Algebraic Topology

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The main book for this course is Armstrong's "Basic Topology"; there is also a useful set of lecture notes from a previous version of this course available online. Allen Hatcher's Algebraic Topology, available freely on the internet, is good for the homotopy part of this course, but uses a different definition of homology to that we will be using.

We shall write $\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$, real n-dimensional Euclidean (or affine) space, $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ the n-sphere, $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ the n-ball, $I = \{x \in \mathbb{R}^1 : 0 \leq x \leq 1\}$ the unit interval, $T = S^1 \times S^1$ the torus, and $S^1 \times I$ the annulus.

Definition: for a set X with a collection τ of subsets of X, X is a <u>topological space</u> if $\emptyset, X \in \tau$, $U, U' \in \tau \Rightarrow U \cap U' \in \tau$, and $U_j \in \tau[\forall j] \Rightarrow \bigcup_{j \in J} U_j \in \tau$. The elements of τ are called open subsets of X; the complements of open subsets are called closed subsets.

Suppose $f : X \to Y$ is a function of topological spaces; then f is continuous if $f^{-1}(U)$ is open in X for any open $U \subset Y$. Clearly the same definition with "closed" in place of "open" is equivalent. Such an f is called a "map".

f is called a homeomorphism if it has an inverse map (i.e. a map g such that $fg = 1_Y, gf = 1_X$. If such an f exists we say X and Y are homeomorphic.

Definition: for X, Y topological spaces, the product space is $X \times Y = \{(x, y) : x \in X, y \in Y\}$ with open subsets unions of sets of the form $U \times V$ where $U \subset X, V \subset Y$ open.

Definition: for X a set and $\sim$ an equivalence relation, the <u>quotient</u> $\frac{X}{\sim}$ is the set of equivalence classes of elements of X under $\sim$. We have a natural function $p : X \to \frac{X}{\sim}$ by $p(x) = [x]$, the equivalence class of x. If X is a topological space then we define a topology on $\frac{X}{\sim}$ by $U \subset \frac{X}{\sim}$ is open if $p^{-1}(U)$ is open.

Definition: $\mathbb{R}P^n$, real n-dimensional projective space, is $\frac{S^n}{\sim}$ where $\sim$ is the identification of antipodal points; there is no useful picture of this for $n > 1$.

Definition: let $f, g : X \to Y$ be maps of topological spaces. A <u>homotopy</u> from f to g is a map $F : X \times I \to Y$ such that $F(x, 0) = f(x), F(x, 1) = g(x)$; we write $f \overset{F}{\simeq} g$ or just $f \simeq g$. For $A \subset X$, F is a <u>homotopy relative to A</u> if $F(a, t) = F(a, 0) \forall t \in I, a \in A$.

Lemma: Homotopy (and homotopy relative to some A) is an equivalence relation on the space of maps $X \to Y$: we have $f \simeq f$ (relative to any A), than $f \simeq g \Leftrightarrow g \simeq f$ is trivial, and that $f \simeq g, g \simeq h \Rightarrow f \simeq h$ the lecturer did not feel worth proving.

Lemma: suppose $f, f' : X \to Y, g, g' : Y \to Z$; if $f \simeq f', g \simeq g'$ then $gf \simeq gf' \simeq g'f \simeq g'f'$: by assumption we have a homotopy $G : Y \times I \to Z$ such that $G(x, 0) = g(x), G(x, 1) = g'(x)$. Then we have a map $\phi : X \times$

$I \to Y \times I$ by $(x,t) \mapsto (f(x),t)$; then $H = G\phi : X \times I \to Z$ is a homotopy: $H(x,0) = G(\phi(x,0)) = G(f(x),0) = g(f(x)), H(x,1) = G(\phi(x,1)) = G(f(x),1) = g'(f(x))$, so $gf \simeq g'f$ [the lecturer again didn't deign to prove the rest of this].

Lemma/definition: suppose $f, g : X \to Y \subset \mathbb{R}^n$ where Y is convex, or more generally where the line segment joining $f(x), g(x)$ lies in Y for every $x \in X$. Then $f \simeq g$: define $: X \times I \to Y$ by $F(x,t) = (1-t)f(x) + tg(x)$. Note that this same proof is valid for homotopy relative to some $A \subset X$.

Definition: topological spaces X, Y are <u>homotopy equivalent</u> or of the same homotopy type if $\exists f : X \to Y, g : Y \to X$ with $fg \simeq 1_Y, gf \simeq 1_X$; in this case f or g is called a homotopy equivalence. We write $X \simeq Y$.

Lemma: this is an equivalence relation on the class of topological spaces: clearly $X \simeq X$ and $X \simeq Y \Leftrightarrow Y \simeq X$. given $X \simeq Y$ by $f : X \to Y, g : Y \to X$ and $Y \simeq Z$ by $h : Y \to Z, e : Z \to Y$ we have $(hf)(ge) \simeq 1_Z, (ge)(hf) \simeq 1_X$ so we have the result.

Definition: X is <u>contractible</u> if $X \simeq$ a single point.

Example: $X \simeq \mathbb{R}^n$ is contractible: fix $p \in X$, let $Y = \{p\}$, then define $f : x \in X \mapsto p \in Y, g : p \in Y \mapsto p \in X$. Then $fg = 1_Y$, and $gf = x \mapsto p$ is $\simeq 1_X$ by linear homotopy. This proof holds for any convex subset of $\mathbb{R}^n$

Definition: For $A \subset X$ a map $r : X \to X$ is called a <u>retraction</u> of X onto A if $r|_A = 1_X$ and $r(X) \subset A$. If $r \simeq 1_X$ then we say r is a deformation retraction of X onto A.

Definition: A path in X from x_0 to x_1 is a map $\alpha : I \to X$ with $\alpha(0) = x_0, \alpha(1) = x_1$. When $x_0 = x_1$ α is called a loop based at x_0. X is callex path-connected if $\forall x_0, x_1 \in X \exists$ a path between them; path-connected components are defined in the obvious way.

Example: the well-known topologist's sin curve.

Definition: Let $\alpha_i : I \to X$ be paths from x_{i-1} to x_i. The product $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n : I \to X$ is defined as $\alpha_1 \cdot \dots \cdot \alpha_n(t) = \alpha_1(nt)$ for $0 \le t \le \frac{1}{n}$, $\alpha_2(nt-1)$ for $\frac{1}{n} \le t \le \frac{2}{n}$, and so on until $\alpha_n(nt-(n-1))$ for $\frac{n-1}{n} \le t \le 1$.

Definition: The inverse α^{-1} is defined by $\alpha^{-1}(t) = \alpha(1-t)$; the reader should verify $(\alpha \cdot \beta)^{-1} = \beta^{-1} \cdot \alpha^{-1}$.

Lemma: For $\alpha, \beta, \alpha', \beta'$ paths in X, such that $\alpha \simeq \alpha'$ relative to $\{0,1\}$ and $\beta \simeq \beta'$ relative to $\{0,1\}$, $\alpha \cdot \beta \simeq \alpha' \cdot \beta'$; also $\alpha^{-1} \simeq \alpha'^{-1}$ relative to $\{0,1\}$: Say our homotopies are $F : I \times I \to X$ from α to α', $G : I \times I \to X$ from β to β'. Then we can define $H : I \times I \to X$ from $\alpha\beta$ to $\alpha'\beta'$ by $H(x,t) = F(2x,t)$ for $0 \le x \le \frac{1}{2}$, $G(2x-1,t)$ for $\frac{1}{2} \le x \le 1$; for the inverse we define $E : I \times I \to X$ by $E(x,t) = F(1-x,t)$.

Lemma: Assume α, β, γ paths in X from x_{i-1} to X_i, then firstly $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma) \simeq \alpha \cdot \beta \cdot \gamma$ relative to $\{0,1\}$, secondly if e_{x_0} is the constant path at x_0 then $e_0 \cdot \alpha \simeq \alpha$ relative to $\{0,1\}$ and $\alpha \cdot e_{x_1}$ (where e_{x_1} is the obvious thing) $\simeq \alpha$ relative to $\{0,1\}$, and finally $\alpha \cdot \alpha^{-1} \simeq e_{x_0}$ relative to $\{0,1\}$: For the first, we will only show $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot \beta \cdot \gamma$, the other parts being similar: we have $(\alpha \cdot \beta) \cdot \gamma = \alpha(4t)$ for $0 \le t \le \frac{1}{4}$, $\beta(4t-1)$ for $\frac{1}{4} \le t \le \frac{1}{2}$ and $\gamma(2t-1)$ for $\frac{1}{2} \le t \le 1$ and $\alpha \cdot \beta \cdot \gamma = \alpha(3t)$ for $0 \le t \le \frac{1}{3}$, $\beta(3t-1)$ for $\frac{1}{3} \le t \le \frac{2}{3}$ and $\gamma(3t-2)$ for $\frac{2}{3} \le t \le 1$, so define a map $f : I \to I$ by $f(t) = \frac{4}{3}t$ for $0 \le t \le \frac{1}{2}, \frac{2}{3}t + \frac{1}{3}$ for $\frac{1}{2} \le t \le 1$. Then we have $f \simeq 1_I$ relative to $\{0,1\}$ as I is convex, so $(\alpha \cdot \beta) \cdot \gamma = (\alpha \cdot \beta \cdot \gamma)f \simeq (\alpha \cdot \beta \cdot \gamma)1_I = \alpha \cdot \beta \cdot \gamma$ relative to $\{0,1\}$. For the second part, we define $f : I \to I$ by $f(t) = 0$ for $0 \le t \le \frac{1}{2}$, $2t-1$ for $\frac{1}{2} \le t \le 1$, then $\alpha f(t) = e_{x_0} \cdot \alpha(t)$. But $1_I \simeq f$ relative to $\{0,1\}$ so

$e_{x_0} \cdot \alpha = \alpha f \simeq \alpha 1_I = \alpha$ relative to $\{0,1\}$. For the final part, let $g = 1_I$; we have $g \cdot g^{-1}$ (considering g as a path $I \to I \simeq$ the constant map at 0 relative to $\{0,1\}$. Now $\alpha \cdot \alpha^{-1} = \alpha(g \cdot g^{-1}) \simeq \alpha(0) = e_{x_0}$ relative to $\{0,1\}$; the other cases are similar.

Definition (fundamental group): the fundamental group of $x_0 \in X$, $\pi_1(X, x_0)$, is the set of equivalence classes of loops based at x_0 under homotopy relative to $\{0,1\}$, with group operation $[\alpha][\beta] = [\alpha \cdot \beta]$.

Theorem: This is a group: we have associativity from that of paths, above; the identity is $[e_{x_0}]$, and the inverse of $[\alpha]$ is $[\alpha^{-1}]$.

Example: For $X \subset \mathbb{R}^n$ corvex, any loop at x_0 is homotopic to e_{x_0}, so $[\alpha] = [e_{x_0}] \forall \alpha$ and $\pi_1(X, x_0) = 0$.

[I have begun rewriting a lot of this, as the lecturer appears unable to remember all the bits of many theorems]

Theorem: Any [continuous?] $f : X \to Y$ with $f(x_0) = y_0$ induces a homomorphism $f_\star : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ such that $f \simeq f'$ rel $\{0,1\} \Rightarrow f_\star = f'\star$, $(1_X)_\star$ is the identity, and for $f : X \to Y, g : Y \to Z$, $(gf)_\star = g_\star f_\star$: Define $f_\star(\alpha) = f\alpha$ for any path α on X. Then, defining $f_\star([\alpha]) = [f_\star(\alpha)]$, this is a homomorphism: $f_\star[\alpha\beta] = [f_\star(\alpha\beta)] = [f_\star\alpha f_\star\beta] = [f_\star\alpha][f_\star\beta]$; we have $f_\star[e_{x_0}] = [fe_{x_0}] = [e_{y_0}]$, and $f_\star[\alpha^{-1}]f_\star[\alpha] = [f_\star(\alpha \cdot \alpha^{-1})] = f_\star[e_{x_0}] = f_\star[\alpha \cdot \alpha^{-1}]$.

Now we have those properties: if $f \simeq f'\text{rel}\{x_0\}$ then $f_\star(\alpha) \simeq f'_\star(\alpha)\text{rel}\{0,1\}$ so $f_\star[\alpha] = f'_\star[\alpha]$, the second property is trivial, and for the final one $(gf)_\star(\alpha) = gf(\alpha) = g(f(\alpha)) = g(f_\star\alpha) = g_\star f_\star(\alpha)$.

Theorem: For $\gamma : I \to X$ a path from x_0 to x_1, we have an isomorphism $\gamma_\# : \pi_1(X, x_0) \to \pi_1(X, x_1)$ such that $\gamma \simeq \gamma'\text{rel}\{0,1\} \Rightarrow \gamma_\# = \gamma'_\#$, $(e_{x_0})_\#$ is the identity, $(\gamma \cdot \lambda)_\# = \lambda_\#\gamma_\#$ where λ is a path from x_1 to x_2, and if $f : X \to Y$ is a map with $f(x_0) = y_0, f(x_1) = y_1$ then $(f_\star\gamma)_\# f_\star = f_\star\gamma_\#$: define $\gamma_\#$ by $\gamma_\#(\alpha) = \gamma^{-1} \cdot \alpha \cdot \gamma$; this is clearly a homomorphism. $[\gamma^{-1}\alpha\gamma] = [e_{x_1}] \Rightarrow \gamma^{-1} \cdot \alpha \cdot \gamma \simeq e_{x_1}\text{rel}\{0,1\} \Rightarrow \gamma \cdot \gamma^{-1} \cdot \alpha \cdot \gamma \cdot \gamma^{-1} \simeq \gamma e_{x_1}\gamma^{-1} = e_{x_0}\text{rel}\{0,1\}$, i.e. $\alpha \simeq e_{x_0}\text{rel}\{0,1\}$, so the homomorphism is injective. For any $[\beta] \in (X, x_1)$ we have $\gamma_\#(\gamma\beta\gamma^{-1}) = \gamma^{-1}\gamma\beta\gamma^{-1}\gamma \simeq \beta\text{rel}\{0,1\}$, so the homomorphism is surjective and thus an isomorphism as required.

For the properties, $\gamma_\#(\alpha) = \gamma^{-1} \cdot \alpha \cdot \gamma \simeq \gamma'^{-1} \cdot \alpha \cdot \gamma' = \gamma'(\alpha)\text{rel}\{0,1\}$, the second property is clear, for the third $(\gamma \cdot \lambda)_\#(\alpha) = (\gamma \cdot \lambda)^{-1}\alpha \cdot \gamma \cdot \lambda = \lambda^{-1} \cdot \gamma^{-1} \cdot \alpha \cdot \gamma \cdot \lambda = \lambda^{-1} \cdot \gamma_\#(\alpha) \cdot \lambda = \lambda_\#\gamma_\#(\alpha)$. For the fourth let $\alpha : I \to X$ be a loop based at x_0; $(f_\star\gamma_\#)[\alpha] = f_\star(\gamma^{-1} \cdot \alpha \cdot \gamma) = f_\star\gamma \cdot f_\star\alpha \cdot f_\star\gamma^{-1} = (f_\star\gamma)_\#[f_\star\alpha] = (f_\star\gamma_\#)f_\star[\alpha]$.

Theorem: suppose $f \overset{F}{\simeq} g$ [the F should be *below* the $\simeq$ throughout this course, but I have insufficient latex skillz], $f, g : X \to Y$. Let γ be the path from $f(x_0)$ to $g(x_0)$, $\gamma(t) = F(0,t)$. Then $\gamma_\# f_\star = g_\star$: let $\alpha_I \to X$ be a loop based at x_0. Define $G(s,t) = F(\alpha(s), t)$¡ then we want to show $[\gamma^{-1}(f_\star\alpha)\gamma] = [g_\star\alpha]$. Define $H : I \times I \to Y$ from $\gamma^{-1} \cdot f_\star\alpha \cdot \gamma$ to $g_\star\alpha$ by $H(s,t) = \gamma^{-1}(3s)$ for $0 \le s \le \frac{1-t}{3}$, $G(\frac{3s+t-1}{2t+1}, t)$ for $\frac{1-t}{3} \le s \le \frac{2+t}{3}$, $\gamma(3s-2)$ for $\frac{2+t}{3} \le s \le 1$, and this is a homotopy (the idea behind this map is to distort our existing homotopy between $f_\star\alpha$ and $g_\star\alpha$ to one relative to $\{0,1\}$ by smoothly adding γ and γ^{-1} to the $f_\star\alpha$ end. We have $H(s,0) = \gamma^{-1}(3s)$ for $0 \le s \le \frac{1}{3}$, $G(3s-1, 0)$ (i.e. $f_\star\alpha(3s-1)$) for $\frac{1}{3} \le s \le \frac{2}{3}$, and $\gamma(3s-2)$ for $\frac{2}{3} \le s \le 1$ and $H(s,1) = \gamma^{-1}(0)$ for $s = 0$, $(g_\star\alpha)(s)$ for $0 \le s \le 1$, and $\gamma(1)$ for $s = 1$, so we have a homotopy as required.

Corollary: let $f : X \to Y$ be a homotopy equivalence. Then $\pi_1(X, x_0) \simeq \pi_1(Y, y_0)$ for $f(x_0) = y_0$: let f' be the inverse of f. We have $f^\star : \pi_1(X, x_0) \to$

$\pi_!(Y, y_0)$ and $f'_\star : \pi_1(Y, y_0) \to \pi_1(X, f'(y_0))$. By definition $f'f \simeq 1_X, ff' \simeq 1_Y$, so $(f'f)_\star = f'_\star f_\star$ is an isomorphism so $f'_\star$ is surjective; similarly $f_\star f'_\star$ is an isomorphism so $f'_\star$ is injective, so $f'_\star$ is an isomorphism; we cannot directly use the same argument for $f_\star$ as we have two slightly different $f_\star$ in play [lol lecturer], but the situation is symmetric in f, f' so $f_\star$ is also an isomorphism.

Example: Suppose X is contractible, i.e. X is of the same homotopy type as a single point. Then by this corollary $\pi_1(X, x_0) = 0 \forall x_0 \in X$.

As an aside, we usually write fundamental groups multiplicatively, but we still sometimes use 0 for their identities [lol us]. Also, we write π_1 because other homotopy groups $\pi_2, \ldots$ exist, though we will not see them in this course.

Remark: If X is path-connected then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ and we sometimes write $\pi_1(X)$ for this common value.

Definition: X is <u>simply connected</u> if X is path-connected and $\pi_1(X) = 0$.

Covering Spaces

Assume X is path-connected. A covering space is another space $\tilde{X} \neq \emptyset$ with a covering map $p : \tilde{X} \to X$ such that $\forall x \in V \exists U \ni x$ open such that $p^{-1}(U)$ is a disjoint union of open subsets V such that $p|_V : V \to U$ is a homeomorphism; such a U is called elementary.

Examples: $p : \mathbb{R}^1 \to S^1$ $p(t) = e^{2\pi i t}$, $p : S^1 \to S^1$ $p(z) = z^n$ for an integer n, $p : S^n \to \mathbb{RP}^n$ $p(x) = [x] = \{x, -x\}$, and extensions of these e.g. $p : \mathbb{R}^1 \times S^1 \to S^1 \times S^1$ $p(t, z) = (e^{2\pi i t}, z)$.

Lemma (path-lifting): Let $p : \tilde{X} \to X$ be a covering map, $x_0 \in X$, $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$, and α a path in X with $\alpha(0) = x_0$. Then $\exists!$ path $\tilde{\alpha} \in \tilde{X}$ with $\tilde{\alpha}(0) = \tilde{x}_0, p_\star(\tilde{\alpha}) = \alpha$: for each $x \in X$ take U_x a corresponding elementary open set; thus $X = \bigcup_{x \in X} U_x$. Then $I = \bigcup_{x \in X} \alpha^{-1}(U_x)$; I is compact so we have a finite subcover $I = \bigcup_{i=1}^n \alpha^{-1}(U_i)$. We can divide I into $[\frac{k-1}{m}, \frac{k}{m}]$ for some m in such a way that $\alpha([\frac{k-1}{m}, \frac{k}{m}]) \subset U_i$, for some i, for each m; this is valid though nontrivial, induct from the case $n = 2$. Since each of these U_i is elementary we can find a path $\tilde{\beta}_1$ which is a lift of β_1 where β_1 is the path given by $[0, \frac{1}{m}]$ (we have $\tilde{U}_i$ homeomorphic to U_i); then inductively we can find $\tilde{b}_k$ similarly. Then $\tilde{\alpha} = \tilde{\beta}_1 \cdot \cdots \cdot \tilde{\beta}_m$ is as required.

Lemma (homotopy-lifting): let $p : \tilde{X} \to X$ be a covering map, $F : I \times I \to X$ and $\tilde{F} : I \times \{0\} \to \tilde{X}$ such that $p(\tilde{F}(s, 0)) = F(s, 0)$. Then there is a unique extension of $\tilde{F}$ to $I \times I$ such that $p\tilde{F} = F$: cover X by elementary open sets as above, getting a finite subcover $I \times I = \bigcup_{i=1}^n F^{-1}(U_i)$; again find m such that $F(\sigma_{k,l}) \subset U_i$ for some $i \forall l, m$ where $\sigma_{k,l} = [\frac{k-1}{m}, \frac{k}{m}] \times [\frac{l-1}{m}, \frac{l}{m}]$. Now $F(\sigma_{1,1}) \subset U_i$ so since U_i is elementary we can extend $\tilde{F}$ over $\sigma_{1,1}$ uniquely; continuing inductively we can extend $\tilde{F}$ over all of $I \times I$.

Now the remainder of both lemmas follows from this:

Claim: Let $p : \tilde{X} \to X$ be a covering map and Y a connected space; suppose $f, g : Y \to \tilde{X}$ are maps such that $pf = pg$. Then the set $A = \{y \in Y : f(y) = g(y)\}$ is $\emptyset$ or Y: it is enough to prove A is open and closed, then we are done by connectedness of Y. Take $y \in \bar{A}$; put $x = pf(y) = pg(y)$, take $U \ni x$ elementary. There are components V, W of $p^{-1}(U)$ such that $f(y) \in V, g(y) \in W$, so $f^{-1}(V), g^{-1}(W)$ both intersect A; by the definition of A this means they intersect [I'm sure the lecturer is wrong here - they intersect because the place where they intersect A is the same. This whole proof is very perverse] Now

$p\mid_V\colon V \to U$ is a homeomorphism so $f(y) = g(y)$; thus $y \in A$ so A is closed; similarly A is open.

Corollary: Let $p : \tilde{X} \to X$ be a covering map, $x_0 \in X, \tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$. If $\alpha \simeq \beta$ (relative to $\{0,1\}$) are paths in X such that $\alpha(0) = \beta(0) = x_0$ then the lifts $\tilde{\alpha} \simeq \tilde{\beta}$ relative to $\{0,1\}$ such that $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}_0$: this is immediate by the homotopy lifting lemma (the endpoint is the same since it is the same in a basic neighbourhood in X).

Example: $\pi_1(S^1) = \mathbb{Z}$: set $x_0 = (1,0) \in S^1$, let $p : \mathbb{R}^1 \to S^1$ be the covering map $p(t) = e^{2\pi i t}$. Take $\tilde{x}_0 = 0 \in \mathbb{R}$. Then for a loop α based at x_0, there is a unique lift $\tilde{\alpha}$ of α; we know that $\tilde{\alpha}(1) = n \in \mathbb{Z}$. If $n = 0$ then $\alpha \simeq$ the constant loop at x_0; if $n > 0$ take $\tilde{\beta}$ to be a "simple" path in $\mathbb{R}^1$ from 0 to n; since $\mathbb{R}^1$ is contractible, $\tilde{\alpha} \simeq \tilde{\beta}\mathrm{rel}\{0,1\}$ so $\alpha = p_\star\tilde{\alpha} = p_\star\tilde{\beta} =: \beta$. If γ is a "simple" path from 0 to 1 then $[\alpha] = [p_\star\tilde{\alpha}] = [p_\star\beta] = [p_\star\gamma]^n$; the rest of the example is left as an exercise.

Example: $T = S^1 \times S^1$; by example sheet this implies $\pi_1(T) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$.

Example: $\pi_1(\mathbb{R}^1 \times S^1) = \pi_1(\mathbb{R}^1) \times \pi_1(S^1) = \mathbb{Z}$

Example: $\pi_1(S^1 \times I) = \mathbb{Z}$

Example: $\pi_1(S^n) = 0$ if $n \geq 2$ (outline): take $x_0 \in S^n$ and $\alpha : I \to S^n$ a loop based at x_0. If α is not surjective, by the example sheet $\alpha \simeq e_{x_0}\mathrm{rel}\{0,1\}$ so $[\alpha]$ is trivial in $\pi_1(S^n)$; if α is surjective, take $x \neq x_0 \in S^n$, and let U be a small open neighbourhood of x homeomorphic to an n-ball in $\mathbb{R}^n$. Now $\alpha^{-1}(U)$ is open in I so is the disjoint union of intervals, but $\alpha^{-1}(\{x\})$ is compact so must be contained in finitely many of these intervals. Thus we can construct a loop $\beta \simeq \alpha\mathrm{rel}\{0,1\}$ which is non-surjective [by perturbing α finitely many times to run along the boundary of U rather than through x], and then we have the result by the previous case.

Example: the Möbius strip M has $\pi_1(M) = \mathbb{Z}$, as seen on the example sheet; likewise $\pi_1(\mathbb{RP}^n) = \frac{\mathbb{Z}}{2}$ for $n \geq 2$ (we cannot use the theorem below since S^n is not simply connected)

Action of groups

$\mathrm{Aut}(X)$, the set of homeomorphisms $X \to X$, is a group (under composition) which acts on X. Let $G \subset \mathrm{Aut}(X)$ be a subgroup. Define $\frac{X}{G}$ to be the set of classes $\{f(x) : f \in G\}$ for $x \in X$.

Theorem: suppose $G \subset \mathrm{Aut}(X)$ ("G acts on X as homeomorphisms"), and X is simply connected, and assume $\forall x \in X \exists U \ni x : U \cap g(U) = \emptyset \forall g \neq 1$ with U open. Then $\pi_1(\frac{X}{G}) = G$: let $x_0 \in X$ and $p : X \to \frac{X}{G}$ be the quotient map. For any $g \in G$ choose a path γ_g from x_0 to $g(x_0)$. Now define a map $\theta : G \to \pi_1(\frac{X}{G}, p(x_0))$ by $g \mapsto [p_\star\gamma_g]$. We see the path $g(\gamma_{g'})$ starts at $g(x_0)$ and ends at $gg'(x_0)$; we can take $\gamma_{gg'}$ to be the product $\gamma_g \cdot g(\gamma_{g'})$, so $gg' \mapsto [p_\star\gamma_{gg'}] = [p_\star(\gamma_g \cdot g(\gamma_{g'})]$. So $\theta(gg') = \theta(g)\theta(g')$, and θ is a homomorphism. For injectivity suppose $\theta(g) = [p_\star\gamma_g] =$ the identity of $\pi_1(\frac{X}{G}, p(x_0))$. From our assumption p is a covering map; we can lift $p_\star\gamma_g$ (uniquely) to a path starting at x_0, which will be the same as γ_g; by homotopy lifting this must be homotopic to e_{x_0} relative to $\{0,1\}$; this means γ_g must be a loop based at x_0; in particular $\gamma_g(1) = \gamma_g(0) = x_0$ so $g(x_0) = x_0$ and $g = 1$.

Suppose α is a loop based at $p(x_0)$; by path lifting we have a lift $\tilde{\alpha}$ such that $\tilde{\alpha}(0) = x_0$; since α is a loop $\tilde{\alpha}(1) \in [x_0]$ and $\exists g \in G : \tilde{\alpha}(1) = g(x_0)$. By construction $\tilde{\alpha} \simeq \gamma_g \mathrm{rel}\{0,1\}$, so θ is surjective. Thus θ is an isomorphism.

Example: $G = \mathbb{Z}$ acts on $\mathbb{R}^1$ by $gx = x + g$; $\frac{\mathbb{R}^1}{\mathbb{Z}} = S^1$ so $\pi_1(S^1) = \mathbb{Z}$.

Example: $G = \frac{\mathbb{Z}}{2} = \{e, g\}$ acts on S^n by $ex = x, gx = -x$; by definition $\frac{S^n}{\frac{\mathbb{Z}}{2}} = \mathbb{RP}^n$. If $n \geq 2$, $\pi_1(\mathbb{RP}^n) = \frac{\mathbb{Z}}{2}$.

Example: $G = \mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{R}^2$ by $g = (m,n) : (x,y) \mapsto (x+m, y+n)$; $\frac{\mathbb{R}^2}{\mathbb{Z}\times\mathbb{Z}} = T$ so $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$.

Free groups, generators, relations

Let X be a set. Define a <u>word</u> to be a finite "product" $x_1^{n_1} \ldots x_m^{n_m}$ with $x_i \in X, n_i \in \mathbb{Z}$ (including the empty word). We say a word is <u>reduced</u> if $x_1 \neq x_{i+1}, n_i \neq 0, \forall i$. The set of reduced words forms a group with the empty word as the identity and the product being the reduced word of the concatenation of two words. We call this group $f(X)$, the free group generated by X.

Example: $f(\{x\}) = \mathbb{Z}$.

If we have a bijection $X \to Y$ this gives an isomorphism $f(X) \to f(Y)$.

If G is a group and $\exists X \subset G$ with $f(X) = G$ then we call G a free group.

Theorem: A group G is the free group generated by $X \subset G$ iff for any function $f : X \to H$ where H is a group, there is a unique extension $\bar{f} : G \to H$ which is a (group) homomorphism.

Most groups are, alas, not free groups, e.g. $\frac{\mathbb{Z}}{3} = \{0,1,2\}$.

Let G be a group and $X \subset G$. We say X generates G if there is a surjective homomorphism $f(X) \to G$. Let N be the kernel of this homomorphism, then $N \triangleleft f(X)$; if $R \subset N$ is such that N is the smallest normal subgroup containing R then R [with X] determines N, and X, R together determine G.

Elements of X are called generators; elements of R are called relations. X, R is a presentation of G. When X, R are finite $X = \{x_1, \ldots, x_m\}, R = \{r_1, \ldots, r_l\}$ we can write $G = \{x_1, \ldots, x_m | r_1, \ldots, r_l\}$ or $\{x_1, \ldots, x_m | r_1 = 1, \ldots, r_l = 1\}$.

Example: $G = \{x : x^n = 1\}$ is $\mathbb{Z}_n$.

Free product of groups: suppose G, H are groups, then the free product $G \star H$ is the set of reduced words $x_1^{n_1} \ldots x_m^{n_m}$ such that $x_i \in G \dot{\cup} H$ (where $\dot{\cup}$ denotes disjoint union) and x_i, x_{i+1} are never in the same group, x_i is never the identity, and n_i is never 0, with product as before and identity the empty word.

Theorem: for G, H, K groups, $K = G \star H$ iff there are homomorphisms $\alpha : G \to K, \beta : H \to K$ such that for any group E with homomorphisms $\gamma : G \to E, \lambda : H \to E$ there is a unique $\theta : K \to E$ such that $\gamma = \theta \circ \alpha, \lambda = \theta \circ \beta$ (we would like to use this as the definition of a free product of groups, but then we would need to prove that it exists, which is nontrivial).

Example: $G = \mathbb{Z} = H$, $\mathbb{Z} \star \mathbb{Z} = f(\{x_1, x_2\})$; note this is $\neq \mathbb{Z} \times \mathbb{Z} = \{x_1, x_2 : x_1 x_2 x_1^{-1} x_2^{-1} = 1\}$.

If $G = \{x_1, \ldots, x_m | r_1 = 1, \ldots, r_l = 1\}, H = \{y_1, \ldots, y_s | t_1 = 1, \ldots, t_p = 1\}$ then $G \star H = \{x_1, \ldots, x_m, y_1, \ldots, y_s : r_1 = 1, \ldots, r_l = 1, t_1 = 1, \ldots, t_p = 1\}$.

[A seemingly arbitrary aside from the lecturer: Given the square, if we identify two opposite edges in parallel we have a cylinder; if we identify them in the other direction we have a Möbius strip. If we take the Möbius strip and then identify the other two edges in the same direction this gives the Klein bottle; in the opposite direction we obtain $\mathbb{RP}^2$].

Let $G = \{x, y : xyxy^{-1} = 1\}$. Then G acts on $\mathbb{R}^2$ by $x(s,t) = (s+1,t), y(s,t) = (1-s, t+1)$. Now $\frac{\mathbb{R}^2}{G} = K$ the Klein bottle; by our theorem above, $G = \pi_1(K)$.

Van Kampen Theorem: If $X = U \cup V$ for U, V open and $U, V, U \cap V$ path-connected, then consider the natural homomorphisms $\pi_1(U) \to \pi_1(X), \pi_1(V) \to \pi_1(X), \pi_1(U \cap V) \to \pi_1(U), \pi_1(U \cap V) \to \pi_1(V)$. Then $\pi_1(X) = \pi_1(U) \star \pi_1(V)$ relative to $\pi_1(U \cap V)$: take the smallest normal subgroup N of $\pi_1(U) \star \pi_1(V)$ such that the homomorphisms $\theta : \pi_1(U \cap V) \to \pi_1(U)$, $\pi_1(U) \to \frac{\pi_1(U)\star\pi_1(V)}{N}$, $\frac{\pi_1(U)\star\pi_1(V)}{N} \to X$, $\sigma : \pi_1(U \cap V) \to \pi_1(V)$, $\pi_1(V) \to \frac{\pi_1(U)\star\pi_1(V)}{N}$ all commute. N is the normal subgroup generated by elements of the form $(\theta[\alpha])(\sigma[\alpha])^{-1}$ and vice versa for $[\alpha] \in \pi_1(U \cap V)$; as before we could define our free product in terms of homomorphisms to another group E.

Now the theorem is that $\frac{\pi_1(U)\star\pi_1(V)}{N} \simeq \pi_1(X)$.

Corollary: Under the assumptions of the theorem, if $U \cap V$ is simply connected then $\pi_1(X) = \pi_1(U) \star \pi_1(V)$.

Definition: for X_i spaces and $x_i \in X_i$, the "wedge sum" $\vee X_i$ is the disjoint union of the X_i under identification of all the x_i

Example: $X = S^1 \vee S^1$. Take x, y points on the two different circles, neither being the "special" intersection point. Set $U = X \setminus \{x\}, V = X \setminus \{y\}$. Then, up to homotopy, $U = S^1 = V$. $U \cap V$ is contractible so simply connected, so $\pi_1(X) = \mathbb{Z} \star \mathbb{Z}$. More generally for $X = S^1 \vee \cdots \vee S^1$ n times, a "flower petal" space, let x_i be in "S^1 number i" and $\neq x$, the point of intersection; let $U = X \setminus \{x_1\}, V = X \setminus \{x_2, \ldots, x_n\}$. Then $U \cap V$ is simply connected so $\pi_1(X) = \pi_1(U) \star \pi_1(V) = \pi_1(U) \star \mathbb{Z}$; inductively this is $\mathbb{Z} \star \cdots \star \mathbb{Z}$ n times, which is the free group generated by n elements.

Example: $Y = \mathbb{R}^2 \setminus \{x\}, X = \mathbb{R}^2 \setminus \{x, y\}$ for $x \neq y$. By projections (in the exam, arguing from geometry like this is valid, but we must specify precisely which projections we are doing) $X \simeq S^1 \vee S^1$.

Example: $X = S^1 \vee S^2$; as before take $x \in S^1, y \in S^2$, and set $U = X \setminus \{x\}, V = X \setminus \{y\}$, then $\pi_1(U) = \pi_1(S^2) = 0, \pi_1(V) = \pi_1(S^1) = \mathbb{Z}, \pi_1(U \cap V) = 0$ so $\pi_1(X) = \mathbb{Z}$.

Example: $X = S^n$ for $n \geq 2$; take $x \neq y \in S^n$ and let $U = X \setminus \{x\} \simeq \mathbb{R}^n \simeq X \setminus \{y\} = V$, then $\pi_1(X) = \pi_1(U) \star \pi_1(V) \mathrm{rel} \pi_1(U \cap V) = 0$ (irrelevant what $\pi_1(U \cap V)$ is).

Example: $X =$ the punctured torus. T is the quotient of a square by identifying opposite edges in the same direction (alternatively $T = S^1 \times S^1$. Take D (the puncture, a closed disc) in the interior of $I \times I$, then X is the quotient of $I \times I \setminus D$ by the same relation as before. By projection from the centre of D, which is a deformation retraction, $I \times I \setminus D$ is homotopic to $\partial(I \times I)$. Now X is of the same homotopy type as the quotient of the boundary of $I \times I$, since removing D does not affect the boundary or the quotient. But this is just $S^1 \vee S^1$, so $\pi_1(X) = \mathbb{Z} \star \mathbb{Z}$.

Sketch proof of theorem: we already have a natural homomorphism $\Phi : \frac{\pi_1(U)\star\pi_1(V)}{N} \to \pi_1(X)$. Surjectivity of Φ is equivalent to surjectivity of the natural homomorphism $\phi : \pi_1(U) \star \pi_1(V) \to \pi_1(X)$. Let $[\alpha] \in \pi_1(X)$ such that α is a loop based at x_0. We can write $[\alpha] = [\alpha_1] \cdot \cdots \cdot [\alpha_m]$ with each α_i a path inside U or inside V (as we did when proving path lifting). We can choose a path γ_i from x_0 to $\alpha_i(1)$; if $\alpha_i(1) \in U \cap V$ then take γ_i to be contained in $U \cap V$ (which we can do by path-connectedness). Then $[\alpha] = [\alpha_1 \cdot \gamma_1^{-1}][\gamma_1 \alpha_2 \gamma_2^{-1}] \ldots [\gamma_{m-1} \alpha_m]$, and this is a word in $\pi_1(U)\star\pi_1(V)$ (since all the elements are equivalence classes of loops), and its image under ϕ is $[\alpha]$.

Now injectivity of Φ; this is equivalent to $\ker\phi = N$. Recall $\phi([\alpha_1][\alpha_2]\dots[\alpha_n]) = [\alpha_1]\dots[\alpha_n]$ where the α_i are loops based at x_0. We define two operations and their inverses: if $[\alpha_i],[\alpha_{i+1}]$ come from the same group then we can replace them by $[\alpha_i\cdot\alpha_{i+1}]$ without changing the free product, and if $[\alpha_i]\in\pi_1(U)$ and α_i lies inside $U\cap V$ then we can consider $[\alpha_i]$ as an element of $\pi_1(V)$, without changing anything under Φ.

Now suppose $\phi_([\alpha_1]\dots[\alpha_n]) = \phi([\beta_1]\dots[\beta_{n'}]) = [\alpha]$. We have $[\alpha_1\cdot\dots\cdot\alpha_n] = [\beta_1\cdot\dots\cdot\beta_{n'}] = [\alpha]$, so $\alpha_1\cdot\dots\cdot\alpha_n \overset{F}{\simeq} \beta_1\cdot\dots\cdot\beta_{n'} \simeq \alpha$ relative to $\{0,1\}$; we have a homotopy $F: I\times I\to X$. For a suitabl large m we can have $F(\sigma_{k,l})\subset U$ or V (possibly both) for each k,l, where $\sigma_{k,l} = [\frac{k-1}{m},\frac{k}{m}]\times[\frac{l-1}{m},\frac{l}{m}]$, and also have $nn' \mid m$; thus the "joins" in the product $\alpha_1\dots\alpha_n$ are at vertices of $\sigma_{k,l}$. We relabel the $\sigma_{k,l}$ as $R_1,R_2,\dots$ going first along and then up the square $I\times I$ [I *think* the first coordinate is horizontal; lecture was vastly unclear]. For any l we choose a path γ_l in $I\times I$ such that $\gamma(0)\in\{0\}\times I, \gamma(1)\in\{1\}\times I$ such that γ_l separates $R_1,\dots,R_l$ and the other squares; we also take $\gamma_0 = I\times\{0\}, \gamma_{m^2} = I\times\{1\}$; then each F_{γ_l} defines a path in X which is in fact a loop based at x_0. Write $F\mid_{\gamma_l} = \epsilon_1\cdot\epsilon_2\cdot\dots\cdot\epsilon_r$ such that each ϵ_i is a path corresponding to the side of a single square; by construction ϵ_i is a path in U or V (or possibly both). Now for any vertex v of a small square, we choose a path λ_v from x_0 to $F(v)$ such that if $F(v)\in U\cap V$, λ_v lies inside $U\cap V$, and if $F(v) = x_0$ we take λ_v to be the constant path at x_0. We see $F\mid_{\gamma_l} = \epsilon_1\dots\epsilon_r = (\epsilon_1\cdot\lambda_\star^{-1})(\lambda^\star\cdot\epsilon_2\cdot\lambda_\star^{-1})\dots$, where each $\lambda_\star$ is some λ_v. The bracketed terms are loops, so (technically taking classes []) this gives us a word in $\pi_1(U)\star\pi_1(V)$. Note that $[F\mid_{\gamma_l}] = [F\mid_{\gamma_{l+1}}]$ and the difference between the words corresponding to the two paths corresponds precisely to one of our operations; thus they define the same word in $\frac{\pi_1(U)\star\pi_1(V)}{N}$. But γ_0 gives exactly $[\alpha_1]\dots[\alpha_n]$ and γ_{m^2} gives precisely $[\beta_1]\dots[\beta_{n'}]$.

Example: T, the torus (in fact we know $\pi_1(T) = \pi_1(S^1\times S^1) = \mathbb{Z}\times\mathbb{Z}$, and it is much more work to find it by this method, but ultimately this method will generalize better). Recall that the punctured torus U has $\pi_1(U) = \mathbb{Z}\star\mathbb{Z}$; let V be the image of the interior of the square that T is a quotient of, take our puncture D to be a disc inside the square. We have $U\cap V\simeq S^1$ so $\pi_1(U\cap V) = \mathbb{Z}$, and clearly $\pi_1(V) = 0$. Thus $\pi_1(T) = \frac{\pi_1(U)}{\langle \mathrm{Im}\theta\rangle}$ where $\langle\mathrm{Im}\theta\rangle$ is the normal subgroup generated by the image of θ, where θ is the natural homomorphism $\pi_1(U\cap V)\to\pi_1(U)$.

Let A be a corner of the square, a,b,c,d the edges of the square taken clockwise from A, B a point inside the square not in D, and h be a loop clockwise around D based at B. Let p be the quotient from the square to T, $x_0 = p(B), x_1 = p(A), \alpha = p_\star a, \beta = p_\star b$, then $p_\star c = \alpha^{-1}, p_\star d - \beta^{-1}, \gamma = p_\star[AB], p_\star h = \lambda$. Then we have $a\cdot b\cdot c\cdot d\simeq [AB]\cdot h\cdot[AB]^{-1}$ so $\alpha\cdot\beta\cdot\alpha^{-1}\cdot\beta^{-1}\simeq \gamma\cdot\lambda\cdot\gamma^{-1}$ relative to $\{0,1\}$, so this is a loop based at x_1 and $[\gamma^{-1}\alpha\beta\alpha^{-1}\beta^{-1}\gamma] = [\lambda]$. SO $[\lambda] = [\gamma^{-1}\alpha\gamma][\gamma^{-1}\beta\gamma][\gamma^{-1}\alpha^{-1}\gamma][\gamma^{-1}\beta^{-1}\gamma] =: \alpha'\beta'\alpha'^{-1}\beta'^{-1}$, so the commutator $[[\alpha'],[\beta']] = \theta([\lambda])$. So $\pi_1(X) = \langle[\alpha'],[\beta'] : [\alpha'][\beta'][\alpha'^{-1}][\beta'^{-1}] = 1\rangle = \langle t,u : tut^{-1}u^{-1} = 1\rangle = \mathbb{Z}\times\mathbb{Z}$.

Definition (Manifold): for $n\in\mathbb{N}$, an n-dimensional manifold or n-manifold is a Hausdorff space X such that each $x\in X$ has a neighbourhood homeomorphic to $\mathbb{R}^n$; a 2-manifold is called a surface.

For S_1,S_2 surfaces, the connected sum $S_1\#S_2$ is obtained by: choose "discs" (i.e. open sets homeomorphic to discs) D_1, D_2 in the respective surfaces and consider $S_1 \setminus \mathrm{In}D_1 \sqcup S_2 \setminus \mathrm{In}D_2$ (where $\mathrm{In}D$ is the interior $D \setminus \partial D$); $S_1 \# S_2$ is the quotient of this when we identify ∂D_1 and ∂D_2 via some homeomorphism. We will see later that this space is independent both of our choice of discs and our choice of homeomorphism between the boundaries of the discs. Note that we did not need to embed the surfaces into some larger space, and there may be multiple, qualitatively different ways of doing this.

Example: $S_1 = T, S_2 = T$. More generally, $X_g = T \# \ldots \# T$ g times is called a closed surface of genus g. We can obtain this as the quotient of a $4g$-gon P: going clockwise around the boundary, label the edges as $a_1, b_1, a_1, b_1, a_2, b_2, a_2, b_2, \ldots, a_g, b_g, a_g, b_g$ and identify each a_i with the other, where one is taken in the clockwise direction and the other in the anticlockwise direction, similarly the b_i. As before, we define U to be the image of $P \setminus D$ where D is an interior disc, and $V = P \setminus \partial P$. We have $U \cap V \simeq S^1$, and $U \simeq$ the quotient of ∂P , which is $S^1 \vee \cdots \vee S^1$ $2g$ times, so $\pi_1(U) = \mathbb{Z} \star \cdots \star \mathbb{Z}$ $2g$ times. V is contractible, so $\pi(V) = 0$. Now if λ is a loop going once around D in P, then let θ bet the natural homomorphism $\pi_1(U \cap V) \to \pi_1(U)$, and then $\theta(\lambda) = \prod_{i=1}^{g} [\alpha'_i, \beta'_i]$, the product of commutators; this generates the image of θ, so by the van kampen theorem $\pi_1(X_g) = \{\alpha'_1, \ldots, \alpha'_g, \beta'_1, \ldots, \beta'_g : \prod_{i=1}^{g} [\alpha^{|}_i prime, \beta'_i] = 1\}$.

Covering Spaces

Take X path-connected, $p : \tilde{X} \to X$ a covering map, $p(\tilde{x}_0) = x_0 \in X$, then:

Theorem: $p_\star : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective: if $p_\star[\alpha] = [e_{x_0}]$ take a lift of $p_\star \alpha$ starting at $\tilde{x}_0$; this is the same α, but by homotopy lifting then $[\alpha] = [e_{\tilde{x}_0}]$.

Theorem: For $p : \tilde{X} \to X$ a covering map, $x_0 \in X$, $\{p_\star \pi_1(\tilde{X}, \tilde{x}_0) : \tilde{x}_0 \in p^{-1}(x_0)\}$ forms a conjugacy class of subgroups of $\pi_1(X, x_0)$: for $\tilde{x}, \tilde{y} \in p^{-1}\{x_0\}$ choose a path γ from $\tilde{x}$ to $\tilde{y}$. We have maps $p_\star : \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x_0)$, $\gamma_\# : \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(\tilde{X}, \tilde{y})$ and $p_\star : \pi_1(\tilde{X}, \tilde{y}) \to \pi_1(X, x_0)$; to "close up" this and make a commuting diagram our fourth map is $(p_\star \gamma)_\# : \pi_1(X, x_0) \to \pi_1(X, x_0)$.

$p_\star \gamma$ is a loop based at x_0. If $[\alpha] \in \pi_1(\tilde{X}, \tilde{x})$ then $[\alpha] \mapsto p_\star[\alpha] \mapsto [p_\star \gamma]^{-1}[p_\star \alpha][p_\star \gamma]$ or $[\alpha] \mapsto [\gamma^{-1} \alpha \gamma] \mapsto [p_\star \gamma^{-1}][p_\star \alpha][p_\star \gamma]$, so we really do have a commutative diagram, and $p_\star \pi_1(\tilde{X}, \tilde{y}) = [p_\star \gamma]^{-1} p_\star \pi_1(\tilde{X}, \tilde{x})[p_\star \gamma]$ [in the lecture the middle term was just $\pi_1(X, x_0)$, but this is ambiguous].

Now let H be a subgroup of $\pi_1(X, x_0)$ conjugate to $p_\star \pi_1(\tilde{X}, \tilde{x})$ (with $\tilde{x}$ fixed). Then by definition there is $[\beta] \in \pi_1(X, x_0)$ such that $H = [\beta^{-1} p_\star \pi_1(\tilde{X}, \tilde{x})[\beta]$; by path lifting we have a path $\tilde{\beta}$ starting at $\tilde{x}$, the endpoint of which must be some $\tilde{y} \in p^{-1}\{x_0\}$. Then choose $\gamma = \tilde{\beta}$. Thus we have the theorem.

We have an action of $\pi_1(X, x_0)$ on $p^{-1}\{x_0\}$, acting from the right, defined by for $[\alpha] \in \pi_1(X, x_0), \tilde{x} \in p^{-1}\{x_0\}$, $\tilde{x} \cdot [\alpha] = \tilde{\alpha}(1)$ where $\tilde{\alpha}$ is a lift of α starting at $\tilde{x}$. We clearly have $\tilde{x} \cdot [e_{x_0}] = \tilde{x}$, and the reader may verify that $\tilde{x} \cdot ([\alpha][\beta]) = (\tilde{x} \cdot [\alpha]) \cdot [\beta]$, so this really is an action.

Note that the orbit of $\tilde{x}$ under this action is $p^{-1}\{x_0\}$ by the above; in fact there is a bijection between elements of this orbit and right cosets of $p_\star \pi_1(\tilde{X}, \tilde{x})$ in $\pi_1(X, x_0)$. Note $p_\star \pi_1(\tilde{X}, \tilde{x}) = \{[\alpha] \in \pi_1(X, x_0) : \tilde{x} \cdot [\alpha] = \tilde{x}$.

Corollary: $p^{-1}\{x_0\}$ bijects with $p^{-1}\{y_0\}$ for all $x_0, y_0 \in X$.

Definition: a space X is <u>locally path-connected</u> if $\forall x \in X \forall$ neighbourhoods $U \ni x \exists$ open $V \subset U$, $V \ni x$ such that V is open and path-connected; note that unlike most "locally" properties this is a much stronger condition than being path-connected.

Suppose $p : \tilde{X} \to X$ is a covering map, $f : Y \to X$ a map and $f(y_0) = x_0 = p(\tilde{x}_))$. We call $\tilde{f} : Y \to \tilde{X}$ a lift of f if $f = p\tilde{f}$ and $\tilde{f}(y_0) = \tilde{x}_0$.

Theorem: for $p : \tilde{X} \to X$ a covering map, X path connected and $f : Y \to X$ a map with $f(y_0) = p(\tilde{x}_0) = x_0$, Y connected and locally path-connected, there is a lift of f iff $f_\star\pi_1(Y, y_0) \subset p_\star\pi_1(\tilde{X}, \tilde{x}_0)$: if there is a lift, then clearly $f_\star\pi_1(Y, y_0) \subset p_\star\pi_1(\tilde{X}, \tilde{x}_0)$ since we have $p_\star : \pi_1(Y, y_0) \to \pi_1(X, x_0) = p_\star \circ f$. Conversely if we have $f_\star\pi_1(\dots) = p_\star\pi_1(\dots)$, for any $y \in Y$ choose a path α from y_0 to y. Now $f_\star\alpha$ is a path in X from x_0 to $f(y)$; lift this path to $\tilde{X}$ starting at $\tilde{x}_0$, obtaining $\widetilde{f_\star\alpha}$. Define $\tilde{f}(y) = \widetilde{f_\star\alpha}(1)$; this is well defined, since if β is another path from y_0 to y then $\alpha\beta^{-1}$ is a loop based at y_0, so $[\alpha\beta^{-1}] \in \pi_1(Y, y_0)$ and by assumption $f_\star[\alpha\beta^{-1}] \in p_\star\pi_1(\tilde{X}, \tilde{x}_0)$. So $f_\star[\alpha\beta^{-1}]$ is the image of some loop based at $\tilde{x}_0$, so its lift $\widetilde{f_\star[\alpha\beta^{-1}]}$ is a loop based at $\tilde{x}_0$; $\widetilde{f_\star[\alpha\beta^{-1}]} = \widetilde{f_\star\alpha}\widetilde{f_\star\beta}$ so $\widetilde{f_\star\alpha}(1) = \widetilde{f_\star\beta}(1)$ and $\tilde{f}$ is well defined; we have $p\tilde{f} = f$ by construction. We now just need continuity of $\tilde{f}$:

Assume $V' \subset \tilde{X}$ is open; to prove $\tilde{f}^{-1}V'$ is open, it is enough to prove this for $V' \subset U'$ where U' is a component of $p^{-1}U$ for some elementary $U \subset X$ (so $U' \simeq U$): let $V = p(V') \simeq V'$. Now for any $y \in \tilde{f}^{-1}V'$, since f is a map $f^{-1}V$ is open, so there is an open $W \subset f^{-1}V$ with $W \ni y$ and W path-connected. Now $f(W) \subset V$, so $\tilde{f}(W) \subset V'$; $\tilde{f}(y) \in V'$¡ and $\tilde{f}(W) \subset p^{-1}V$, but W is path-connected, so the image of W is path-connected: for γ a path in V starting at y, take a lift of $f\gamma$ starting at $\tilde{f}(y)$. So $\tilde{f}(W) \subset V'$, since p^{-1} is a disconnected union of copies of V (and we get the right one because $f(y) \in V$), so $\tilde{f}^{-1}V'$ is open and $\tilde{f}$ is a map.

Addendum: This $\tilde{f}$ is unique, which follows from our much earlier claim that if Y is connected, $f : Y \to X$, $p : \tilde{f} \to f$ a covering, $\tilde{f}, \tilde{\tilde{f}}$ both maps $Y \to \tilde{X}$ with $p\tilde{f} = p\tilde{\tilde{f}} = f$, then $\{y : \tilde{f}(y) = \tilde{\tilde{f}}(y)\}$ is either $\emptyset$ or Y (then we have the result as $\tilde{f}(y_0) = \tilde{\tilde{f}}(y_0) = \tilde{x}_0$).

Definition: for X path-connected, a covering space $p : \tilde{X} \to X$ is called a <u>universal cover</u> if $\tilde{X}$ is simply connected.

Suppose $\tilde{X}$ is a universal cover of X, and $\tilde{X}'$ any covering of X, $p(\tilde{x}_0) = x_0 = p'(\tilde{x}'_0)$. By the theorem there is a lift $\tilde{p} : \tilde{X} \to \tilde{X}'$. Now if $\tilde{X}'$ is also universal, we also have a lift $\tilde{p}' : \tilde{X}' \to \tilde{X}$; in this case $\tilde{p}\tilde{p}' = 1_{\tilde{X}'}, \tilde{p}'\tilde{p} = 1_{\tilde{X}}$. So there is a sense in which universal covers are unique [a better lecturer might perhaps have told us what this sense is; ah well].

Assume $p : \tilde{X} \to X$ is universal; suppose $U \subset X$ is elementary, $V \subset p^{-1}(U)$ homeomorphic to U (by $p\mid_V$). Take $\tilde{x}_0 \in V, x_0 \in U$; then we have maps communing maps $\pi_1(V, \tilde{x}_0) \to \pi_1(\tilde{X}, \tilde{x}_0), \pi_1(U, x_0) \to \pi_1(X, x_0), p_\star : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ and $\pi_1(V, \tilde{x}_0) \xrightarrow{\sim} \pi_1(U, x_0)$, but $\pi_1(\tilde{X}, \tilde{x}_0) = 0$, so each $x_0 \in X$ has a neighbourhood U such that the natural map $\pi_1(U, x_0) \to \pi_1(X, x_0)$ is trivial.

Example: Take Q_n to be the circle in $\mathbb{R}^2$ of centre $(\frac{1}{n}, 0)$ of radius $\frac{1}{n}$ (i.e. through 0) [set $X = \bigcup_{n\in\mathbb{N}} Q_n$, with the topology inherited from $\mathbb{R}^2$]; then any neighbourhood U of $x_0 = (0, 0)$ will contain a loop going aroung one of the circles (in fact, infinitely many such), so the image of $\pi_1(U, x_0) \to \pi_1(X, x_0)$ is not 0. By the above result this means X cannot have any universal cover.

Theorem: Let X be path-connected and locally path-connected, and $p : \tilde{X} \to X$ a universal cover. Then for any conjugacy class of subgroups of $\pi_1(X)$, there is a covering space corresponding to it: let H be a subgroup of $\pi_1(X)$. Then there is a bijection between the set of $[\gamma]$ for γ paths in X starting at x_0 and points of $\tilde{X}$ by $[\gamma] \mapsto \tilde{\gamma}(1)$: the inverse of this is $\tilde{x} \mapsto [p_\star\alpha]$ where α is a path from $\tilde{x}_0$ to $\tilde{x}$; this is well defined as $\tilde{X}$ is simply connected.

Now define an equivalence relation by $\tilde{x} \sim \tilde{y}$ if $p(\tilde{x}) = p(\tilde{y})$ and $[\gamma_{\tilde{x}} \cdot \gamma_{\tilde{y}}^{-1}] \in H$; this is an equivalence relation: $\tilde{x} \sim \tilde{x}$ since $p(\tilde{x}) = p(\tilde{x})$ and $[\gamma_{\tilde{x}} \cdot \gamma_{\tilde{x}}^{-1}] = [e_{x_0}] \in H$; if $\tilde{x} \sim \tilde{y}$ then $p(\tilde{y}) = p(\tilde{x})$ and $[\gamma_{\tilde{y}} \cdot \gamma_{\tilde{x}}^{-1}] = [\gamma_{\tilde{x}} \cdot \gamma_{\tilde{y}}^{-1}]^{-1} \in H$, so $\tilde{y} \sim \tilde{x}$. If $\tilde{x} \sim \tilde{y}$ and $\tilde{y} \sim \tilde{z}$ then $p(\tilde{x}) = p(\tilde{y}) = p(\tilde{z})$ and $[\gamma_{\tilde{x}}\gamma_{\tilde{z}}^{-1}] = [\gamma_{\tilde{x}}\gamma_{\tilde{y}}^{-1}][\gamma_{\tilde{y}}\gamma_{\tilde{z}}^{-1}] \in H$ so $\tilde{x} \sim \tilde{z}$.

Now put $X_H = \frac{\tilde{X}}{\sim}$; we have natural maps $r : \tilde{X} \to X_H$, the quotient map, and $q : X_H \to X$ such that $p = qr$. Take the quotient topology on X_H; r is then continuous so X_H is path-connected. For any $x \in X$, U an elementary open set containing x, $p^{-1}U$ is a disjoint union of "copies" of (i.e. regions homeomorphic to) U; take V and W to be two components of $p^{-1}U$.Assume $r(V) \cap r(W) \neq \emptyset$; take $\tilde{x} \in V, \tilde{y} \in W$ such that $r(\tilde{x}) = r(\tilde{y})$; by the definition of X_H $\tilde{x} \sim \tilde{y}$, i.e. $[\gamma_{\tilde{x}} \cdot \gamma_{\tilde{y}}^{-1}] \in H$. Let α, β be lifts of $\gamma_{\tilde{x}}, \gamma_{\tilde{y}}$ starting at $\tilde{x}_0$; let $\tilde{z} \in V, \tilde{t} \in W$ such that $p(\tilde{z}) = p(\tilde{t})$. Choose a path λ in V from $\tilde{x}$ to $\tilde{z}$, and let θ be the corresponding path from $\tilde{y}$ to $\tilde{t}$ in W, so $p_\star\lambda = p_\star\theta$. Now $[p_\star(\alpha\lambda) \cdot p_\star(\beta\theta)^{-1}] = [p_\star\alpha \cdot p_\star\lambda \cdot p_\star\theta^{-1} \cdot p_\star\beta^{-1}] = [p_\star\alpha \cdot p_\star\beta^{-1}] \in H$, but this is $[\gamma_{\tilde{z}} \cdot \gamma_{\tilde{t}}^{-1}]$, so $\tilde{z} \sim \tilde{t} \therefore r(\tilde{z}) = r(\tilde{t})$, so $r(V) = r(W)$. Therefore $r : X_H \to X$ is a covering map, because $q^{-1}(U)$ is a disjoint union of "copies" of U.

Put $x_0^H = r(\tilde{x}_0)$; if μ is a loop in X_H based at x_0^H then $q_\star\mu$ is a loop based at x_0 in X, so $\tilde{x}_0 \sim \widetilde{q_\star\mu}(1) \Rightarrow [p_\star\widetilde{q_\star\mu} \cdot e_{\tilde{x}_0}^{-1}] \in H$, i.e. $[q_\star\mu] \in H$, so $q_\star\pi_1(X_H, x_0^H) \subset H$.

Now let $[\gamma] \in H$, then $\tilde{\gamma}(1) \sim \tilde{x}_0$ so $r_\star\tilde{\gamma}$ is a loop in X_H based at x_0^H. But $[q_\star(r_\star\tilde{\gamma})] = [\gamma] \in H$, so $q_\star\pi_1(X_H, x_0^H) = H$.

If $H' = h^{-1}Hh$ with $h = [\gamma]$, take $\tilde{\gamma}$ a lift of Y to X_H, going from x_0^H to some point y_0^H; the image of the same $q : X_H \to X$ taken as a map from $\pi_1(X_H, y_0^H) \to \pi_1(X, x_0)$ rather than $\pi_1(X_H, x_0^H) \to \pi_1(X, x_0)$ has image H'.

Take X path-connected and locally path-connected (this is not really part of this definition, but many books like to take all surfaces path-connected and locally path-connected when covering these subjects). Suppose that $\forall x \in X$ there is a neighbourhood $U \subset X$, $U \ni x$ such that the natural map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial. Then we say X is semilocally simply connected

Theorem: A space X which is path-connected, locally path-connected and semilocally simply connected has a universal cover; we shall not prove this theorem, but the cover is given by setting $\tilde{X}$ to be the set of $[\gamma]$ for all paths γ starting at x_0.

Example: this theorem applies for any connected manifold X.

Homology Theory

In contrast to $\pi_1(X)$, the groups $H_i(X, \mathbb{Z})$ which we will study in this section are easy to calculate but quite hard to define; we shall do so over the course of the next through lectures.

Simplicial complexes: Suppose $a_0, \ldots, a_n \in \mathbb{R}^n$. We say these are independent (in the affine sense) if $a_1 - a_0, \ldots, a_N - a_0$ are independent vectors in $\mathbb{R}^n$; equivalently if there are λ_i with $\sum \lambda_i a_i = 0$ and $\sum \lambda_i = 0$ then $\lambda_i = 0 \forall i$.

Suppose we have $a_0, \ldots, a_n \in \mathbb{R}^N$ independent. We define the n-dimensional simplex $\sigma = (a_0 \ldots a_n)$ to be $\{\sum \lambda_i a_i : \sum \lambda_i = 1, \lambda_i \geq 0\}$. This is the convex hull of the a_i, i.e. the smallest convex set containing all the a_i. In fact for a point in this set the λ_i are unique (and this is even true in the plane generated by the a_i); they are called the barycentric coordinates.

If τ is a complex such that the vertices of τ are a subset of the vertices of σ then we write $\tau \subset \sigma$; we can write $\tau < \sigma$ to mean $\tau \leq \sigma$ and $\tau \neq \sigma$. If $\tau < \sigma$ we call τ a proper face of σ. By convention we take $\emptyset$ to be a simplex, with $\emptyset \leq \sigma$ for any simplex σ.

The interior of σ is $\sigma^0 := \{\sum \lambda_i a_i : \sum \lambda_i = 1, \lambda_i > 0\}$. The barycentre of σ is $\hat{\sigma} := \frac{1}{n+1}(a_0 + \cdots + a_n)$.

A (finite) complex K is a collection of simplexes such that if $\sigma \in K$ and $\tau \leq \sigma$ then $\tau \in K$, and if $\sigma, \tau \in K$ then $\sigma \hat{\tau} \in K$ [by which we mean the intersection of the interiors of σ, τ, not the intersection of their vertices. I think].

For a simplex $\sigma = (a_0 \ldots a_n)$ we call n the dimension of σ; for a complex K, $\dim K = \max_{\sigma \in K} \dim \sigma$.

A subcomplex of a complex K is a complex L such that $L \subset K$.

For a complex K we define $|K|$ to be the polyhedron of K, i.e. the union of all the georemtric points of K $\bigcup_{\sigma \in K} |\sigma|$, where $|\sigma|$ is the set of points of σ in $\mathbb{R}^N$. We take the induced topology from $\mathbb{R}^N$ on $|K|$; clearly under this $|K|$ is compact.

We want to "approximate" all "nice" spaces by deforming them into complexes, e.g. S^1 is homeomorphic to a triangle (without interior), S^2 to a tetrahedron. This is nice because complexes are, in some piecewise sense, "linear"; then maps can be deformed into "linear" maps between complexes.

Suppose K, L are complexes. A simplicial map $s : K \to L$ is a function from the vertices of K to the vertices of L such that if $\sigma = (a_0 \ldots a_n)$ then $\{s(a_0), \ldots, s(a_n)\}$ are precisely the vertices of some simplex of L (but we don't say $(s(a_0) \ldots s(a_n))$ is a simplex of L because we want to allow the case where some of the $s(a_i)$ are equal to each other). We can extend S to a map $s : |K| \to |L|$: for any $\sigma \in K$ we extend s from the vertices of $\sigma = (a_0 \ldots a_n)$ to all points of σ by $s(\sum \lambda_i a_i) = \sum \lambda_i s(a_i)$. We call s an isomorphism if it is bijective.

Lemma: For K a complex, $|K| = \bigsqcup_{\sigma \in K} \sigma^0$: clearly $|K| \supset \bigsqcup \sigma^0$; now if $x \in |K|$ then take $\sigma \in K$ such that $x \in \sigma$, then $x \in \tau^0$ for some $\tau \leq \sigma$.

Definition: For K a complex, a a vertex of K, we define $\mathrm{star}(a, t)$ by $\bigcup_{\sigma \in K, \sigma \ni a} \sigma^0$

Definition: Suppose $f : X \to Y$ is a map with $X = |K|, Y = |L|$. A simplicial map $s : K \to L$ is a simplicial approximation of f if $f(\mathrm{star}(a, K)) \subset \mathrm{star}(s(a), L)$ [$\forall a \in |K|$?].

Examples: 1) $f : I \to I$ $f(x) = x$; consider the first space $X = I$ as the complex K with vertices $0, \frac{1}{2}, 1$; similarly Y is the complex K with vertices $0, \frac{1}{3}, 1$. Define $s(0) = 0, s(\frac{1}{2}) = \frac{1}{3}, s(1) = /1$, but this is not a simplicial approximation because $f(\mathrm{star}(0, K)) = f([0, \frac{1}{2})), \mathrm{star}(s(0), L) = [0, \frac{1}{3})$. 2) $f : X = I \to Y = I$ $f(x) = x^2$. Let $K =$ the complex with vertices $0, \frac{1}{2}, 1$, L have vertices $0, \frac{1}{4}, 1$, then $s : 0 \mapsto 0, \frac{1}{2} \mapsto \frac{1}{4}, 1 \mapsto 1$ is a simplicial approximation.

In the first example, we cannot directly get a simplicial approximation. However, if we add the points $\frac{1}{4}, \frac{3}{4}$ to K then $0 \mapsto 0, \frac{1}{4} \mapsto 0, \frac{1}{2} \mapsto \frac{1}{3}, \frac{3}{4} \mapsto \frac{1}{3}, 1 \mapsto 1$ is a simplicial approximation.

Definition: 1) We say a topological space X is triangulated if there is a (simplicial) complex K and homeomorphism $f : |K| \to X$. 2) For K a complex,

$\sigma \in K$ a simplex we define $\mathrm{star}(\sigma, K) = \bigcup_{\sigma \leq \tau \in K} \tau^0$ (note this agrees with our earlier definition of the star of a vertex).

Lemma: For K a complex and $a_0, \ldots, a_n$ vertices of K, $\bigcap_i \mathrm{star}(a_i, K)$ is non-empty only when the a_i are all vertices of some simplex $\sigma \in K$. In this case, $\mathrm{star}(\sigma, K) = \bigcap_{i=0}^n \mathrm{star}(a_i, K)$: Suppose $x \in \bigcap_{i=0}^n \mathrm{star}(a_i, K)$, so $\exists! \sigma' \in K$ such that $x \in \sigma'^0$ and $a_0, \ldots, a_n$ are vertices of σ'. Put $\sigma = (a_0 \ldots a_n) \leq \sigma'$; clearly $\mathrm{star}(\sigma, K) \supset \bigcap_i \mathrm{star}(a_i, K)$. Now for any $x \in \mathrm{star}(\sigma, K)$, $\exists! \tau \in K$ with $x \in \tau^0, \sigma \leq \tau$. Since τ contains all the vertices $a_0, \ldots, a_n$, $x \in \mathrm{star}(a_i, K) \forall i$, so $x \in \bigcap_i \mathrm{star}(a_i, K)$.

Theorem: Let K, L be complexes, $f : |K| \to |L|$ and $s : K \to L$ a simplicial approximation of f. Then $f \simeq s \mathrm{rel}\{x \in |K| : f(x) = s(x)\}$: Let $x \in |K|$. $\exists! \sigma \in K$ with $\sigma^0 \ni x$, say $\sigma = (a_0 \ldots a_n)$. So $x \in \mathrm{star}(\sigma, K) = \bigcap_i \mathrm{star}(a_i, K)$. Thus by the definition of a simplicial approximation, $f(x) \in f(\bigcap_i \mathrm{star}(a_i, K) \subset \bigcap_i f(\mathrm{star}(a_i, K)) \subset \bigcap_i \mathrm{star}(s(a_i), L) \neq \emptyset$. Thus $\{s(a_0), \ldots, s(a_n)\}$ are the vertices of a simplex $\tau \in L$, so $f(x) \in \mathrm{star}(\tau, K)$. Again since s is simplicial, $s(\sigma) = \tau$; in particular $s(x) \in \tau$. Now $\exists! \tau' \in L$ such that $f(x) \in \tau'^0$ and $\tau \leq \tau'$, so $s(x), f(x)$ are both $\in \tau'$. Since τ' is a convex set, we can apply linear homotopy.

Remark: We have proved that the smallest simplex in L that contains $f(x)$ also contains $s(x)$. Suppose that we have this property (but don't assume s is simplicial); then we want to show $f(\mathrm{star}(a, K)) \subset \mathrm{star}(s(a), L)$ and we will have that this is equivalent to s being simplicial: suppose $x \in \mathrm{star}(a, K)$, then $\exists! \sigma \in K$ such that $x \in \sigma^0$, so $s(x) \in s(\sigma)^0$. But also $\exists! \mu \in L$ such that $f(x) \in \mu^0$. Now by the property above $s(x) \in \mu$, and by the definition of a complex $s(\sigma) \cap \mu \leq \mu$, and $s(\sigma) \cap \mu \leq s(\sigma)$; since $s(\sigma) \cap \mu$ contains an interior point of $s(\sigma)$, $s(\sigma) \cap \mu = s(\sigma)$. Therefore $s(\sigma) \leq \mu$. Now $s(a)$ is a vertex of $s(\sigma)$ and of μ; since $f(x) \in \mu^0$, $f(x) \in \mathrm{star}(s(a), L)$ and we have the result. In practice it is often more convenient to use this rather than the "official" definition of being simplicial.

Definition: For K a complex, the first derived barycentric subdivision of K is $K^{(1)} := \{(\hat{\sigma}_0 \hat{\sigma}_1 \ldots \hat{\sigma}_n) : \sigma_0 < \sigma_1 < \cdots < \sigma_n \in K\}$ (recall $\hat{\sigma}$ is the barycentre of σ). Then we define inductively $K^{(r)} = (K^{(r-1)})^{(1)}$.

Definition: For K a complex, the <u>mesh</u> of K is the maximum diameter of any simplex of K.

Lemma: For any complex K, for any $\epsilon > 0$ there is an r such that the mesh of $K^{(r)}$ is smaller than ϵ; the proof of this is an exercise in elementary geometry.

Theorem: For K, L complexes and any $f : |K| \to |L|$, for some r there is a simplicial approximation $s : |K^{(r)}| \to |L|$ of f (this is meaningful because $|K^{(r)}| = |K|$): Take r to be large enough that $f(\mathrm{star}(a, K^{(r)}) \subset \mathrm{star}(b, L)$ for some vertex b of L (not necessarily unique) [presumably we take r such that this holds for all a?] Choose one such b and call it $s(a)$; this gives us a function $s : \{\text{vertices of } K^{(r)}\} \to \{\text{vertices of } L\}$. Now for any $\sigma \in K^{(r)}$, $\sigma = (a_0 \ldots a_n)$, choose $x \in \sigma^0$. By assumption $f(xx) \in \mathrm{star}(s(a_i), L) \forall i$, i.e. $f(x) \in \bigcap_i \mathrm{star}(s(a_i), L)$, so this intersection is nonempty and by the lemma above (previous to the one immediately preceding this), $\{s(a_0), \ldots, s(a_n)\}$ is the set of all the vertices of some simplex of L. So s is a simplicial approximation of f.

Example: $X = I, Y = S^2$. Let $\gamma : I \to S^2$ be a path of S^2. We can triangulate S^2 by the complex L of a cube with one diagonal on each face, and we can choose this to be done in such a way that $\gamma(0), \gamma(1)$ are vertices of L.

So γ is homotopic to a path along the edges of L, and the fundamental group of S^2 is the edge group of L.

Homology Groups

Definition: A sequence of abelian groups and homomorphisms $\mathcal{G} = \cdots \to G_{-2} \xrightarrow{\partial_2} G_{-1} \xrightarrow{\partial_1} G_0 \xrightarrow{\partial_0} G_1 \xrightarrow{\partial_{-1}} G_{-2} \to \ldots$ is a chain complex if $\partial_i \partial_{i+1} = 0$ $[\forall i]$. It is exact or an exact sequence if $\ker \partial_i = \mathrm{Im}\delta_{i+1}$. An exact sequence $0 \to G' \to G \to G'' \to 0$ is called a short exact sequence.

For a chain complex as above, the homology groups are defined as $H_i(\mathcal{G}) = \frac{\ker \partial_i}{\mathrm{Im}\partial_{i+1}}$; if $\mathcal{G}$ is exact then $H_i(\mathcal{G}) = 0 \forall i$ and vice versa.

An ordered simplex is a simplex $\sigma = (a_0 \ldots a_n)$ where the order of $a_0, \ldots, a_n$ is considered.

Let K be a (simplicial) complex. The ith chain group $C_i(K)$ is defined as the quotiend of the free abelian group generated by ordered i-simplexes by $\langle \sigma - (\mathrm{sgn}\pi)\pi\sigma \rangle$ for (every) $\pi \in S_{i+1}$, the symmetric group on $i+1$ elements (i.e. the free abelian group generated by i-simplexes with two possible orderings), $C_i(K) = 0$ for $i < 0$ and $i > \dim K$. We also define homomorphisms $\partial_i : C_i(K) \to C_{i+1}(K)$ by for σ an i-simplex, $\partial_i(\sigma = (a_0 \ldots a_i)) = \sum_{j=0}^{i} (-1)^j (a_0 \ldots \hat{a}_j \ldots a_i)$ (i.e. the summand is $(a_0 \ldots a_{j-1} a_{j+1} \ldots a_i)$ (so e.g. for $\sigma = (a_0a_1a_2)$, $\partial_2(\sigma) = (a_1a_2) - (a_0a_2) + (a_0a_1) = (a_0a_1) + (a_1a_2) + (a_2a_0)$). ∂_i is called a boundary homomorphism (or operator).

We have a sequence $\mathcal{C} = \cdots \to C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(k) \xrightarrow{\partial_0 = 0} 0$ and this is a chain complex as $\partial_i \partial_{i+1} = 0$: let $\sigma = (a_0 \ldots a_{i+1})$ be an $i+1$-simplex, then $\partial_i \partial_{i+1}(\sigma) = \partial_i(\sum_{j=0}^{i+1} (-1)^j (a_0 \ldots \hat{a}_j \ldots a_{i+1} = \sum_{j=0}^{i+1} (-1)^j \sum_{k=j+1}^{i+1} (-1)^{k-1} (a_0 \ldots \hat{a}_j \ldots \hat{a}_k \ldots a_{i+1}) + \sum_{j=0}^{i+1} (-1)^j \sum_{k=0}^{j-1} (-1)^k (a_0 \ldots \hat{a}_k \ldots \hat{a}_j \ldots a_{i+1})$; each simplex appears twice in this with different signs $(-1)^{j+k-1}$ and $(-1)^{j+k}$, so this is 0. So we can define the homology of K to be $H_i(K) = H_i(\mathcal{C}) = \frac{\ker \partial_i}{\mathrm{Im}\partial_{i+1}}$. We usually call $Z_i(K) := \ker \partial_i$ the group of i-cycles and $B_i(K) := \mathrm{Im}\partial_{i+1}$ the group of i-boundary chains.

Example: For K a complex, $H_0(K)$ is the free abelian group generated by the connected components of K: suppose a, b are vertices of K: 1) in the case a, b belong to the same component, by the example sheet there is an edge path connecting a, b, i.e. there are vertices $a_1, \ldots, a_n$ such that we have a path $(aa_1) + (a_1a_2) + \cdots + (a_nb)$; this is a 1-chain in $G(K)$ and its boundary (applying ∂_1) is $a_1 - a + a_2 - a_1 + \cdots + b - a_n = b - a$, so $b - a \in B_0(K)$, but $C_0(K) = Z_0(K)$, so if a, b are in the same component then $a = b \in H_0(K)$. 2) Suppose a, b are not in the same component, then it is easy to see that $b - a \notin B_0(K)$ so $b \neq a$ is $H_0(K)$. So $H_0(K)$ is a free abelian group with rank equal to the number of connected components.

Remark: By the definition of homology groups $H_i(K)$ is a finitely generated abelian group. So we can write $H_i(K) = F_i \oplus T_i$ where F_i is the free part and T_i is the torsion part. The ith Betti number is defined as $\mathrm{rank}(F_i)$; we sometimes write b_i or β_i.

Example: A cone: suppose K is a complex, $|K| \subset \mathbb{R}^N$; take $a \in \mathbb{R}^{N+1} \setminus \mathbb{R}^N$ and define the cone CK over K as $K \cup \{a\} \cup \{(aa_0 \ldots a_n) : (a_0 \ldots a_n) \in K\}$; obviously CK is connected, so $H_0(CK) = \mathbb{Z}$. We define a homomorphism $d_i : C_i(K) \to C_{i+1}(K)$ by $d_i(\sigma = (a_0 \ldots a_i)) = (aa_0 \ldots a_i)$ if $\sigma \in K$, $d_i(\sigma) = 0$ if $\sigma \notin K$. Now for $i.0$, $(\partial_{i+1} d_i)(\sigma = (a_0 \ldots a_i)) = (a_0 \ldots a_i) +$

$\sum_{j=0}^{i+1}(-1)^{j+1}(aa_0 \dots \hat{a}_j \dots a_i)$, and $\sigma-(d_{i+1}\partial_i)(\sigma) = (a_0 \dots a_i)-d_{i-1}(\sum_j(-1)^j(a_0 \dots \hat{a}_j \dots a_i)) = (a_0 \dots a_i) + \sum_j(-1)^{j+1}(aa_0 \dots \hat{a}_j \dots a)$, so if $\sigma \in K$ then $\partial_{i+1}d_i(\sigma) = \sigma - d_{i-1}\partial_i(\sigma)$; clearly this also holds for $\sigma \notin K$. In particular if $z \in Z_i(CK)$ then $\partial_{i+1}d_i(z) = z$, so $Z_i(CK) = B_i(CK)$ so $H_i(CK) = 0$ for $i > 0$.

Examples: Let $\sigma = (a_0 \dots a_n)$ be an n-simplex and K the set of faces of σ, $L = K \setminus \{\sigma\}$. Then K is a cone (just by picking a vertex and the face opposite it), so $H_i(K) = 0 \forall i > 0$ and $H_0(K) = \mathbb{Z}$. Consider $0 \to C_{n-1}(L) \to C_{n-2}(L) \to \cdots \to C_0(L) \to 0$; take $n \neq 0, 1$. We have a corresponding $C_n(K) \to C_{n-1}(K) \to C_{n-2}(K) \to \cdots \to C_0(K) \to 0$; we have $C_{n-1}(K) = C_{n-1}(L), C_{n-2}(K) = C_{n-2}(L)$ and so on, and the ∂s for these are also equal. So $Z_i(K) = Z_i(L) \forall i \leq n-1$ and $B_i(K) = B_i(L) \forall i \leq n-2$. $B_{n-1}(L) = 0$; we know $H_{n-1}(K) = 0 = \frac{Z_{n-1}(K)}{B_{n-1}(K)}$ so $Z_{n-1}(K) = B_{n-1}(K)$ but this also $= Z_{n-1}(L)$. So $H_{n-1}(L) = \frac{Z_{n-1}(L)}{B_{n-1}(L)} = \frac{B_{n-1}(K)}{0} = B_{n-1}(K)$. Note that $C_n(K) = \langle\sigma\rangle = \mathbb{Z}$, so $B_{n-1}(K) = \mathrm{Im} C_n(K) = \mathbb{Z}$. So $H_{n-1}(L) = \mathbb{Z}$. Thus $H_0(L) = \mathbb{Z}, H_i(L) = 0 \forall 0 < i < n-1, H_{n-1}(L) = \mathbb{Z}$.

Remark: For $n = 2$, $\pi_1(|L|) = H_1(L) = \mathbb{Z}$. In general, $H_1(X) = \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]}$ the abelianization of $\pi_1(X)$.

Remark: $|L|$ is homeomorphic to S^{n-1}; if we accept for now that homology groups are invariant under homeomorphisms, we have that $H_i(S^{n-1}) = H_i(L)$.

Our main goals for the next few lectures are: define $H_i(X)$ for more spaces X, and prove that if maps $f, g : X \to Y$ are homotopic then the induced homomorphisms $f_\star : H_n(X) \to H_n(Y) = g_\star : H_n(X) \to H_n(Y) \forall n$.

Definition: Let K, L be complexes and $s, t : K \to L$ simplicial maps. We say that s and t are <u>contiguous</u> if for any $\sigma \in K$, $\exists \tau \in L$ such that $s(\sigma), t(\sigma)$ are faces of τ. The smallest such τ is called the <u>carrier</u> of σ. By linear homotopy, this implies $s \simeq t$.

Lemma: Suppose $s, t : K \to L$ are simplicial maps which approximate a map $f : |K| \to |L|$. Then s, t are contiguous: let $\sigma \in K$, take $x \in \sigma^0$. We already know that the smallest $\tau \in L$ with $\tau \ni f(x)$ also contains $s(x), t(x)$, so $s(\sigma, t(\sigma) \leq \tau$.

Definition (chain map): Suppose we have twto chain complexes $\mathcal{G} : \cdots \to G_n \xrightarrow{\partial_n^{\mathcal{G}}} G_{n-1} \to \dots, \mathcal{F} : \cdots \to F_n \xrightarrow{\partial_n^{\mathcal{F}}} F_{n-1} \to \dots$. A chain map $d : \mathcal{G} \to \mathcal{F}$ is a collection of homomorphisms $d_n : G_n \to F_n$ which give us a commutative diagram, i.e. $\partial_n^{\mathcal{F}} d_n = d_{n-1}\partial_n^{\mathcal{G}} \forall n$.

Definition (chain homotopy): We say chain maps $d, d' : \mathcal{G} \to \mathcal{F}$ are chain homotopic if there are homomorphisms $e_n : G_n \to F_{n+1}$ such that $d_n - d'_n = e_{n-1}\partial_n^{\mathcal{G}} + \partial_{n+1}^{\mathcal{F}} e_n$ (this definition appears bizzare, but it will turn out to be precisely what we need for the homology groups to be the same).

Lemma: Suppose $d, d' : \mathcal{G} \to \mathcal{F}$ are chain homotopic chain maps. Then they induce the same homomorphisms $d_\star = d'_\star : H_n(\mathcal{G}) \to H_n(\mathcal{F})$: first we explain how a chain map induces homomorphisms $d_\star : H_n(\mathcal{G}) \to H_n(\mathcal{F})$. From the definition of a chain map we have $d_n(Z_n(\mathcal{G})) \subset Z_n(\mathcal{F}), d_n(B_n(\mathcal{G})) \subset B_n(\mathcal{F})$ so we have natural homomorphisms $d_\star : \frac{Z_n(\mathcal{G})}{B_n(\mathcal{G})} \to \frac{Z_n(\mathcal{F})}{B_n(\mathcal{F})}$, i.e. $d_\star : H_n(\mathcal{G}) \to H_n(\mathcal{F})$. Now since d, d' are chain homotopic, $d_n - d'_n = e_{n-1}\partial_n^{\mathcal{G}} + \partial_{n+1}^{\mathcal{F}} e_n$. To see that $d_\star = d'_\star$, it is enough to apply the formula to $z \in Z_n(\mathcal{G})$: $d_n(z) - d'_n(z) = e_{n-1}\partial_n^{\mathcal{G}}(z) + \partial_{n+1}^{\mathcal{F}} e_n(z)$. The first term is 0 since z is a cycle by definition; the second term lies in $B_n(\mathcal{F})$, so the entire thing is 0 in $\frac{Z_n(\mathcal{F})}{B_n(\mathcal{F})} = H_n(\mathcal{F})$. So

$d_\star(z) = d'_\star(z)$.

Definition/Lemma: Suppose $s : K \to L$ is a simplicial map. Then s induces homomorphisms $s_\star : H_n(K) \to H_n(L)$ in the following way: enough to show s induces a chain map $s : \mathcal{C}(K) \to \mathcal{C}(L)$: we want to define homomorphisms $s_n : C_n(K) \to C_n(L)$. For $\sigma \in C_n(K)$ an (oriented) simplex, $\sigma = (a_0 \dots a_n)$ we define $s_n(\sigma) = \{0$ if $\dim s(\sigma) < n$, $(s(a_0) \dots s(a_n))$ if $\dim s(\sigma) = n\}$. To show that this gives a commutative diagram it is enough to show $\partial_n^L s_n(\sigma) = s_{n-1}\partial_n^K(\sigma)$ for $\sigma = (a_0 \dots a_n) \in C_n(K)$ an n-simplex: $\partial_n^L s_n(\sigma) = 0$ if $\dim s(\sigma) < n$, $\sum_{i=0}^n (-1)^i (s(a_0) \dots \widehat{s(a_i)} \dots s(a_n))$ otherwise, $s_{n-1}\partial_n^K(\sigma) = 0$ if $\dim s(\sigma) < n-1$, $(-1)^j(s(a_0) \dots \widehat{s(a_j)} \dots s(a_n)) + (-1)^k(s(a_0) \dots \widehat{s(a_k)} \dots s(a_n))$ if $\dim s(\sigma) = n-1$ and $s(a_j) = s(a_k)$ (taking $j < k$), $\sum_j (-1)^j (s(a_0) \dots \widehat{s(a_j)} \dots s(a_n))$ is $\dim s(\sigma) = n$. $(-1)^k(s(a_0) \dots \widehat{s(a_k)} \dots s(a_n)) = (-1)^k(-1)^{k-j-1}(s(a_0) \dots \widehat{s(a_j)} \dots s(a_n))$ as we have to perform k and then $k-j-1$ swaps to switch a_j and a_k, so the $\dim s(\sigma) = n-1$ case is actually $= 0$, and s gives a chain map as required.

Lemma: Let $s, t : K \to L$ be contiguous simplicial maps, then $s_\star = t_\star$ as maps $H_n(K) \to H_n(L)$: for each simplicial map we get a chain map, $s_n : C_n(K) \to C_n(L)$ and $t_n : C_n(K) \to C_n(L)$, each of which individually gives a commutative diagram together with the $\partial : C_{n+1}(K) \to C_n(K)$ and the same for L. We want to prove that s, t are chain homotopic; by convention we first set $e_{-1} = 0$. Let $\sigma \in C_0(K)$; we define $e_0(\sigma) = 0$ if $s(\sigma) = t(\sigma)$, $(s(\sigma)t(\sigma))$ if $s(\sigma) \neq t(\sigma)$. So we get a homomorphism $C_0(K) \to C_1(L)$; it is easy to see that $\partial_1^L e_0 + e_{-1}\partial_0^K = t_0 - s_0$ (notice $e_0(\sigma) = t_0(\sigma) - s_0(\sigma)$. Now suppose we have already constructed $e_{-1}, e_0, \dots, e_{n-1}$ with 1) $\partial_{i+1}^L e_i + e_{i-1}\partial_i^K = t_i - s_i \forall 0 \leq i \leq n-1$ and 2) if $\sigma \in C_i(K)$ is a simplex, then $e_i(\sigma) \in C_{i+1}(L)$ is a chain in the carrier of σ (i.e. we require $e_i(\sigma) = \alpha = \sum m_k \tau_k$ to have $\tau_k \in$ the carrier of σ $\forall k$; saying τ is the carrier our condition is $e_i(\sigma) \in C_{i+1}(\tau)$) (recall the carrier of σ is the smallest simplex in L containing both $s(\sigma)$ and $\tau(\sigma)$) $\forall 0 \leq i \leq n-1$. Let $Z = t_n(\sigma) - s_n(\sigma) - e_{n-1}\partial_n^K(\sigma) \in C_n(L)$ for $\sigma \in C_n(K)$. $\partial_n^L(Z) = \partial_n^L(t_n(\sigma) - s_n(\sigma) - e_{n-1}\partial_n^K(\sigma)) = \partial_n^L t_n(\sigma) - \partial_n^L s_n(\sigma) - \partial_n^L e_{n-1}\partial_n^K(\sigma) = \partial_n^L t_n(\sigma) - \partial_n^L s_n(\sigma) - t_{n-1}\partial_n^K(\sigma) + s_{n-1}\partial_n^K(\sigma) + e_{n-2}\partial_{n-1}^K\partial_n^K(\sigma)$ (substituting for $\partial_n^L e_{n-1}$). Then $\partial_n^L t_n(\sigma) = t_{n-1}\partial_n^K(\sigma)$ and similarly for s since t, s are chain maps, but $\partial_{n-1}^K \partial_n^k = 0$, so this whole expression is 0. So $Z \in Z_n(L)$, so $Z \in C_n(\tau)$ where τ is the carrier of σ considered as a complex. On the other hand τ is a cone and so $H_n(\tau) = 0 \therefore B_n(\tau) = Z_n(\tau)$, so $Z \in B_n(\tau)$ and there is $x \in C_{n+1}(\tau)$ such that $\partial_{n+1}(x) = Z$ $(C_{n+1}(\tau) \subset C_{n+1}(L))$. So choose such an x and define $e_n(\sigma) = x$. Then $\partial_{n+1}^L e_n + e_{n-1}\partial_n^K = t_n - s_n$. So by induction we can construct $e_i : C_i(K) \to C_i(L)$ which show that s, t are chain homotopic, so we have the result.

Suppose $f : |K| \to |L|$ is a map. Now if $s : K^{(r)} \to L$ is a simplicial approximation of f, we have homomorphisms $s_\star : H_n(K^{(r)}) \to H_n(L)$. Suppose $t : K^{(r')} \to L$ is another simplicial approximation; we also have $t_\star : H_n(K^{(r')}) \to H_n(L)$. Then we have natural isomorphisms $H_n(K) \to H_n(K^{(r)})$ and the same for r', so we have an isomorphism $H_n(K^{(r)}) \to H_n(K^{(r')})$ (wlog take $r' \geq r$), and these form a [commuting] diagram with $s_\star, t_\star$. So $f_\star : H_n(K) \to H_n(L) :=$ $s_\star \circ$ the isomorphism $H_n(K) \to H_n(K^{(r)})$ is well defined.

Now suppose $f : |K| \to |L|, f : |L| \to |M|$ are maps. We then have $f_\star : H_n(K) \to H_n(L), g_\star : H_n(L) \to H_n(M)$. Then for $gf : |K| \to |M|$, $(gf)_\star = g_\star f_\star$ by combinatorics [yes, the lecturer really did just say that]. Also, clearly

$f = I \Rightarrow f_\star = I$ [I here means the identity, not the unit interval as it will shortly mean].

Exercise: Suppose K, L are complexes. Then there is an $\epsilon > 0$ (depending only on L) such that if $f, g : |K| \to |L|$ are maps satisfying $d(f,g) < \epsilon$ (where $d(f,g) = \sup\{d(f(x), g(x)) : x \in |K|\}$) then there is a simplicial map $s : K^{(r)} \to L$ which approximates both f and g.

Theorem: If $f, g : |K| \to |L|$ are homotopic maps, then $f_\star = g_\star : H_n(K) \to H_n(L)$: we have a homotopy $F : |K| \times I \to |L|$. So if m is sufficiently large and $f_k(x) := F(x, \frac{k}{m})$ for $0 \le k \le m$, then $d(f_i, f_{i+1}) < \epsilon \forall i$ with $f_0 = f$ and $f_m = g$, so by the exercise $\exists$ simplicial maps $s_i : K^{(r)} \to L$ which each approximate f_i and f_{i+1}. So both s_i and s_{i+1} approximate f_{i+1}; we prooved that this implies s_i, s_{i+1} are contiguous, so $s_{i\star} = s_{i+1\star}$ as maps $H_n(K^{(r)} = H_n(K) \to H_n(L)$. So $f_\star = g_\star : H_n(K) \to H_n(L)$.

Theorem: Let $f : |K| \to |L|$ be a homotopy equivalence. Then $f_\star : H_n(K) \to H_n(L)$ is an isomorphism $\forall n$: let g be the inverse of f. We have $gf \simeq 1_{|K|}, fg \simeq 1_{|L|}$, so $f_\star$ is injective, $g_\star$ surjective, $g_\star$ injective and $f_\star$ surjective, so $f_\star, g_\star$ are isomorphisms.

If X is a space which has a triangulation (a homeomorphism $|K| \to X$) then we can define homology groups by $H_n(X) := H_n(K)$. If $|L| \to X$ is another triangulation then $|K|, |L|$ are homeomorphic, so this is well defined.

If $f : X \to Y$ is a map of triangulable spaces, then $f_\star : H_n(X) \to H_n(Y)$ can be defined by taking triangulations $|K| \to X$ and $|L| \to Y$ and take $f' : |K| \to |L|$ to be the induced map, then define $f_\star = f'_\star : H_n(K) \to H_n(L)$.

Suppose $\sigma = (a_0 \dots a_m)$ is an m-simplex. We already know that $H_n(|\sigma|) = \mathbb{Z}$ if $n = 0, 0$ if $n > 0$. $H_n(|\partial\sigma|) = \mathbb{Z}$ if $n = 0$, 0 for $0 < n < m-1$, and $\mathbb{Z}$ for $n = m-1$. We have $|\partial\sigma| \simeq S^{m-1}$, so $H_n(S^{m-1}) = \mathbb{Z}$ if $n = 0$, 0 if $0 < n < m-1$ and $\mathbb{Z}$ if $n = m-1$. $H_n(S^{m-1}) = 0$ if $n > m-1$, because $\dim \partial\sigma = m-1$.

Corollary: S^m and $S^{m'}$ are not of the same homotopy type unless $m = m'$.

Corollary: $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^{m'}$ exactly when $m = m'$, as if $f : \mathbb{R}^m \to \mathbb{R}^{m'}$ is a homeomorphism then $\mathbb{R}^m \setminus \{0\}$ is homeomorphic to $\mathbb{R}^{m'} \setminus \{f(0)\}$.

Corollary (Brower fixed point theorem): Any map $f : B^m \to B^m$ has a fixed point: suppose $f : B^m \to B^m$ has no fixed point. Define a map $g : B^m \to \partial B^m = S^{m-1}$ by letting $g(x)$ be the intersection of the line joining $x, f(x)$ [with ∂B^m] in the direction of x. We have that if $x \in S^{m-1}$ then $g(x) = x$. So we have a diagram: inclusion $i : S^{m-1} \to B^m$, $g : B^m \to S^{m-1}$ and $1 : S^{m-1} \to S^{m-1}$, and have corresponding maps $i_\star : H_{m-1}(S^{m-1}) \to H_{m-1}(B^m)$, $g_\star$ and $1_\star = 1$. But $H_{m-1}(B^m) = 0$ and $H_{m-1}(S^{m-1}) = \mathbb{Z} \not\subseteq 0$, so we have a contradiction.

Example: consider K to be a "bowtie" of two triangles meeting at a single vertex a_0, not including the interiors. It is homeomorphic to $S^1 \vee S^1$. $H_0(K) = \mathbb{Z}$ since K is connected. $H_1(K) = \frac{Z_1(K)}{B_1(K)}$; we have $0 = C_2(K) \to C_1(K) \xrightarrow{\partial} C_0(K) \to 0$ so $B_1(K) = 0$ and $H_1(K) = Z_1(K)$. If we let the two triangles be $a_0a_1a_2$ and $a_0a_3a_4$ then a typical element $z \in G(K)$ is written as $z = m_1(a_0a_1) + m_2(a_0a_2) + m_3(a_0a_3) + m_4(a_0a_4) + m_5(a_1a_2) + m_6(a_3a_4)$ so $\partial_1(z) = m_1a_1 - m_1a_0 + m_2a_2 - \cdots + m_6a_4 - m_6a_3$; for this to be =0 we must have $m_1 + m_2 + m_3 + m_4 = 0, m_1 = m_5, m_2 = -m_5, m_3 = m_6, m_4 = -m_6$. So there are only two degrees of freedom; given e.g. m_1, m_3 the rest are determined, so $Z_1(K) = \mathbb{Z}^2$ so $H_1(K) = \mathbb{Z}^2$. Now $\pi_1(K) = \mathbb{Z} \star \mathbb{Z}$ [so $H_1(K)$ is the abelianization of this, as we expect].

Similarly, $H_1(S^1 \vee \cdots \vee S^1 \text{ m times}) = \mathbb{Z}^m$.

Example: $K =$the same thing but including the interior of the second triangle. $H_0(K) = \mathbb{Z}, H_1(K) = H_1(\partial\sigma)$ where $\sigma = (a_0a_1a_2) = H_1(S^1) = \mathbb{Z}$, since $K \simeq$ a triangle.

Theorem (Mayer-Vietoris Sequence)

Let K be a complex and $L, M \subset K$ subcomplexes. Then there is an exact sequence $\cdots \to H_n(L \cap M) \overset{\phi_\star}{\to} H_n(L) \oplus H_n(M) \overset{\psi_\star}{\to} H_n(K) \overset{\Delta_\star}{\to} H_{n-1}(L \cap M) \to \ldots$, where $\phi_\star(z) = (i_{1\star}z, -i_{2\star}z), \psi_\star(x, y) = i_{3\star}(x) + i_{4\star}(y)$ for the obvious inclusions i_1, i_2, i_3, i_4.

For example, in the case of the "bowtie" above, taking L, M to be the two triangles so $L \cap M = \{a_0\}$, we have $0 \to H_1(L \cap M) \to H_1(L) \oplus H_1(M) \to H_1(K) \to H_0(L \cap M) \to H_0(L) \oplus H_0(M) \to H_0(K) \to 0$, i.e. $0 \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(K) \overset{\Delta_\star}{\to} \mathbb{Z} \overset{\phi_\star}{\to} \mathbb{Z} \oplus \mathbb{Z} \overset{\psi_\star}{\to} \mathbb{Z} = 0$. $\psi_\star : (x, y) \mapsto x + y$ has $\ker \psi_\star = \{(x, -x)\} = \{(la_0, -la_0) : l \in \mathbb{Z}\} \simeq \mathbb{Z}$, so $\mathrm{Im}\phi_\star \simeq \mathbb{Z}$, so $\ker \phi_\star = 0$. Thus $\mathrm{Im}\Delta_\star = \ker \phi_\star = 0$. So we get an exact sequence $0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(K) \to 0$. So $H_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$.

Proof of the theorem: We have an exact sequence $0 \to C_n(L \cap M) \overset{\phi_n}{\to} C_n(L) \oplus C_n(M) \overset{\psi_n}{\to} C_n(K) \to 0$ by $\phi_n(z) = (i_{i\star}(z), -i_{2\star}(z)), \psi_n(x, y) = i_{3\star}(x) + i_{4\star}(y)$. Clearly $\psi_n\phi_n = 0 : z \mapsto (z, -z) \mapsto z + (-z) = 0$. Everything is clear; we just need to check that $\ker \psi_n = \mathrm{Im}\phi_n$: suppose $\psi_n(x, y) = 0$. Write $x = x' + x'', y = y' + y''$, with x' an n-chain in $L \cap M$ but every term in x'' coming from $L \setminus (L \cap M)$ [and similarly for y]. Then $\psi_n(x, y) = \psi_n(x' + x'', y' + y'') = i_{3\star}(x') + i_{3\star}x'' + i_{4\star}(y') + i_{4\star}(y'')$. If this $= 0$ we thus have that $i_{3\star}(x') + i_{4\star}(y') = 0$ (since this is the only part in $L \cap M$) and similarly $i_{3\star}(x'') = i_{4\star}(y'') = 0$ so $x'' = 0, y'' = 0, y' = -x'$ i.e. $(x, y) = (x', -x') \in \mathrm{Im}\phi_n$.

We thus have a commutative diagram in which rows, but not columns, are exact:

$$
\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \to & C_{n+1}(L \cap M) & \to & C_{n+1}(L) \oplus C_{n+1}(M) & \to & C_{n+1}(K) & \to & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
\ldots & \to & C_n(L \cap M) & \to & C_n(L) \oplus C_n(M) & \to & C_n(K) & \to & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
\ldots & \to & C_{n-1}(L \cap M) & \to & C_{n-1}(L) \oplus C_{n-1}(M) & \to & C_{n-1}(K) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}
$$

Another way of writing this is $0 \to \mathcal{C}(L \cap M) \to \mathcal{C}(L) \oplus \mathcal{C}(M) \to \mathcal{C}(K) \to 0$; there is a general theorem in homological algebra which says that for any short exact sequence of chain complexes we get an exact sequence like that in the statement of this theorem.

We define $\Delta_\star$ as follows: suppose $z \in H_n(K) = \frac{Z_n(K)}{B_n(K)}$, and identify z with some representative $z \in Z_n(K) \subset C_n(K)$. Take $z' \in C_n(L) \oplus C_n(M)$ such that $\psi_\star(z') = z$ (we can do this because our rows are exact), so $\partial z' \in C_{n-1}(L) \oplus C_{n-1}(M)$. We see that $\psi_{n-1}\partial(z') = \partial\psi_n(z') = \partial z = 0$ as $z \in Z_n(K)$, so $\partial z' \in \ker \psi_{n-1}$; by exactness, $\partial z' \in \mathrm{Im}\phi_{n-1} \therefore \exists! z'' \in C_{n-1}(L \cap M)$ such that $\phi_{n-1}(z'') = \partial z'$ (this is unique since ϕ_{n-1} is injective). We define $\Delta_\star(z)$ to be this z'', considered as an element of $H_{n-1}(L \cap M)$.

We have $\phi_{n-2}\partial z'' = 0 \Leftrightarrow \partial z'' = 0$. $\phi_{n-2}\partial z'' = \partial\phi_{n-1}z'' = \partial\partial z' = 0$, so $z'' \in Z_{n-1}(L\cap M)$. So this definition is valid - we can consider z'' as an element of $H_{n-1}(L\cap M) = \frac{Z_{n-1}(L\cap M)}{B_{n-1}(L\cap M)}$. We need to prove that $\Delta_\star(z)$ is independent of our choice of z, z', z''; assume we have chosen $\bar{z}, \bar{z}', \bar{z}''$ instead. Then $z = \bar{z} \in H_n(K)$, i.e. $z - \bar{z} \in B_n(K)$ as elements of $Z_n(K)$, so $\exists u \in C_{n+1}(K)$ such that $z - \bar{z} = \partial u$. Again by exactness of rows in our diagram, $\exists v \in C_{n+1}(L) + C_{n+1}(M)$ such that $u = \psi_{n+1}(v)$; $\partial\psi_{n+1}(v) = z - \bar{z}$, so $\psi_n\partial v = z - \bar{z}$. Now if we calculate $\psi_n(z' - \bar{z}' - \partial v) = \psi_n(z') - \psi_n(\bar{z}' - \psi_n\partial v) = z - \bar{z} - (z - \bar{z}) = 0$, so $z' - \bar{z}' - \partial v \in \ker\psi_n \Rightarrow \exists w \in C_n(L\cap M)$ such that $\phi_n(w) = z' - \bar{z}' - \partial v$. $\partial\phi_n(w) = \partial(z' - \bar{z}' - \partial v) = \partial z' - \partial\bar{z}' - \partial\partial v = \partial z' - \partial\bar{z}' = \phi_{n-1}(z'') - \phi_{n-1}(\bar{z}'')$, but also $\partial\phi_n w = \phi_{n-1}\partial w = \phi_{n-1}(z'') - \phi_{n-1}(\bar{z}'') - \phi_{n-1}(z'' - \bar{z}'')$ (ϕ is linear). Therefore, since ϕ_{n-1} is injective, $\partial w = z'' - \bar{z}'' \Rightarrow z'' - \bar{z}'' \in B_{n-1}(L\cap M)$, so $z'' = \bar{z}''$ as elements of $H_{n-1}(L\cap M)$, so $\Delta_\star$ is well defined.

This proof method is called "diagram chasing". To prove that the sequence we get is actually exact, we can use the same arguments - the reader may finish this as an exercise, or just accept it as a fact.

Examples: $X = S^1 \times I$ has $X \simeq S^1 \therefore H_n(X) = H_n(S^1) = \mathbb{Z}$ for $n = 0, 1$, 0 for $n \geq 2$.

Example: $X = T = S^1 \times S^1$; take L to be a (triangulation of a) "strip", a cylinder made of one section of the "tube" of the torus, and M to be the tube the rest of the way around, so the intersection $|L| \cap |M|$ is two disjoint circles. By MV we have an exact sequence $0 \to H_2(L\cap M) \to H_2(L) \oplus H_2(M) \to H_2(X) \to H_1(L\cap M) \to H_1(L) \oplus H_1(M) \to H_1(X) \to H_0(L\cap M) \to H_0(L) \oplus H_0(M) \to H_0(X) \to 0$, which since $|L| \simeq S^1, |M| \simeq S^1$ is $0 \to 0 \to 0 \oplus 0 \to H_2(X) \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to H_1(X) \xrightarrow{\Delta_\star} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\phi_\star} \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to 0$. Then $\mathrm{Im}\phi_\star = \ker(\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}) \simeq \mathbb{Z} \therefore \ker\phi_\star \simeq \mathbb{Z} \simeq \mathrm{Im}\Delta_\star$ so we get $0 \to H_2(X) \to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}\oplus\mathbb{Z} \to H_1(X) \to \mathbb{Z} \to 0$. α comes from $H_1(L\cap M) \to H_1(L) \oplus H_1(M)$. $L \cap M$ is two circles, so $H_1(L\cap M)$ is generated by the cycles x, y around them; then L is a cylinder with ends the two circles, so we have $H_1(L\cap M) \to H_1(L)$ by $mx + ny \mapsto (m+n)x$, and similarly for M. So $\mathrm{Im}\alpha \simeq \mathbb{Z}$ so $0 \to H_2(X) \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \to H_1(X) \to \mathbb{Z} \to 0$ are exact, so $H_2(X) = \mathbb{Z}, H_1(X) = \mathbb{Z}\oplus\mathbb{Z} = \pi_1(X)$ as always.

Example: $X = \mathbb{RP}^2$, the disc S^2 where we identify opposite boundary points. We can quite easily take a triangulation and calculate the homology groups directly from this, but we want to use M-V. Set $L \subset X$ to be (the image of) a triangle inside S^2, $M = X \setminus L^0$. We have $L \simeq$ a single point, $M \simeq$ the quotient of the boundary of the disc (by projection from a point in L). So we have a deformation retraction $M \to \frac{S^1}{\sim} = \mathbb{RP}^1 \simeq S^1$. $L\cap M =$ the (topological) boundary of $L \simeq S^1$. So M-V exact sequence is $0 \to 0 \to 0 \oplus 0 \to H_2(X) \xrightarrow{\Delta_\star} \mathbb{Z} \xrightarrow{\phi_\star} 0 \oplus \mathbb{Z} \to H_1(X) \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to 0$. So $\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$ is injective so $H_1(X) \to \mathbb{Z}$ must be surjective, so we can write the end of the sequence as $\cdots \to H_1(X) \to 0$ (this will always be the case if all the spaces are connected). $\phi_\star : H_1(L\cap M) \to H_1(L) \oplus H_1(M) = H_1(M)$ i.e. $H_1(S^1) \to H_1(\mathbb{RP}^1)$. This will be the same as the corresponding homomorphism $\pi_1(S^1) \to \pi_1(\mathbb{RP}^1)$, which we saw on the example sheet is $n \mapsto 2n$. So $\mathrm{Im}\phi_\star = 2\mathbb{Z}$ and we have a short exact sequence $0 \to H_2(X) \to \mathbb{Z} \to \mathrm{Im}\phi_\star \simeq \mathbb{Z} \to 0$. Now since $\mathrm{Im}\phi_\star = 2\mathbb{Z}$, $\Delta_\star$ is injective. But $\mathrm{Im}\Delta_\star = \ker(\mathbb{Z} \to \mathrm{Im}\phi_\star) = 0$, so $H_2(X) = 0$. Also we have $\mathrm{Im}\phi_\star \to \mathbb{Z} \to H_1(X) \to 0$; $\mathrm{Im}\phi_\star = 2\mathbb{Z}$ so $H_1(X) = \frac{\mathbb{Z}}{2\mathbb{Z}} = \mathbb{Z}_2 = \pi_1(X)$.

We know $H_0(T) = \mathbb{Z}, H_1(T) = \mathbb{Z}^2, H_2(T) = \mathbb{Z}$, and $H_0(\mathbb{RP}^2) = \mathbb{Z}, H_1(\mathbb{RP}^2) = \mathbb{Z}_2, H_2(\mathbb{RP}^2) = 0$.

Reminder: An n-manifold is a Hausdorff space X such that each $x \in X$ has a neighbourhood homeomorphic to $\mathbb{R}^n$. A surface is a 2-manifold. A closed surface is [the lecturer was utterly incomprehensible here].

Theorem (Classification of closed surfaces): Every connected closed surface is homeomorphic to one of 1) S^2 2) $M_g = T\#\dots\#T$ g times (recall $X\#Y$ is given by removing the interior of a small disc from each of X, Y, then identify the boundaries of the removed discs by $x \sim h(x)$ for some homeomorphism $h : S^1 \to S^1$) or 3) $N_g = \mathbb{RP}^2\#\dots\#\mathbb{RP}^2$ g times. 1) and 2) are orientable, but 3) is not orientable (a closed surface X is orientable if we can triangulate X and orient each triangle in a compatible way)

Note: $T\#\mathbb{RP}^2 = \mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2, \mathbb{RP}^2\#\mathbb{RP}^2 = K$ the Klein bottle.

Example: $H_0(S^2) = \mathbb{Z}, H_1(S^2) = 0, H_2(S^2) = \mathbb{Z}$.

Example: M_g can be realized as the quotient of a 4g-gon where we identify edge n in the clockwise direction with edge $n+2$ in the anticlockwise direction for $n \equiv 1, 2 \bmod 4$. Take α an interior circle, γ a path from the "first" vertex to a point of α. Then $\gamma\alpha\gamma^{-1} \simeq$ the boundary of the 4g-gon, so $\frac{\gamma\alpha\gamma^{-1}}{\sim}$ is the quotient of the boundary. Take a triangulation of our 4g-gon which gives a triangulation of the quotient (this is always possible). To calculate the homology groups we take L to be a triangle in the interior of P, $K = M_g \setminus L^0$. $L \simeq$ a single point, $L \cap K \simeq S^1$, $K \simeq S^1 \vee \cdots \vee S^1$ $2g$ times (see the similar exercise when we were applying the Van Kampen theorem). Then we have an exact sequence $0 \to 0 \to 0 \oplus 0 \to H_2(M_g) \xrightarrow{\Delta_\star} \mathbb{Z} \xrightarrow{\phi_\star} 0 \oplus \mathbb{Z}^{2g} \xrightarrow{\psi_\star} H_1(M_g) \to 0$. Since M_g is orientable, we can orient all the triangles in a compatible way, i.e. such that if σ_i are our triangles then $\partial(\sum \sigma_i) = 0$. So $Z_2(M_g) \neq 0$ since $\sum \sigma_i \in Z_2(M_g)$. But $B_2(M_g) = 0$ as teh surface is 2D, so $H_2(M_g) = \frac{Z_2(M_g)}{B_2(M_g)} \neq 0$. So $\mathrm{Im}\Delta_\star \neq 0 \Rightarrow \ker\phi_\star \neq 0$. Since $\mathbb{Z}^{2g}$ is a free group, $\mathrm{Im}\phi_\star = 0$ or $\mathrm{Im}\phi_\star \simeq \mathbb{Z} \Rightarrow \ker\phi_\star = 0$, so $\mathrm{Im}\phi_\star = 0 \Rightarrow \Delta_\star$ is surjective, so an isomorphism. So we se $H_2(M_g) = \mathbb{Z}$; similarly $\psi_\star$ is an isomorphism and $H_1(M_g) = \mathbb{Z}^{2g}$.

Facts: Suppose that $0 \to G' \to G \to G'' \to 0$ is a short exact sequence of finitely generated abelian groups. Then $\mathrm{rank}G = \mathrm{rank}G' + \mathrm{rank}G''$. Let $0 \to G_m \to G_{m-1} \xrightarrow{\theta} \cdots \to G_1 \to 0$ be an exact sequence of finitely generated abelian groups, then $\mathrm{rank}G_m - \mathrm{rank}G_{m-1} + \mathrm{rank}G_{m-2} - \cdots = 0$; we have $0 \to G_m \to G_{m-1} \to \mathrm{Im}\theta \to 0$ and $0 \to \mathrm{Im}\theta \to G_{m-2} \to G_{m-3} \to \cdots \to G_1 \to 0$, so this follows from the previous fact by induction.

Recall $N_g = \mathbb{RP}^2\#\dots\#\mathbb{RP}^2$ g times. We will use the MV exact sequence again; we have $N_g = \mathbb{RP}\#N_{g-1}$. To aid computation, set $N'_g = N_g \cup$ the interior of the disc which was removed when taking the $\#$, so $N_g = N'_g \setminus \mathrm{disc}^0$. Set $L = \mathbb{RP}^2, K = N_{g-1}$ so $N'_g = L \cup K, L \cap K = B^2 \simeq$ a single point. By MV we have an exact sequence $0 \to 0 \to 0 \oplus H_2(N_{g-1}) \to H_2(N'_g) \to \mathbb{Z}_2 \oplus H_1(N_{g-1}) \to H_1(N'_g) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$. Suppose that we have computed $H_1(N_{g-1}) = \mathbb{Z}_2 \oplus \mathbb{Z}^{g-2}, H_2(N_{g-1}) = 0$ (this is true for $g = 1$, so the induction starts). Then we get $0 \to H_2(N'_g) \to 0 \to \mathbb{Z}_2^2 \oplus \mathbb{Z}^{g-2} \to H_1(N'_g) \to 0$. Thus $H_2(N'_g) = 0$ and $H_1(N'_g) = \mathbb{Z}_2^2 \oplus \mathbb{Z}^{g-2}$.

Again apply MV, now to $L = \mathbb{RP}^2, K = N_g$ so $L \cap K = \mathbb{RP}^2 \setminus (B^2)^0, L \cup K = N'_g$. $\mathbb{RP}^2 \setminus (B^2)^0$ is the quotient of a square with the interior of a disc removed by identifying opposite sides in a clockwise direction, so $\simeq$ the quotient of the boundary of said square which is $\mathbb{RP}^1$. So the sequence is $0 \to 0 \to 0 \oplus H_2(N_g) \to 0 \to \mathbb{Z} \to \mathbb{Z}_2 \oplus H_1(N_g) \to \mathbb{Z}_2^2 \oplus \mathbb{Z}^{g-2} \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$. So $H_2(N_g) = 0$ and $H_1(N_g)$ must be $\mathbb{Z}^{g-1}$ or $\mathbb{Z}_2 \oplus \mathbb{Z}^{g-1}$. To distinguish between these two we could calculate the homomorphism $\mathbb{Z}_2^2 \oplus \mathbb{Z}^{g-2} \to \mathbb{Z}$ explicitly and find its kernel, but instead we will apply MV again, this time to $L = B^2, K = N_g, K \cap L = S^1, K \cup L = N_g'$. Then by MV we have $0 \to 0 \to 0 \oplus 0 \to 0 \to \mathbb{Z} \to 0 \oplus H_1(N_g) \to \mathbb{Z}_2^2 \oplus \mathbb{Z}^{g-2} \to 0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$. So combining these, $H_1(N_g) = \mathbb{Z}_2 \oplus \mathbb{Z}^{g-1}$.

Homology of closed surfaces: 1) $H_0(S^2) = \mathbb{Z}, H_1(S^2) = 0, H_2(S^2) = \mathbb{Z}$. 2) $H_0(M_g) = \mathbb{Z}, H_1(M_g) = \mathbb{Z}^{2g}$, $H_2(M_g) = \mathbb{Z}$. 3) $H_0(N_g) = \mathbb{Z}, H_1(N_g) = \mathbb{Z}_2 \oplus \mathbb{Z}^{g-1}, H_2(N_g) = 0$. Recall 1, 2 are orientable, but 3. non-orientable, so we have:

Corollary: a closed surface is orientable iff $H_2 = \mathbb{Z} \Leftrightarrow H_2 \neq 0$.

Relation of homolo with the fundamental group

Euler characteristic:

Definition: Let K be a complex. The Euler characteristic is defined as $\chi(K) = \#\text{vertices} - \#\text{edges} + \#\text{triangles} - \dots$.

Theorem (Euler-Poincaré): Let $X = |K|$, K a complex. Then $\chi(X) = \sum_0^n (-1)^i \beta_i$ where $n = \dim K, \beta_i = \operatorname{rank} H_i(X)$. In particular, since the H_i are invariant under homotopy equivalence, $\chi(X) = \chi(Y)$ if $X \simeq Y$ (and in particular if $X = |K| = |L|$ then $\chi(K) = \chi(L)$). Example: $X = S^1$; $\chi(X) = \beta_0 - \beta_1 = 1 - 1 = 0$. $X = S^2$ has $\chi(X) = \beta_0 - \beta_1 + \beta_2 = 1 - 0 + 1 = 2$. $X = S^1 \vee \dots \vee S^1$ n times has $\chi(X) = \beta_0 - \beta_1 = 1 - n$.

Definition: A graph Γ is a connected complex of dimension ≤ 1. A graph Γ is a tree if it is contractible.

Theorem: Let Γ be a graph. Then $\chi(\Gamma) \leq 1$ with equality exactly when Γ is a tree: by definition $\chi(\Gamma) = \beta_0 - \beta_1 = 1 - \beta_1 \leq 1$. If Γ is a tree then $\chi(\Gamma) = 1 - 0 = 1$. If Γ is a graph such that $\chi(\Gamma) = 1$ then $\beta_0 - \beta_1 = 1 = v - e$ where v is the number of vertices, e the number of edges. So there is a vertex $x \in \Gamma$ which belongs to at most one edge; define Γ' to be Γ with x and any edge containing it removed. Put v^prime to be the number of vertices of Γ', e' to be the obvious thing, then $\chi(\Gamma') = v' - e' = v - 1 - (e - 1)$ (now we neglect the silly case where x is not in an edge [lolecturer]) $= v - e = 1$, by induction Γ' is a tree so Γ is a tree.

We have now finished the examinable part of this course.