^2} dz \right|$ (assuming $\overline{D(a,r)} \subset U$) which is $\leq r \sup_{|z-a|=r} \frac{|f(z)-f_n(z)|}{\frac{r^2}{4}} \to 0$ as $n \to \infty$ since $f_n \to f$ uniformly on the circle $\{|z-a| = r\}$, so $(f_n') \to f'$ uniformly on $D(a, \frac{r}{2})$.

This result is complemented by a theorem of Weierstrauss: every analytic function is a locally uniform limit of <u>rational functions</u>. This is very much not the case for the reals, since on $[0,1]$ another theorem of Weierstrauss tells us that every <u>continuous</u> function is a uniform limit of polynomials.

Applications

1) $f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ converges for $z \in \mathbb{C} \setminus \mathbb{Z}$ (by comparison with $\sum_1^\infty \frac{1}{n^2}$); more precisely $f(z) = \sum_{n=-N}^{N} \frac{1}{(z-n)^2} + f_N(z)$ where for $|z| \le R$, $f_N(z)$ converges uniformly on $\{z : |z| \le N\}$ by comparison with $\sum_{n=N}^\infty \frac{1}{(n-N)^2}$. So $f(z)$ represents a meromorphic function on $\mathbb{C}$ with a pole at each $z = n \in \mathbb{Z}$ with principal part $\frac{1}{(z-n)^2}$. Now consider $g(z) = \frac{\pi^2}{\sin^2 \pi z}$ which has the same poles as f; as $z \to 0$, $z^2 g(z) = \frac{\pi^2 z}{\sin^2 \pi z} \to 1$ and g is <u>even</u> so the principal part of g at $z = 0$ is $\frac{1}{z^2}$. Also $g(z+1) = g(z)$ so $\forall n \in \mathbb{Z}$ the principal part of g at $z = n$ is $\frac{1}{(z-n)^2}$. We claim $f = g$, i.e. $(\star)$ $\pi^2 \operatorname{cosec}^2 \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$. Note that $f = g + h$ where h is <u>entire</u>; also $f(z+1) = f(z), g(z+1) = g(z)$ so it suffices to show that $|h(z)|$ is bounded on $\{z = x+iy : x \in [-\frac{1}{2}, \frac{1}{2}]\}$ and $|h(x+iy)| \to 0$ as $|y| \to \infty$ uniformly in $x \in [-\frac{1}{2}, \frac{1}{2}]$, then we can apply Liouville's theorem to h. In fact both f and g satisfy this: $|f(x+iy)| \le \frac{1}{y^2} + 2\sum_{n=1}^\infty \frac{1}{|y|^2 + (n-\frac{1}{2})^2} \to 0$ as $|y| \to \infty$.(by comparing the sum with $\int \frac{1}{y^2+t^2} dt$) and $|g(x+iy)| \le \frac{4\pi^2}{|e^{\pi i z} - e^{-\pi i z}|^2} \le \frac{4\pi^2}{(e^{\pi y} + e^{-\pi y})^2} \to 0$ as $|y| \to \infty$. Some related formulae (with proofs left as exercises for the reader) are $\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^\infty (\frac{1}{z-n} + \frac{1}{z+n}) = \frac{1}{z} + \sum_{n=1}^\infty \frac{2z}{z^2-n}$ (this converges by comparison with $\sum \frac{1}{n^2}$; note we do not write $\sum_{n=-\infty}^\infty \frac{1}{z-n}$ since this does not converge (a sketch of the proof is to differentiate both sides to obtain $(\star)$ showing that the difference is constant, then the constant must be 0 since both sides are odd), and $\sin \pi z = z \prod_{n=1}^\infty (1 - \frac{z}{n})(1 + \frac{z}{n}) = z \prod_{n=1}^\infty (1 - \frac{z^2}{n^2})$ (this is proven by computing the logarithmic derivative $\frac{f'}{f}$ of each side).

2) The <u>Γ function</u> is defined by $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ for $s \in \mathbb{C}, \operatorname{Re}(s) > 0$ (this last condition implies the integral converges). Write $\Gamma_N(s) = \int_{\frac{1}{N}}^N e^{-t} t^{s-1} dt$; this represents an analytic function of $s \in \mathbb{C}$ (by 2.5.4) and our error terms $\int_N^\infty e^{-t} t^{s-1} dt \to 0$ uniformly for $\sigma_1 \le \operatorname{Re}(s) \le \sigma_2$ and $\int_0^{\frac{1}{N}} e^{-t} t^{s-1} dt \to 0$ uniformly provided $\sigma_1 > 0$ [where the σ_i are presumably $\frac{1}{N}, N$ respectively]. So by the theorem in the previous lecture, $\Gamma(s)$ is analytic on $\{\operatorname{Re}(s) > 0\}$. If we integrate by parts, $\Gamma(s) = [e^{-t} \frac{t^s}{s}]_0^\infty + \frac{1}{s} \int_0^\infty e^{-t} t^s ds = \frac{1}{s}\Gamma(s+1)$ for $\operatorname{Re}(s) > 0$, i.e. $\Gamma(s+1) = s\Gamma(s)$. Since $\Gamma(1) = \int_0^1 e^{-t} dt = 1$ this implies $\Gamma(k) = (k-1)! \forall k = 1, 2, \ldots$. So for $\operatorname{Re}(s) > -n$ with $N \ge 0 \in \mathbb{Z}$ we can define $\Gamma(s) = \frac{1}{s}\Gamma(s+1) = \cdots = \frac{1}{s(s+1)\ldots(s+N-1)}\Gamma(s+N)$ which defines a <u>meromorphic</u> function with simple poles at $s = 0, -1, -2, \ldots$ and $\operatorname{Res}_{s=-k} \Gamma(s) = \frac{\Gamma(k)}{-1 \times -2 \times \cdots \times -k} = \frac{(-1)^k}{k!}$, so $\Gamma(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ with poles at $s = 0, -1, -2, \ldots$. We can also show that $\Gamma(s) \neq 0$ for $s \in \mathbb{C}$ and in fact $\frac{1}{\Gamma(s)}$ is entire with zeroes at $0, -1, -2, \ldots$; it $= e^\gamma s \prod_{n=1}^\infty (1 + \frac{s}{n}) e^{-\frac{s}{n}}$ where $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n)$, Euler's constant.

By this point we have almost certainly reached the end of the examinable portion of this course, but the remainder is retained for interest:

3) The <u>Riemann ζ function</u>: $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$. $|n^{-s}| = n^{-\operatorname{Re} s}$ so this converges for $\operatorname{Re}(s) > 1$ and converges uniformly on $\{s : \operatorname{Re}(s) \ge \sigma\}$ for any $\sigma > 1$ so represents an analytic function on the half-plane $\operatorname{Re}(s) > 1$. This ζ has two particularly nice properties:

a) ζ can be extended to an analytic function on $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$: we use that $\Gamma(s)\zeta(s) = \int_0^\infty \sum_{n=1}^\infty n^{-s} t^{s-1} e^{-t} dt$ (we have interchanged integration and summation, but this is valid). We substitute by $t \to nt$, then this is $\int_0^\infty \sum_{n=1}^\infty e^{-nt} t^{s-1} dt = \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$. We split this as $\int_0^1 \frac{t^{s-1}}{e^t-1} dt + \int_1^\infty \frac{t^{s-1}}{e^t-1} dt$. $\int_1^\infty \frac{t^{s-1}}{e^t-1} dt$ defines an analytic function on $s \in \mathbb{C}$ by the same argument as for $\Gamma(s)$; for the other integral we expand $\frac{t}{e^{t-1}} = \frac{t}{t+\frac{1}{2!}t^2+\dots} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$ where the B_k are the "Bernoulli numbers"; these are $\in \mathbb{Q}$ with $b_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = B_5 = b_7 = \dots = 0$, and the even B_i continue ($B_{12} = -\frac{691}{2370}$; informally, this is the only place 691 appears in mathematics, so if 691 pops out of an equation then it is likely the Bernoulli numbers are somehow involved). This is $\sum_{k=0}^{N} \frac{B_k t^k}{k!} + t^{N+1} F_N(t)$ so our integral $\int_0^1 \frac{t^{s-1}}{e^{t-1}} dt = \sum_{k=0}^{N} \int_0^1 \frac{B_k t^k}{k!} t^{s-2} + \int_0^1 t^{N+s-1} F_N(t) dt = \sum_{k=0}^{N} \frac{B_k}{k!} \frac{1}{k+s-1} +$ a remainder which is analytic for $\mathrm{Re}(s)\ 1 - N$. So $\Gamma(s)\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$ with poles of order ≤ 1 at $s = 1, 0, -1, \dots$; the residue at $1-k$ is $\frac{B_k}{k!}$. But $\Gamma(s)$ has poles at $s = 0, -1, \dots$ and no zeroes so $\zeta(s)$ extends to a function on $\mathbb{C}$ with a pole at $s = 1$ with residue 1 and $\forall k \geq 1, \zeta(1-k) = \frac{B_k}{k!} \frac{(-1)^k - 1}{(k-1)!} = (-1)^{k-1} \frac{B_k}{k}$. $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots)$ where $n = p_1^{r_1} + \dots + p_m^{r_m}$ for the p_i prime, which is $\prod_p \frac{1}{1-\frac{1}{p^s}}$. This implies the number of primes is infinite (as otherwise we would have no pole at $s = 1$) and (though the proof of this result takes an entire course in part II) that the number of primes $\leq X$ as a function of $X \sim \frac{X}{\log X}$, the prime number theorem; it also leads to many other important results in number theory.